$\pi/8$ by way of ∞

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(with thanks to Hal Canary, Matt Cook, Ander Holroyd, Dan Hoey, Michael Kleber, Lionel Levine, Ed Pegg, Yuval Peres, and Francis Wong)

Note: My email address in the program book is wrong; my correct email address is "jamespropp at gmail dot calm".

1. *The first ten minutes of my half-hour talk, in one slide.*

You'll find my Eight Rotors Puzzle in your gift bags, accompanied by an article written for this occasion, explaining how to "turn **M** into **G**".

I plan to submit an expanded version of this article to the next G4G book; the new version will contain (among other things) your comments and ideas.

In particular, I'd like to know:

(a) What's the hardest eight-rotors puzzle of this kind?

(b) Can one build the puzzle in a way that enforces the rules mechanically?

3. *The middle ten minutes, in five slides (plus a Java applet).*

A bug crawls in an infinite grid, visiting different sites (i, j), each equipped with a rotor pointing north, south, east, or west (initially pointing away from $(0, 0)$).

Each time the bug arrives at the black square (1,1) (the **sink**), it goes to (0,0) next.

Otherwise, the bug decides where to go using the **rotor-router rule**: the arrow at the current site is rotated 90 degrees clockwise, and the bug moves one step in the direction of the arrow.

Whenever the bug arrives at (0,0) from one of its neighbors, or arrives at (1,1), we say a **stage** has ended.

Run the procedure for n stages, and let m be the number of times a stage ended with the bug being "captured at the sink" (landing on the black square).

Then m/n goes to $\pi/8$ as n goes to ∞.

In fact, $m - n\pi/8$ isn't just small compared to n; it's at most a constant times $\log n$.

Indeed, for n up through 10^4, m/n is (more often than not) the **best** approximation to $\pi/8$ with denominator n.

Example: When n is 904, m is 355 (so $m/n = (1/8) \times (355/113)$).

So this "computer" is computing $\pi/8$, with increasing precision as its running time goes to ∞.

My proof that m/n goes to $\pi/8$ uses NO structural understanding of the picture; instead, it uses numerical arguments inspired by probability theory.

(In fact, the whole design of the machine was inspired by properties of **random** walk in the grid; the rotor-router mechanism is a way of **derandomizing** random walk.)

The four-colored picture is all the memory the system has; in some mysterious sense, the picture represents how this strange sort of computer is “thinking”.

Indeed, the settings of the rotors is EVERYTHING, since it’s the only thing that changes from one stage to the next.

On the other hand, the settings of the rotors is NOTHING!

After any number of stages, “lobotomize” this computer: reset any finite subset of the rotors any way you like.

It’s still the case that m/n will go to $\pi/8$!

More specifically, resetting all the rotors out to distance r from $(0, 0)$ will have the effect of perturbing each m by at most a constant times $\log r$. Since r is fixed while m and n go to ∞, this won’t change the limit of m/n.

3. *The last ten minutes, in one slide (plus some pictures and a Java applet).*

See `http://www.mathpuzzle.com/29Jun2003.html`.

The 8-petaled flower in the middle becomes less pretty, but starts to hint at the nature of its secrets, if we treat the big circle as the unit circle in the complex plane and then apply the conformal map $z \mapsto 1/z^2$; see `http://math.berkeley.edu/~levine/gallery/invertedrotor1m15x.png`.

See `http://jamespropp.org/quasirandom.html` for more info on rotor-routers and related things.