A formal approach to Jim Propp's *Self-Referential Aptitude Test*

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Problem statement Below is the test by Jim Propp (`propp@math.wisc.edu`) from *Math Horizons*, (Feb. 2005)[1]; we renumbered the questions to range from 0 to 19.

0. The first question whose answer is B is question
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4
1. The only two consecutive questions with identical answers are questions
   (A) 5 and 6 (B) 6 and 7 (C) 7 and 8 (D) 8 and 9 (E) 9 and 10
2. The number of questions with the answer E is
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4
3. The number of questions with the answer A is
   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8
4. The answer to this question is the same as the answer to question
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4
5. The answer to question 16 is
   (A) C (B) D (C) E (D) none of the above (E) all of the above
6. Alphabetically, the answer to this question and the answer to the following one are
   (A) 4 apart (B) 3 apart (C) 2 apart (D) 1 apart (E) the same
7. The number of questions whose answers are vowels is
   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8
8. The next question with the same answer as this one is question
   (A) 9 (B) 10 (C) 11 (D) 12 (E) 13
9. The answer to question 15 is
   (A) D (B) A (C) E (D) B (E) C
10. The number of questions preceding this one with the answer B is
    (A) 0 (B) 1 (C) 2 (D) 3 (E) 4
11. The number of questions whose answer is a consonant is
    (A) even (B) odd (C) a perfect square (D) a prime (E) divisible by 5
12. The only even-numbered problem with answer A is
    (A) 8 (B) 10 (C) 12 (D) 14 (E) 16
13. The number of questions with answer D is
    (A) 6 (B) 7 (C) 8 (D) 9 (E) 10
14. The answer to question 11 is
    (A) A (B) B (C) C (D) D (E) E
15. The answer to question 9 is
    (A) D (B) C (C) B (D) A (E) E
16. The answer to question 5 is
    (A) C (B) D (C) E (D) none of the above (E) all of the above
17. The number of questions with answer A equals the number of questions with answer
    (A) B (B) C (C) D (D) E (E) none of the above
18. The answer to this question is:
    (A) A (B) B (C) C (D) D (E) E
19. Standardized test is to intelligence as barometer is to
    (A) temperature (B) wind-velocity (C) latitude (D) longitude (E) all of the above

[1] See `http://www.maa.org/mathhorizons/Puzzles/Feb05-SRAT/original_puzzle.htm`

J. Propp further comments: (Quote) The solution to [this] puzzle is unique; in some cases the knowledge that the solution is unique may actually give you a short-cut to finding the answer to a particular question, but it's possible to find the unique solution even without making use of the fact that the solution is unique. (Thanks to Andy Latto for bringing this subtlety to my attention.)

I should mention that if you don't agree with me about the answer to #19, you will get a different solution to the puzzle than the one I had in mind. But I should also mention that if you don't agree with me about the answer to #19, you are just plain wrong. :-) (End quote)

Formalization in Funmath We encode the letters A .. E for the answers by 0 .. 4 to avoid the unnecessary clutter of explicit conversion mappings. Formally, the answer to the test is a string $a : \square\, 20 \to \square\, 5$ satisfying the following system of equations.

$$
\begin{array}{lcl}
a\ 0 &=& \bigwedge i : \square\,5 \mid a\,i = 1 \\
a\ 1 &=& \bigwedge i : \square\,5 \mid a\,(i+5) = a\,(i+6) \qquad \text{Extra eqn.:} \quad \exists!\, i : \square\,18 \,.\, a\,i = a\,(i+1) \\
a\ 2 &=& 4\,\$\,a \\
a\ 3 &=& 0\,\$\,a - 4 \\
a\ 4 &=& (a_{<5})^{-}\,(a\,4) \\
a\ 5 &=& (3,3,0,1,2)\,(a\,16) \\
a\ 6 &=& 4 - \mathsf{abs}\,(a\,7 - a\,6) \\
a\ 7 &=& (0\,\$\,a + 4\,\$\,a) - 4 \\
a\ 8 &=& \bigwedge i : \square\,5 \mid a\,(i+9) = a\,8 \\
a\ 9 &=& (3,0,4,1,2)^{-}\,(a\,15) \\
a\,10 &=& 1\,\$\,a_{<10} \\
a\,11 &=& ((\mathsf{Evn}, \mathsf{Odd}, \mathsf{Sqr}, \mathsf{Prm}, \mathsf{Mof})^{\mathsf{T}}\,(1\,\$\,a + 2\,\$\,a + 3\,\$\,a))^{-}\,1 \\
a\,12 &=& ((a_{\mathsf{Evn}})^{-}\,0 - 8)/2 \\
a\,13 &=& 3\,\$\,a - 6 \\
a\,14 &=& a\,11 \\
a\,15 &=& (3,2,1,0,4)^{-}\,(a\,9) \\
a\,16 &=& (3,3,0,1,2)\,(a\,5) \\
a\,17 &=& \forall\,(i : 1\,..\,4 \,.\, 0\,\$\,a \neq i\,\$\,a)\ ?\ 4 \nmid ((\$\,a) \rceil (1\,..\,4))^{-}\,(0\,\$\,a) - 1 \\
a\,18 &=& a\,18 \\
a\,19 &=& 4
\end{array}
$$

Most operators are basic Funmath [1]. We just mention $m\,..\,n = \{k : \mathbb{Z} \mid m \leq k \leq n\}$ and $\square\,n = 0\,..\,n-1$. A property of $\bigwedge$ is that, if S is a subset of $\mathbb{N}$ and P a predicate on $\mathbb{N}$ satisfying $\exists\, n : S \,.\, P\,n$, then $m = \bigwedge\,(n : S \mid P\,n) \equiv P\,m \wedge \forall\, n : S \,.\, P\,n \Rightarrow m \leq n$. Also, $n\,\$\,a = \sum i : \mathcal{D}\,a \,.\, a\,i = n$ counts occurrences of n in a. Ad hoc operators: abs is the absolute value operator, and Evn etc. are appropriate predicates of type $\mathbb{N} \to \mathbb{B}$.

Note: we provide some extra information by stating that none of the equations contain out-of-domain applications or type violations (e.g., a right-hand side outside $\square\,5$). This is ensured by the designer of the test and captured in the formalization.

Calculating the solution(s) We shall use very few words; justifications are written between $\langle\ \rangle$, equation references between [ ], using [20] for the "extra eqn.". Heuristic: we scan the list various times; first looking for equations yielding an answer by themselves, then extracting the maximum of information out of single equations, then in combination etc.. The numbering indicates how many answers are still left.

20. [19] $\boldsymbol{a\,19 = 4}$

19. [4] $a\,4 = (a_{<5})^-\,(a\,4) \wedge a\,4 \in \mathcal{D}\,(a_{<5})^-$
$\equiv\langle \text{Def.}^- \rangle\ \boldsymbol{a\,4 = 4} \wedge \forall\, i : \square 4\,.\,a\,i \neq 4$

18. [0] $a\,0 = \bigwedge i : \square 5 \mid a\,i = 1$
$\equiv\ \langle \text{Prop.} \bigwedge \rangle\ a\,(a\,0) = 1$ $[\alpha]$
$\wedge\ \forall\, i : \square 5\,.\,a\,i = 1 \Rightarrow a\,0 \le i$ $[\beta]$
$[\alpha] \Rightarrow\langle \text{Substitute } a\,0 = 0 \rangle\ a\,0 \neq 0$ $[\alpha']$
$[\beta] \Rightarrow\langle \text{Instantiate } i := 0 \rangle\ a\,0 \neq 1$ $[\beta']$
[2] $a\,2 = 4\,\$\,a$
$\Rightarrow\langle a\,4 = a\,19 = 4 \rangle\ a\,2 \geq 2$ $[\gamma'']$
$\Rightarrow\langle \text{Put } a\,0 = 2 \text{ in } [\alpha] \rangle\ a\,0 \neq 2$ $[\gamma']$
[4] $\Rightarrow\langle \text{Step \#19.} \rangle\ \forall\, i : \square 4\,.\,a\,i \neq 4$
$\Rightarrow\langle \text{Instantiate } i := 0 \rangle\ a\,0 \neq 4$
$\Rightarrow\langle [\alpha', \beta', \gamma'], a\,0 \in \square 5 \rangle\ \boldsymbol{a\,0 = 3}$

17. $[\alpha] \Rightarrow\langle a\,0 = 3 \rangle\ \boldsymbol{a\,3 = 1} \Rightarrow\langle [3] \rangle\ 0\,\$\,a = 5$

16. [9] $a\,9 = \text{‘30412’}^-\,(a\,15)$
$\Rightarrow\langle \text{Prop. }^- \rangle\ a\,15 = \text{‘30412’}\,(a\,9)$ $[\delta]$
[15] $a\,15 = \text{‘32104’}^-\,(a\,9)$, hence:
$a\,9 = \langle \text{Prop. }^- \rangle\ \text{‘32104’}\,(a\,15)$
$= \langle [\delta] \rangle\ \text{‘32104’}\,(\text{‘30412’}\,(a\,9))$
$= \langle \text{Def. } \circ \rangle\ (\text{‘32104’} \circ \text{‘30412’})\,(a\,9)$
$= \langle \text{Calcul. } \circ \rangle\ \text{‘03421’}\,(a\,9)$
$= \langle \text{Fixpoints} \rangle\ 0$ Hence $\boldsymbol{a\,9 = 0}$

15. $a\,15 = \langle [\delta], a\,9 = 0 \rangle\ 3$ Hence $\boldsymbol{a\,15 = 3}$

14. [16] $a\,16 = \text{‘33012’}\,(a\ \ 5)$
$\Rightarrow\langle [5] \rangle\ a\,16 = \text{‘33012’}^2\,(a\,16)$
$\Rightarrow\langle \text{Calcul. } \circ \rangle\ a\,16 = \text{‘11330’}\,(a\,16)$
$\Rightarrow\langle \text{Fixpoints} \rangle\ a\,16 = 1 \vee a\,16 = 3$
[1] $\Rightarrow\langle \text{Note} \rangle\ \exists\, i : \square 5\,.\,a\,(i+5) = a\,(i+6)$
$\Rightarrow\langle \text{Ch. var.} \rangle\ \exists\, i : 5\,..\,9\,.\,a\,i = a\,(i+1)$
$\Rightarrow\langle [20] \rangle\ \forall\, i : 10\,..\,18\,.\,a\,i \neq a\,(i+1)$
$\Rightarrow\langle \text{Instantiate } i := 15 \rangle\ a\,15 \neq a\,16$
$\Rightarrow\langle a\,15 = 3 \rangle\ a\,16 \neq 3$
$\Rightarrow\langle a\,16 = 1 \vee a\,16 = 3 \rangle\ \boldsymbol{a\,16 = 1}$

13. $a\,5 = \langle [5], a\,16 = 1 \rangle\ 3$ Hence $\boldsymbol{a\,5 = 3}$

12. [6] $a\,6 = 4 - \mathsf{abs}\,(a\,7 - a\,6)$
$\Rightarrow\langle \text{Arithmetic} \rangle\ a\,6 = 4 - (a\,7 - a\,6)$
$\vee\ a\,6 = 4 - (a\,6 - a\,7)$
$\Rightarrow\langle \text{Arithmetic} \rangle\ a\,7 = 4$
$\vee\ a\,7 = 2 \cdot (a\,6 - 2)$
$\Rightarrow\langle \text{Weaken} \rangle\ \mathsf{Evn}\,(a\,7)$
[7] $a\,7 = (0\,\$\,a + 4\,\$\,a) - 4$
$\Rightarrow\langle [2,\ 3],\ a\,3 = 1 \rangle\ a\,7 = a\,2 + 1$ $[\sigma]$
$\Rightarrow\langle [\gamma'']\ a\,2 \geq 2 \rangle\ a\,7 \geq 3$
$\Rightarrow\langle \mathsf{Evn}\,(a\,7) \rangle\ \boldsymbol{a\,7 = 4}$

11. $[\sigma] \Rightarrow\langle a\,7 = 4 \rangle\ \boldsymbol{a\,2 = 3}$

We make a halfway inventory. Remark: no more answers can be 4 (all used up).

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$a\,i$	3		3	1	4	3		4		0						3	1			4

10. [12] $a\,12 = ((a_{\mathsf{Evn}})^-\,0 - 8)/2$
$a\,12 = 0 \equiv\langle \text{Def. }^- \rangle\ (a_{\mathsf{Evn}})^-\,0 = 12$
$\equiv\langle \text{Eq. 12} \rangle\ a\,12 = 2$
$a\,12 = 1 \equiv\langle \text{Eq. 12} \rangle\ (a_{\mathsf{Evn}})^-\,0 = 10$
$\equiv\langle \text{Def. }^- \rangle\ a\,10 = 0$
$\equiv\langle [10],\ \text{a}\,3 = 1 \rangle\ 0$
So $a\,12 \notin \{0,1,2\}$, hence $\boldsymbol{a\,12 = 3}$

9. [12] $\Rightarrow\langle a\,12 = 3,\ \text{def. }^- \rangle\ \boldsymbol{a\,14 = 0}$

8. [14] $a\,14 = a\,11 \Rightarrow\langle \text{a}\,14 = 0 \rangle\ \boldsymbol{a\,11 = 0}$

7. Lemma: $\forall\, i : 6\,..\,9\,.\,a\,i \neq a\,(i+1)$
Proof: use $\{a\,6, a\,8, a\,10\} \cap \{0,4\} = \emptyset$
[1] $\Rightarrow\langle \text{Step 14.} \rangle\ \exists\, i : 5\,..\,9\,.\,a\,i = a\,(i+1)$
$\Rightarrow\langle \text{Lemma} \rangle\ a\,5 = a\,6$
$\Rightarrow\langle a\,5 = 3 \rangle\ \boldsymbol{a\,6 = 3}$

6. [1] $a\,1 = \bigwedge i : \square 5 \mid a\,(i+5) = a\,(i+6)$
$\Rightarrow\langle \text{As before, } a\,5 = a\,6 \rangle\ \boldsymbol{a\,1 = 0}$

5. [8] $a\,8 = \bigwedge i : \square 5 \mid a\,(i+9) = a\,8$
$\Rightarrow\ \langle \text{Prop.} \bigwedge \rangle\ a\,(a\,8 + 9) = a\,8$ $[\kappa]$
$a\,8 = 0\ \Rightarrow\langle [12], a\,12 \neq 0 \rangle\ 0$
$a\,8 = 1\ \Rightarrow\langle [\kappa, 10] \rangle\ a\,10 = 1 \wedge a\,10 = 2$
$a\,8 = 2\ \Rightarrow\langle [\kappa], 8. \rangle\ a\,11 = 2 \wedge a\,11 = 0$
So $a\,8 \notin \{0,1,2\}$, hence $\boldsymbol{a\,8 = 3}$

4. [10] $a\,10 = 1\,\$\,a_{<10}$
$\Rightarrow\langle a_{<10} = \text{‘3031433430’} \rangle\ \boldsymbol{a\,10 = 1}$

3. Table for $b := a_{\not\in\{13,17,18\}}$:

i	0	1	2	3	4
$i\,\$\,b$	4	3	0	7	4

Exactly one answer 0 remains ($0\,\$\,a = 5$).
[17] $\Rightarrow\langle a\,17 \neq 4, 0\,\$\,a = 5 \rangle$
$a\,17 = ((\$\,a)\,\rceil\,(1\,..\,4))^-\,5 - 1$ $[\mu]$
$\Rightarrow\langle \text{Note, prop.}^- \rangle\ \exists!\, i : 1\,..\,4\,.\,i\,\$\,a = 5$
$\Rightarrow\langle \text{Arith.} \rangle\ 1\,\$\,a = 5\ \ \Rightarrow\langle [\mu], 1\,\$\,b = 3 \rangle$
$\boldsymbol{a\,17 = 0} \wedge \boldsymbol{a\,13 = a\,18 = 1}$

0. Result: $a = \text{‘30314334301031031014’}$.

References

[1] R. Boute, "Concrete Generic Functionals: Principles, Design and Applications", in: Jeremy Gibbons, Johan Jeuring, eds., *Generic Programming*, pp. 89–119, Kluwer (2003)