Gale-Robinson sequences and the octahedron and cube recurrences

URL for slides:
`math.wisc.edu/~propp/atlanta.pdf`

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First Gale-Robinson Conjecture (1991), paraphrased: Fix distinct positive integers a, b, and k with $k > \max(a, b)$. Then the infinite sequence of rational numbers $r_1, r_2, \ldots$ defined by the initial conditions

$$r_1 = r_2 = \ldots = r_k = 1$$

and the recurrence relation

$$r_n = (r_{n-a}\, r_{n-k+a} + r_{n-b}\, r_{n-k+b})/r_{n-k}$$

(for $n > k$) is an infinite sequence of *integers*.

Examples:

$a = 1, b = 2, k = 4$:

$$r_n r_{n-4} = r_{n-1} r_{n-3} + r_{n-2} r_{n-2}$$

1,1,1,1,2,3,7,23,59,314,. . .
(the Somos-4 sequence)

$a = 1, b = 2, k = 5$:

$$r_n r_{n-5} = r_{n-1} r_{n-4} + r_{n-2} r_{n-3}$$

1,1,1,1,1,2,3,5,11,37,83,. . .
(the Somos-5 sequence)

Second Gale-Robinson Conjecture (1991), paraphrased: Fix positive integers a, b, and c, and let $k = a + b + c$. Then the infinite sequence of rational numbers $r_1, r_2, \ldots$ defined by the initial conditions

$$r_1 = r_2 = \ldots = r_k = 1$$

and the recurrence relation

$$r_n = (r_{n-a}\, r_{n-k+a} + r_{n-b}\, r_{n-k+b} + r_{n-c}\, r_{n-k+c}) \,/\, r_{n-k}$$

(for $n > k$) is an infinite sequence of *integers*.

Examples:

$a = 1, b = 2, c = 3$:

$$r_n r_{n-6} = r_{n-1} r_{n-5} + r_{n-2} r_{n-4} + r_{n-3}^2$$

1,1,1,1,1,1,3,5,9,23,75,421,. . .
(the Somos-6 sequence)

$a = 1, b = 2, c = 4$:

$$r_n r_{n-7} = r_{n-1} r_{n-6} + r_{n-2} r_{n-5} + r_{n-3} r_{n-4}$$

1,1,1,1,1,1,1,3,5,9,17,41,. . .
(the Somos-7 sequence)

If you start such a sequence with terms other than 1, you can get denominators, but the only primes that can occur in the denominators are primes that belong to the finite set S consisting of those primes that divide one or more of the first k terms ("S-integrality").

The first proof of the Gale-Robinson conjectures was given by Fomin and Zelevinsky in 2002. They proved a stronger claim, versions of which had previously been conjectured by Somos, myself, and possibly others.

Fomin–Zelevinsky (2002): Fix positive integers a, b, and k with $k > \max(a, b)$. Then the infinite sequence of rational functions $r_1(x_1, \ldots, x_k), r_2(x_1, \ldots, x_k)$, $\ldots$ defined by the initial conditions

$$r_1 = x_1,\ r_2 = x_2,\ \ldots, r_k = x_k$$

and the recurrence relation

$$r_n = (r_{n-a}\, r_{n-k+a} + r_{n-b}\, r_{n-k+b}) \,/\, r_{n-k}$$

(for $n > k$) is an infinite sequence of *Laurent polynomials* in $x_1, \ldots, x_k$; that is, each rational function $r(x_1, \ldots, x_k)$ is a polynomial in $x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}$.

S-integrality follows from Laurentness.

A similar sort of story holds for the second Gale-Robinson recurrence.

Fomin and Zelevinsky's proof uses algebraic methods they developed for their theory of cluster algebras.

A stronger conjecture was that the coefficients in these Laurent polynomials are all *positive*. This was proved in 2002 by David Speyer (for the first Gale-Robinson conjecture) and by Gabriel Carroll and David Speyer (for the second Gale-Robinson conjecture) using combinatorial methods, with guidance from me, as part of the two-year program Research Experiences in Algebraic Combinatorics at Harvard (REACH).

A "conditionally independent" proof of the former result was found by Bousquet-Mélou, Propp, and West at about the same time.

Speyer and Bousquet-Mélou–Propp–West (2002): Fix positive integers a, b, and k with $k > \max(a,b)$. Then the rational functions $r_{n,i,j}(\ldots)$ $(n,i,j \in \mathbf{Z})$ defined by the initial conditions

$$r_{n,i,j} = x_{n,i,j} \quad (1 \leq n \leq k)$$

and the recurrence relation

$$\begin{aligned} r_{n,i,j} &= (r_{n-a,i-1,j}\, r_{n-k+a,i+1,j} \\ &+ r_{n-b,i,j-1}\, r_{n-k+b,i,j+1}) \,/\, r_{n-k,i,j} \end{aligned}$$

(for $n > k$) are Laurent polynomials with positive coefficients, and in fact, with all coefficients equal to 1. Specializing away the dependence on i and j gives a proof of the first Gale-Robinson conjecture.

Example: $a = 1, b = 1, k = 2$:

$$r_{n,i,j} = x_{n,i,j} \quad (1 \leq n \leq 2);$$

$$r_{n,i,j} = (r_{n-1,i-1,j}\, r_{n-1,i+1,j} + r_{n-1,i,j-1}\, r_{n-1,i,j+1}) \,/\, r_{n-2,i,j}$$

(for $n > 2$). This is the octahedron recurrence recurrence studied by David Robbins in 1986, in his work with Rumsey on generalizations of Dodgson's method for calculating determinants by condensation. Robbins and Rumsey showed that the $r_{n,i,j}$'s are Laurent polynomials with all coefficients equal to 1.

Elkies, Kuperberg, Larsen, and Propp (1992): These Laurent polynomials are weight enumerators of perfect matchings of ("Aztec diamond") graphs. That is, for each (n, i, j) there is a finite graph $G(n, i, j)$ whose perfect matchings are in one-to-one correspondence with the monomials in $r_{n,i,j}$, where each monomial can be interpreted as the weight of the corresponding perfect matching.

Bousquet-Mélou and West showed that the same is true for the multivariate Laurent polynomials given by the three-dimensional version of the first Gale-Robinson recurrence.

Speyer's result is more general, and applies to many recurrences of type similar to the first Gale-Robinson recurrence. See Speyer, "Perfect Matchings and the Octahedron Recurrence", `math.CO/0402452`.

Cube recurrence:

$$s_{i,j,k}\, s_{i-1,j-1,k-1} = s_{i-1,j,k}\, s_{i,j-1,k-1}$$
$$+ s_{i,j-1,k}\, s_{i-1,j,k-1} + s_{i,j,k-1}\, s_{i-1,j-1,k}$$

Carroll and Speyer studied solutions to this equation with various sorts of boundaries and initial conditions. The same sort of Laurentness property applies, yielding a combinatorial proof of the second Gale-Robinson conjecture, with a bonus (positivity of the coefficients of the Laurent polynomials).

Their proof gives enumerative significance to Gale-Robinson sequences of the second kind in terms of a new kind of combinatorial object called a "grove". See Carroll and Speyer, "The Cube Recurrence", `math.CO/0403417`.

Most of you are number theorists, not combinatorialists; why should/might you care about these things?
If you're a number theorist, you care about integrality.
If you care about integrality, you should care about S-integrality.
If you care about S-integrality, you should care about Laurentness.
If you care about Laurentness of the 1-D recurrence, you should care about Laurentness of the 3-D recurrence.
But if you're looking at the 3-D recurrence, you're already doing combinatorics!

What I won't talk (much) about:

1. Markoff numbers

Dana Scott considered the sequence

$$1, 1, 1, 2, 5, 29, 433, 37666, \ldots$$

given by the quadratic recurrence

$$r_n r_{n-3} = r_{n-1}^2 + r_{n-2}^2$$

More generally:
A Markoff triple is any triple of positive integer solutions to $x^2+y^2+z^2 = 3xyz$, e.g., $(x, y, z) = (2, 5, 29)$.
Production rule:

$$(x, y, z) \mapsto (x, y, z' = (x^2 + y^2)/z)$$

(and likewise for replacing x and y).
Every Markoff triple can be gotten from (1,1,1) by this method.
The indexing set is not the integers, or a 3-D lattice, but a 3-regular tree.

The same sort of story that applies to Somos sequences applies also to the Scott sequence and to Markoff numbers: a Laurentness result holds, the coefficients are positive, and all the positive integers we encounter admit an enumerative interpretation.

2. Robbins stability

View the Somos-4 recurrence as a rational map

$$(w, x, y, z) \to (x, y, z, (xz + y^2)/w)$$

from $\mathbf{C}^4$ to itself (with singularities).
How might we compute the Somos-4 sequence modulo 8 (say)?
Can we replace $\mathbf{C}$ by $\mathbf{Z}/8\mathbf{Z}$?
View modular division as a multi-valued function; e.g., 4 / 2 is 2 or 6 mod 8.
If we only keep track of the terms of the Somos-4 sequence mod 8, divergence occurs when we divide an even number by an even number, which is ambiguous mod 8.

```
                  (1,2,3,7)
                      |
                  (2,3,7,7)
                   /      \
            (3,7,7,3)   (3,7,7,7)
                |           |
            (7,7,3,2)   (7,7,7,6)
                |           |
            (7,3,2,1)   (7,7,6,5)
                |           |
            (3,2,1,1)   (7,6,5,1)
                |           |
            (2,1,1,1)   (6,5,1,1)
           /    |           |     \
(1,1,1,1)   (1,1,1,5)   (5,1,1,5)   (5,1,1,1)
    |           |           |           |
(1,1,1,2)   (1,1,5,6)   (1,1,5,6)   (1,1,1,2)
```

But note that re-convergence occurs too. This happens more often than we can explain.

See Kedlaya and Propp, "In search of Robbins stability", `math.NT/0409535`.

3. Degree-sequences

View the Somos-4 recurrence as a rational map from $\mathbf{CP}^4$ to itself:

$$(t : w : x : y : z) \to (tw : wx : wy : wz : xz+y^2)$$

Let d_n be the degree of the nth iterate. The degree-sequence is 2, 3, 5, 8, 10, 14, 18, 22, 28, 33, 39, 46, 52, 60, ...
This sequence satisfies a linear recurrence and exhibits quadratic growth.
More specifically:
The 1st differences $d_n - d_{n-1}$ are 1, 2, 3, 2, 4, 4, 4, 6, 5, 6, 7, 6, 8, 8, 8, 10, 9, 10, 11, ...
The 2nd differences $d_n - 2d_{n-1} + d_{n-2}$ are 1, 1, −1, 2, 0, 0, 2, −1, 1, 1, −1, 2, 0, 0, 2, −1, 1, 1, −1, ... (period 8).

Quadratic or sub-quadratic degree-growth apparently holds for all the Gale-Robinson recurrences and seems to be a common feature of many "integrable" recurrences. I would be very interested in knowing about recurrences for which the degree-sequence is subexponential but super-quadratic.