

Alternating Sign Matrices and Aztec Diamond Tilings

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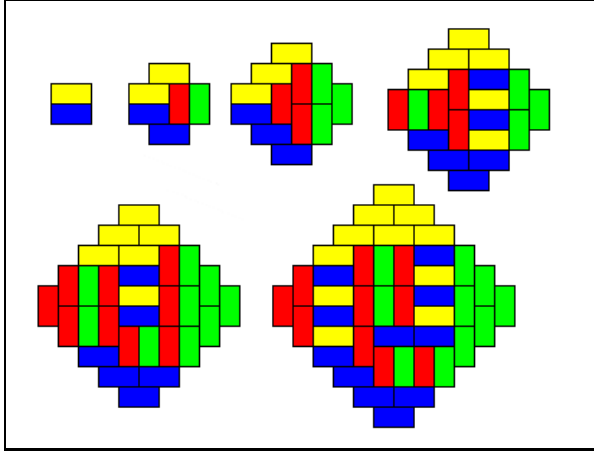


Figure 1: Aztec Diamonds of orders 1 – 6.

1 TOAD Background

1.1 Dominos and Aztec Diamonds

The objects in Figure 1 are called Aztec Diamonds: the first one is of order 1, the second one is of order 2, and the others are of orders 3, 4, 5, and 6. Each of these Aztec Diamonds have been tiled with dominos. A domino is a one-by-two rectangle.

For an order n Aztec Diamond, there are $2^{(n(n+1)/2)}$ different ways to tile it. We call each way a Tiling Of an Aztec Diamond (TOAD). There is a simple algorithm to produce a random TOAD of any size. The idea is to grow a random TOAD from another random TOAD one size smaller.

1.2 TOAD Shuffling

Figure 2 demonstrates this process. First, we start with an order 1 Aztec Diamond. It can be tiled either with two vertical or two horizontal dominos. We tile it with two vertical dominos. We color them red and green. [step 1] To fill a size 2 Aztec diamond, we move the red domino to the left and the green domino to the right. [step 2] This leaves two 2×2 squares

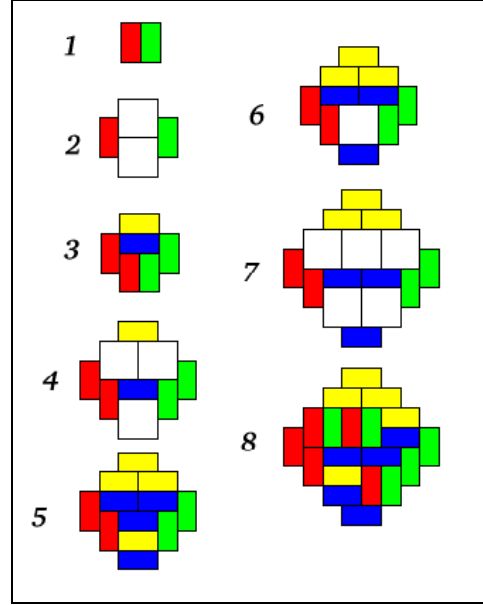


Figure 2: The Domino Shuffling algorithm at work.

open in the resulting order 2 diamond. We make two random choices for how to tile those two squares. In our example, we tile the top one with two horizontal (yellow and blue) dominos and the bottom one with two vertical dominos. [step 3] Move the red dominos to the left, green dominos to the right, yellow dominos up, and blue dominos down. [step 4]

This leaves us with three 2×2 holes to be filled. Make three random decisions about how to fill them, and we have an order 3 TOAD. [step 5] This time, before we try to move the dominos we see that there is a blue domino immediately above a yellow domino. These two dominos will collide the next time we try to move them! To make the algorithm work out correctly, we simply delete any two dominos that are about to collide. [step 6] Now, we can move [step 7] and fill [step 8] to make a size 4 TOAD.

To make an arbitrarily large TOAD, we can simply continue this process: delete, move, and fill. This is called domino shuffling. We have written a Java applet to demonstrate.

2 Alternating Sign Matrices

2.1 Alternating Sign Matrices (ASMs)

An $n \times n$ matrix A is an alternating sign matrix if the following conditions are satisfied:

1. A is a matrix of 0's, 1's, and -1's;
2. The entries in each row or column sum to 1;
3. The nonzero entries in each row or column alternate in sign.

Examples:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The number of ASMs of order n , A_n , is given by the following formula (the sequence A_n starts 1, 2, 7, 42, 429, 7436, ...):

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

2.2 Relationship of ASMs to TOADs

Look at the vertices in an Aztec Diamond that has been rotated 45 degrees, so that it is resting on its side. Look at the vertices in the first row that are not part of the boundary. Count the number of edges that meet at each vertex. This number will be a 2, 3, or 4. Subtract 3 from the number (this will yield a -1, 0, or 1). Do this for each vertex in the first row and then skip a row and repeat the steps. This procedure

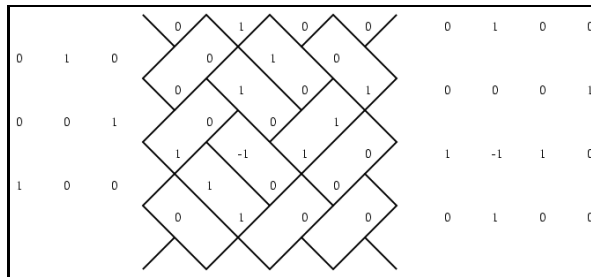


Figure 3: How to produce two ASMs from a TOAD.

will produce the first, larger, outer ASM, which corresponds to the Aztec Diamond. Notice that in the method described we skipped every other row. The skipped rows are used to produce the second, smaller, inner ASM, which also corresponds to the given Aztec Diamond. Use a similar method to find the second ASM. Count the number of edges that meet at each vertex. This number will be a 2, 3, or 4. Subtract 3 from the number (this will yield a -1, 0, or 1). Now notice that for any given Aztec Diamond of order n , there is a pair of ASMs that correspond to that specific tiling: one of size n and another of size $n + 1$. See Figure 3.

2.3 Densely Packed Flux Line Models (DPFLs)

A DPFL consists of a tic-tac-toe graph in which every other external edge is numbered from 1 to $2n$. We color the external edges alternately blue and green, and define a “legal blue-green coloring” as a way of coloring the remaining edges of the graph so that at each vertex there are two edges of each color. (Here we represent different colors with thick and thin lines). It can be shown that these colorings are in one-to-one correspondence with ASMs of order n .

A property of interest to us DPFLs is that their paths (of the same color) never cross each other. See Figure 4.

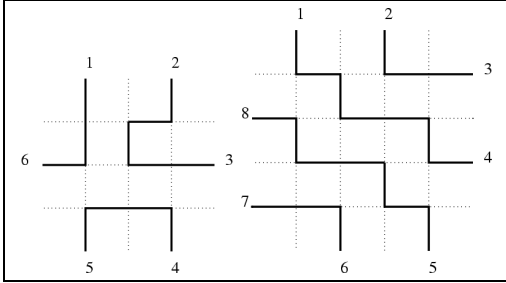


Figure 4: Two DPFLs.

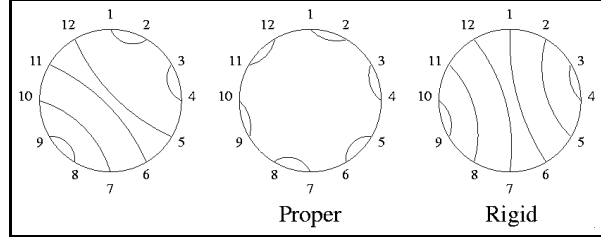


Figure 5: DPFL Linkings.

3 Interesting Things

1234567 \rightarrow 6457123

3.1 DPFL Linking Probabilities

We are exploring how many different DPFLs there are that have a specific pairing of endpoints. There already exists a conjecture concerning the proper pairing of endpoints, that each endpoint is paired with its neighbor. There is also a conjecture about the rigid pairing (the rigid pairing is the pairing in which a given endpoint is paired with the endpoint furthest away from it; it is shown in the following figures). This conjecture states that the rigid pairing corresponds to only one ASM of a specific type. Other types of pairings of endpoints are also being explored, such as shown in the figures below. No conjectures have been formulated yet on these other pairings.

When representing a DPFL it is sometimes easier to draw a circle and number the 'endpoints' around the circle, when the focus lies mainly in which endpoint is paired with which, not in what the exact path looks like between them. The circles in Figure 5 are an example of this.

3.2 Baxter Permutations

Given a set of numbers, we can rearrange, or permute the numbers so they appear in a different order. For example, given the seven numbers 1 2 3 4 5 6 7, we can permute them as follows:

Certain rearrangements have special properties. In the late 1950's and early 1960's, a special type of permutation (called a Baxter permutation) arose in the study of commuting functions. A Baxter permutation can be defined as follows:

Definition: A permutation P of $S_n = 1, 2, 3, \dots, n$ is a Baxter permutation of order n if and only if, for all integers $m \in [n - 1]$, P can be uniquely factorized by:

$$P = P' . m . P^{<} . P^{>} . (m + 1) . P'' \text{ or}$$

$$P = P' . (m + 1) . P^{>} . P^{<} . m . P''$$

where all the numbers in $P^{<}$ (respectively $P^{>}$) are less than m (respectively greater than m_1), and P' and P'' are numbers outside of m and $m + 1$ which we can ignore.

For example, the permutation $1234567 \rightarrow 4326571$ is a Baxter permutation of order 7. To see this, notice that for $m = 4$, we have $P^{>} = 23$ and $P^{<} = 6$. For $m = 1$, we have $P^{>} = 3657$ and $P^{<} = \emptyset$. We can do this for all m from $m = 1$ to $m = 6$.

4 Conjectures

4.1 Proposed Relation between Baxter Permutations and Aztec Diamonds

If a tiling of an Aztec Diamond of order n corresponds to a pair of Alternating Sign Matrices (ASMs) with no -1 's, then we propose that the total number of such tilings is equal to the total number of order $n+1$ Baxter permutations. These tilings are called “well-behaved.”

In other words, we propose that the number of well-behaved tilings of an order n Aztec Diamond is equal to the number of order $n+1$ Baxter permutations, which Chung et. al. proved to be:

$$\sum_{r=0}^n \frac{\binom{n+2}{r} \binom{n+2}{r+1} \binom{n+2}{r+2}}{\binom{n+2}{1} \binom{n+2}{2}}$$

4.2 The ASM Shuffling Conjecture

If we take an order n TOAD, it defines a compatible pair of ASMs, as previously said, one of order n and one of order $n+1$ (small and large). If we then shuffle the tiling into an order $n+1$ TOAD, the small ASM of the resulting tiling is of order $n+1$. It has been observed but not yet proven that the small ASM of the larger tiling matches the large ASM of the smaller tiling.

If true, this means that the small ASM of a TOAD obtained through shuffling is known prior to the shuffling.

4.3 $A(n-1)$ Conjecture

As previously noted, ASMs can be viewed as DPFLs. DPFLs have paths that wander about on a grid of size $n \times n$. If we number the $4n$ tails clockwise in an alternating way, then we can ask the following question: How many ASMs are there whose DPFL has paths connecting 1 to 2, 3 to 4, 5 to 6, and so on.

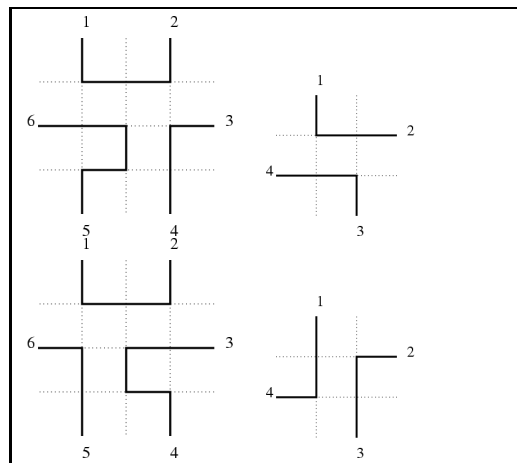


Figure 6: The two properly linking DPFLs of order 3, and the two DPFLs of order 2.

Paths that connect in this way are called “properly linking”. The first several cases seem to suggest that the number of properly linking paths on an $n \times n$ DPFL (or ASM) is the same as the number of DPFLs (or ASMs) of one small size. We write this in the following way: $A^*(n) = A(n-1)$. That is, the number of properly linking ASMs (or DPFLs) of size $n \times n$ is the same as the number of total DPFLs (or ASMs) of size $(n-1) \times (n-1)$. See Figure 6

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