Math 584, Problem Set #4
(due in class Mon., 12/6/10)

You may use computers for any and all of these problems, but you should include explicit description of the steps you took and the code you entered.

Also: You must write down the number of minutes you spent working on each problem. This will help me keep the work-load at an appropriate level.

A. Consider the random walk on $\{0, 1, 2, 3, ...\}$ starting from 1 with absorption at 0, in which $p_{ij} = 1/2$ if $j = i-1$ or $j = i+2$ and $p_{ij} = 0$ otherwise. (a) Using the method we used in class to study one-dimensional random walk, show that the probability of absorption p_a is either 1 or $(\sqrt{5}-1)/2 \approx .608$. (b) Do a pseudorandom simulation of the random walk, with a cut-off at 100. Which value of p_a does your experiment support?

B. In class we saw how we can use the simple random walk on $\{1, 2, 3\}$ (with stationary probability measure $(1/4, 1/2, 1/4)$) to derive a Metropolis chain with stationary measure $(1/3, 1/3, 1/3)$; this derived Markov chain turned out to be nothing more than semi-reflecting random walk on $\{1, 2, 3\}$. Turn this around by using the semi-reflecting random walk on $\{1, 2, 3\}$ (with stationary probability measure $(1/3, 1/3, 1/3)$) to derive a Metropolis chain with stationary measure $(1/4, 1/2, 1/4)$. Check your answer by verifying directly that $(1/4, 1/2, 1/4)$ is a stationary measure for your Metropolis chain.

C. Show that

$$||\pi - \pi'||_{\mathrm{TV}} := \frac{1}{2} \sum_{s \in S} |\pi(s) - \pi'(s)|$$

is also equal to the maximum of $|\pi(E) - \pi'(E)|$ over all subsets E of S.

D. Consider a semireflecting random walk on $\{1, 2, 3, 4\}$ with $p_{i,\max(i-1,1)} = p_{i,\min(i+1,4)} = \frac{1}{2}$ for all i, implemented via a coin toss: when the particle is in site i, toss a coin, and move the particle to site $\max(i-1, 1)$ if the coin comes up heads and move the particle to site $\min(i+1, 4)$ if the coin comes up tails. Suppose two particles execute this walk simultaneously starting from 1 and 4 respectively. Assume the coin used by the first particle is independent of the coin used by the second particle. Let the random variable X be the number of steps each of the two particles takes until they both arrive at the same site for the first time. Use linear algebra to compute the expected value of X exactly, and compare with the results of a 1000-run pseudorandom simulation.

E. Repeat problem D, but this time, assume that the two particles use the *same* coin at each step (instead of *independent* coins). That is, if one particle is at i and the other at j, then with probability $\frac{1}{2}$ the first moves to site $\max(i-1,1)$ and the second moves to site $\max(j-1,1)$ and with probability $\frac{1}{2}$ the first moves to site $\min(i+1,4)$ and the second moves to site $\min(j+1,4)$.

F. Repeat problem E, so that once again, the second particle uses the same coin as the first particle, but this time, the second particle uses the *opposite* side of the coin. That is, if one particle is at i and the other at j, then with probability $\frac{1}{2}$ the first moves to site $\max(i-1,1)$ and the second moves to site $\min(j+1,4)$ and with probability $\frac{1}{2}$ the first moves to site $\min(i+1,4)$ and the second moves to site $\max(j-1,1)$.

G. Consider the pinned steppingstone model on $\{1,2,3,4,5\}$. Site 1 is pinned to color 0 and site 5 is pinned to color 1; sites 2, 3, and 4 can each have color 0 or color 1. At each time step, we roll a 6-sided die to pick one of the six pairs (i,j) with $2 \leq i \leq 4$ and $j = i \pm 1$ and we change the color of site i to the color of site j. Describe this as an 8-state Markov chain. What are the transient states, what are the recurrent states, and what is the stationary distribution?