Cyclic Actions on Catalan Objects

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Brandeis combinatorics seminar
March 15, 2016
(revised April 1, 2016)

Slides at http://jamespropp.org/brandeis16a.pdf

Catalan numbers

The nth Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Catalan numbers

The *n*th Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Its MacMahon q-analogue is

$$C_n(q) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q[k-1]_q\cdots[1]_q}$$

Many enumerative interpretations

The nth Catalan number (see Stanley's new book "Catalan Numbers") counts many things, such as:

+ Noncrossing partitions of n points on a circle
+ Triangulations of an $n + 2$-gon
+ Noncrossing perfect matchings of $2n$ points on a circle

Note that in each case there is a natural cyclic action (of order n in the first case, order $n + 2$ in the second case, order $2n$ in the third case).

There are a lot of actions of order $2n$, featuring such Catalan avatars as parenthesizations, Young tableaux, Dyck paths, plane trees, order ideals, filters, antichains, and noncrossing partitions. They'll be the main focus of this talk.

Basic bijections ($n = 3$)

Noncrossing perfect matchings:

`12,34,56` `16,23,45` `12,36,45` `16,25,34` `14,23,56`

Parenthesizations:

`()()()` `(()())` `()(())` `((()))` `(())()`

2-rowed rectangular standard Young tableaux:

135	124	134	123	125
246	356	256	456	346

Basic bijections ($n = 3$)

Parenthesizations:

```
()()()    (()())    ()(())    ((()))    (())()
```

Dyck paths:

```
                                /\
           /\/\        /\      /  \      /\
/\/\/\    /    \    /\/  \    /    \    /  \/\
```

Plane trees:

```
                                 |
          \   /        |         |        |
\ | /       |         \ |        |        | /
```

Basic bijections ($n = 3$)

Dyck paths:

```
                                /\
           /\/\        /\      /  \      /\
/\/\/\    /    \    /\/  \    /    \    /  \/\
```

Order ideals:

```
   0         0         0         1         0
  0 0       1 1       0 1       1 1       1 0
```

Filters:

```
   1         1         1         0         1
  1 1       0 0       1 0       0 0       0 1
```

Promotion

In each case, there is a natural cyclic action, called promotion, that breaks the set of 5 objects into a 2-cycle and a 3-cycle.

For noncrossing perfect matchings, decrease each number by 1 (mod 6) and re-order as needed.

For 2-by-3 rectangular standard Young tableaux, apply the Bender-Knuth involutions (switch 1 and 2 if possible; switch 2 and 3 if possible; . . . ; switch 5 and 6 if possible).

For plane trees with 3 edges, take the leftmost child of the old root to be the new root, and reorient as needed.

For Dyck paths of length 6, proceed from left to right, replacing $\vee$'s by $\wedge$'s and vice versa wherever possible.

For order ideals and filters in the A_2 root poset, "toggle" bits from left to right, replacing 0's by 1's and vice versa wherever possible ("ties don't matter"). See http://arxiv.org/abs/1108.1172.

Cyclic sieving

The orbit structure of these cyclic actions of order $2n$ is described by the polynomial $C_n(q)$ via the cyclic sieving phenomenon (see Sagan's survey http://arxiv.org/abs/1008.0790).

Let g be the generator of the action, and let ζ be $\exp 2\pi i/2n$.

Then for all integers k, the number of fixed points of g^k is $C_n(\zeta^k)$ (and these data determine the cycle-type of g as a permutation of C_n objects).

E.g., for $n = 3$ (where g has the cycle-type (12)(345)), we have $C_n(q) = 1 + q^2 + q^3 + q^4 + q^6$, so that $C_n(\zeta^1) = 0$, $C_n(\zeta^2) = 2$, and $C_n(\zeta^3) = 3$, corresponding to the fact that g has 0 fixed points, g^2 has 2 fixed points, and g^3 has 3 fixed points.

The resulting orbits-decompositions of the set of Catalan objects of order n gives a partition of C_n into parts dividing $2n$: $2 = 2$, $5 = 2 + 3$, $14 = 2 + 4 + 8$, ...

Cyclic sieving!

Amazingly, $C_n(q)$ describes the orbit structure of all of the actions I've mentioned, even though the cyclic groups involved have different orders!

For the cyclic action of order $n+2$ on triangulations of the $(n+2)$-gon, replace q by powers of a primitive $(n+2)$nd root of unity. We get a partition of C_n into parts dividing $n+2$: $2 = 2$, $5 = 5$, $14 = 2+3+3+6$, $\ldots$

For the cyclic action of order n on noncrossing partitions of n points on a circle, replace q by powers of a primitive nth root of unity. We get a partition of C_n into parts dividing n: $2 = 1+1$, $5 = 1+1+3$, $14 = 1+1+2+2+4+4$, $\ldots$

The Kreweras complement

This last action of order n is actually a subaction (i.e., subgroup action) of a cyclic action of order $2n$, generated by the Kreweras complement, whose square is rotation by $2\pi/n$: see http://arxiv.org/abs/1101.1277. This action too obeys the cyclic sieving phenomenon (and therefore has the same orbit structure as the other cyclic actions of order $2n$ discussed above).

The Panyushev complement

Let $\mathcal{A}_n$ be the set of antichains of the root poset of type A_{n-1}, let $\mathcal{I}_n$ be the set of order ideals of the root poset of type A_{n-1}, and let $\mathcal{F}_n$ be the set of filters of the root poset of type A_{n-1}.

There are natural bijections from $\mathcal{A}_n$ to $\mathcal{I}_n$ (saturate downward), from $\mathcal{I}_n$ to $\mathcal{F}_n$ (complement), and from $\mathcal{F}_n$ to $\mathcal{A}_n$ (take the minimal elements of the filter).

The composition of these bijections is not the identity operation on $\mathcal{A}_n$. It is the Panyushev complement: http://arxiv.org/abs/0711.3353.

The Panyushev complement $(n = 3)$

From antichains:

$$\begin{array}{ccccccccccccccc} & 0 & & & 0 & & & 0 & & & 1 & & & 0 & \\ 0 & & 0 & 1 & & 0 & 0 & & 1 & 0 & & 0 & 1 & & 1 \end{array}$$

to order ideals:

$$\begin{array}{ccccccccccccccc} & 0 & & & 0 & & & 0 & & & 1 & & & 0 & \\ 0 & & 0 & 1 & & 0 & 0 & & 1 & 1 & & 1 & 1 & & 1 \end{array}$$

to filters:

$$\begin{array}{ccccccccccccccc} & 1 & & & 1 & & & 1 & & & 0 & & & 1 & \\ 1 & & 1 & 0 & & 1 & 1 & & 0 & 0 & & 0 & 0 & & 0 \end{array}$$

back to antichains:

$$\begin{array}{ccccccccccccccc} & 0 & & & 0 & & & 0 & & & 0 & & & 1 & \\ 1 & & 1 & 0 & & 1 & 1 & & 0 & 0 & & 0 & 0 & & 0 \end{array}$$

The Panyushev complement ($n = 3$)

So the orbits of the Panyushev complement are

```
 0      0      1
0 0 , 1 1 , 0 0
```

and

```
 0      0
1 0 , 0 1 .
```

First surprise: The Panyushev complement is of order $2n$.

Second surprise: The Panyushev complement, like our other actions of order $2n$, exhibits cyclic sieving governed by the polynomial $C_n(q)$.

Rowmotion

There is a nice way to understand combinatorially why the Panyushev complement has the same orbit-structure as promotion.

Instead of looking at the map $\mathcal{A}_n \to \mathcal{I}_n \to \mathcal{F}_n \to \mathcal{A}_n$, look at the conjugate map $\mathcal{I}_n \to \mathcal{F}_n \to \mathcal{A}_n \to \mathcal{I}_n$ (guaranteed to have the same orbit structure).

This map admits an alternative description, using Striker and William's "toggling" operations that we used when applying promotion to order ideals (turning 0's into 1's and vice versa when possible): toggle from top to bottom instead of from left to right. (When two elements of the poset have the same rank, the associated toggle operations commute.)

Rowmotion ($n = 3$)

The orbits of rowmotion are

```
 0     0     1
0 0 , 1 1 , 1 1
```

and

```
 0     0
1 0 , 0 1 .
```

Promotion and rowmotion

Striker and Williams gave an algebraic proof showing that rowmotion and promotion are conjugate.

Einstein gave a combinatorial construction of a conjugacy, via a "recombination map" that decomposes each triangular array in a rowmotion orbit into strips, and then recombines the strips to yield a promotion orbit.

Rowmotion orbit:

```
 b      e      h
a c -> d f -> g i -> ...
```

Promotion orbit:

```
 e      h
a f -> d i -> ...
```

Recombination ($n = 3$)

Rowmotion orbit:

```
 0        0        1        0        0        1
0 0 -> 1 1 -> 1 1 -> 0 0 -> 1 1 -> 1 1 -> ...
```

Promotion orbit:

```
 0        1        0        0        1        0
0 1 -> 1 1 -> 1 0 -> 0 1 -> 1 1 -> 1 0 -> ...
```

Rowmotion orbit:

```
 0        0        0        0        0        0
1 0 -> 0 1 -> 1 0 -> 0 1 -> 1 0 -> 0 1 -> ...
```

Promotion orbit:

```
 0        0        0        0        0        0
1 1 -> 0 0 -> 1 1 -> 0 0 -> 1 1 -> 0 0 -> ...
```

Kreweras = Panyushev

Establishing an equivariant bijection between the Kreweras complement and the Panyushev complement is harder.

This was accomplished by Armstrong, Stump, and Thomas in all types (not just type A): http://arxiv.org/abs/1101.1277.

The CSP as an exploratory tool

The Cyclic Sieving Phenomenon doesn't just predict the behavior of known actions; it sometimes predicts the existence of "unknown" actions!

Vic Reiner and I both noticed that $C_n(q)$, evaluated at the powers of a primitive $n-1$st root of unity, "looks like" it's counting fixed points of powers of some Catalan action of order $n-1$.

I asked the Dynamical Algebraic Combinatorics mailing list for such an action, and Marko Thiel found one: http://arxiv.org/abs/1601.03999. (It turned out not to be new; it's related to unpublished work of Krattenthaler and Stump.)

There are other roots of unity at which $C_n(q)$ has "nice" behavior, so there may be other cyclic actions on Catalan objects, different from the ones mentioned above, that manifest the CSP.

Cyclic actions without the CSP

It is easily checked on a computer that $|C_7(e^{4\pi i/10})|$ is not an integer. It follows that there cannot be an action of $\mathbb{Z}/10\mathbb{Z}$ on the set of C_7 Catalan objects of order 7 that exhibits the loosest form of the CSP.

Is there nonetheless a uniform and natural way to define a fixed-point-free action of $\mathbb{Z}/m\mathbb{Z}$ on the C_n Catalan objects of order n for all m between $n+2$ and $2n$?

This would give a “bijective” proof of the fact that C_n is divisible by p whenever p is a prime between $n+2$ and $2n$.

Coffee-hour puzzle: For p between n and $2n$, construct a fixed-point-free action of $\mathbb{Z}/p\mathbb{Z}$ on the set of n-element subsets of $\{1, 2, \ldots, 2n\}$, and deduce a bijective proof that $\binom{2n}{n}$ is divisible by p.

The order polytope

Promotion of filters is a special case of a piecewise-linear action on the order polytope of the root poset of type A_{n-1}.

Note that $\mathcal{F}_n$ (the set of filters of the poset) can be naturally identified with with the set of order-preserving maps from the A_{n-1} root poset to the two-point (ordered) set $\{0, 1\}$.

In a similar way, the order polytope $\mathcal{O}_n$ (see Stanley's article Two Poset Polytopes) can be naturally identified with the set of order-preserving maps from the A_{n-1} root poset to the (ordered) interval $[0, 1]$.

The vertices of the order polytope are precisely the indicator functions of the filters.

Piecewise-linear promotion ($n = 3$)

For $n = 3$, we get $\mathcal{O}_3 = \{(x, y, z) : 0 \le x, y, z \le 1,\ x \le y \ge z\}$.

We define piecewise-linear toggle operations

$\tau_1((x, y, z)) = (x', y, z)$ with $x + x' = 0 + y$,

$\tau_2((x, y, z)) = (x, y', z)$ with $y + y' = \max(x, z) + 1$,

$\tau_3((x, y, z)) = (x, y, z')$ with $z + z' = 0 + y$;

$\tau_3 \circ \tau_2 \circ \tau_1 : \mathcal{O}_3 \to \mathcal{O}_3$ is piecewise-linear promotion.

Promotion of filters arises from restricting this action to the vertex-set of $\mathcal{O}_n$.

PL promotion is of order $2n$, but I do not know of a nice proof of this.

Generic points in the order polytope have orbits of size $2n$.

Open problems about PL promotion

PL promotion and its powers are piecewise-linear.
Question: What are the (maximal) "pieces"?

Consider the (finite) set of points in $\mathcal{O}_n$ whose coordinates are rational numbers with denominators dividing m (which we can think of as the set of order-preserving maps from the poset to $\{0, 1, \ldots, m\}$; just like Stanley's P-partitions, but with the order reversed). This set is permuted by PL promotion.

There appears to be a CSP for this action.

Cyclic sieving?

Conjecture: For all m and n, the action of PL promotion on the set S of order-preserving maps f from the A_{n-1} root poset P to $\{0, 1, \ldots, m\}$ exhibits the CSP, where the sieving polynomial is given by

$$[\overline{2}][\overline{3}][\overline{4}]/[2][3][4] \text{ for } n = 3,$$

$$[\overline{2}][\overline{3}][\overline{4}]^2[\overline{5}][\overline{6}]/[2][3][4]^2[5][6] \text{ for } n = 4,$$

$$[\overline{2}][\overline{3}][\overline{4}]^2[\overline{5}]^2[\overline{6}]^2[\overline{7}][\overline{8}]/[2][3][4]^2[5]^2[6]^2[7][8] \text{ for } n = 5,$$

etc., where $[k]$ denotes $[k]_q$ and $[\overline{k}]$ denotes $[2m + k]_q$.

The denominator in each formula is a product of the q-integers $[2]_q$ through $[2n - 2]_q$, with exponents that can be obtained from the coefficients of the q-polynomial $[n - 1]_q[n]_q/[2]_q$.

(Chapoton's q-Ehrhart theory may be useful in describing this.)

Birational promotion

PL promotion is the tropicalization of birational promotion. (See http://arxiv.org/abs/1404.3455.)

E.g., for $n = 3$, birational promotion is $\tau_3 \circ \tau_2 \circ \tau_1$, where

$\tau_1((x, y, z)) = (x', y, z)$ with $xx' = ay$,
$\tau_2((x, y, z)) = (x, y', z)$ with $yy' = (x + z)b$, and
$\tau_3((x, y, z)) = (x, y, z')$ with $zz' = ay$.

Likewise $\tau_2 \circ \tau_3 \circ \tau_1$ is birational rowmotion (conjugate to birational promotion, via recombination).

Question: Is there a CSP for birational rowmotion/promotion?

Birational Panyushev

One can also lift Panyushev complementation from the combinatorial realm to the PL realm to the birational realm.

In the PL realm, this involves not just the order polytope (whose vertices correspond to filters) but also the chain polytope (whose vertices correspond to antichains). Stanley's transfer map between the two (generalizing the map $\mathcal{F}_n \to \mathcal{A}_n$) plays a key role.

The transfer map lifts to the birational realm, which allows us to conjugate birational rowmotion to obtain birational Panyushev complementation.

Homomesy

Claim: In a random rooted plane tree with $n+1$ vertices, the expected number of vertices at even (resp. odd) distance from the root is $(n+1)/2$.

Proof: For any plane tree t, let $f(t)$ be the number of vertices in t at even distance from the root minus the number of vertices in t at odd distance from the root. If t' is obtained from t by plane-tree promotion, then $f(t') = -f(t)$, since promotion turns "odd" vertices into "even" vertices and vice versa. So the average of f along each promotion orbit is 0. So the global average of f (in the set of plane trees) is 0.

We say that the statistic f is 0-mesic under promotion.

More generally, we say f is c-mesic under an action if the average of f along each orbit is c. (See http://arxiv.org/abs/1310.5201.)

Some homomesies

Theorem (conjectured by Panyushev, proved by Armstrong, Stump, and Thomas): In any orbit of the Panyushev complement (acting on antichains in the A_n root poset), the average cardinality of the antichains in the orbit is $n/2$.

Theorem (conjectured by Propp, proved by Haddadan): In any orbit of rowmotion (acting on order ideals in the A_n root poset), the average "signed cardinality" of the order ideals (where elements of rank k have sign $(-1)^k$) is $n/2$. (I subsequently showed that this follows from the 0-mesy on the previous slide in conjunction with the recombination lemma.)

Open question: The second of these homomesies applies in the birational setting as well; does the first? (In the birational realm, a statistic f is homomesic under a map $T : X \to X$ of order n if $f(x)f(T(x))f(T^2(x)) \cdots f(T^{n-1}(x))$ is independent of x.)

Homomesy: a few maps, many statistics

Einstein has found that rowmotion and promotion of order ideals, and corresponding maps on antichains, have bewilderingly many statistics satisfying homomesy. (His results are numerically-driven conjectures, not proofs.)

These statistics are linear combinations of indicator functions. These indicator functions span a feature space containing a high-dimensional subspace of homomesies.

Checkmark homomesy for Panyushev complementation

One of Einstein's conjectures sharpens Panyushev's original conjecture in type A.
E.g., for an antichain A in the A_9 root poset, let $f(A)$ be the number of elements of A that are marked with a 1 below:

```
              .
             . .
            1 . .
           . 1 . .
          . . 1 . .
         . . . 1 . 1
        . . . . 1 1 .
       . . 1 . . 1 . .
```

Einstein's conjecture predicts that $f(A)$ has average 1 as A varies over an orbit of Panyushev complementation.

Homomesy: a few statistics, many maps

Work by Einstein et al. has found that for a great many operations on noncrossing partitions generated by “generalized toggles” of the sort suggested by Striker (http://arxiv.org/abs/1601.03710), the number-of-blocks statistic is homomesic, along with a small number of related statistics. See http://arxiv.org/abs/1510.06362.

These operations are compositions of arc-toggling involutions on the set of noncrossing partitions.

Summary

There are lots of actions (some natural, some derived from composing bijections, some obtained by composing involutions).

More often than we deserve, actions have small period and nice orbit structure.

There are lots of statistics (some natural, some obtained as linear combinations of natural ones).

More often than we deserve, statistics have the constant-averages-along-orbits property relative to cyclic actions.

Thank you for listening!

The slides for this talk are at http://jamespropp.org/brandeis16a.pdf .