Symbiosis and Reciprocity

a talk in honor of
Richard A. Brualdi, RAB

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slides on web at
`www.math.wisc.edu/`
`~propp/brualdi.pdf`

Some representative papers by RAB:

5. Permanent of the direct product of matrices, Pacific J. Math. 16 (1966), pp. 471-482.

17. Common transversals and strong exchange systems. J. of Combinatorial Theory 8 (1970), pp. 307-329.

176. Vector majorization via positive definite matrices, (with S. G. Hwang and S.-S. Pyo), Linear Algebra Applics, 257 (1997), 105-120.

193. Maximal nests of subspaces, the matrix Bruhat decomposition, and the Marriage Theorem, with an application to graph coloring, Elec. J. Linear Algebra, 9 (2002), 118-121.

46. The DAD theorem for arbitrary row sums, Proc. Amer. Math. Soc. 45 (1974), pp. 189-194.

67. Matrices of 0's and 1's with total support. J. Comb. Theory, (A) 28 (1980), pp. 249-256.

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Some **non-representative** papers:

46. The DAD theorem for **arbitrary** row sums, Proc. Amer. Math. Soc. 45 (1974), pp. 189-194.

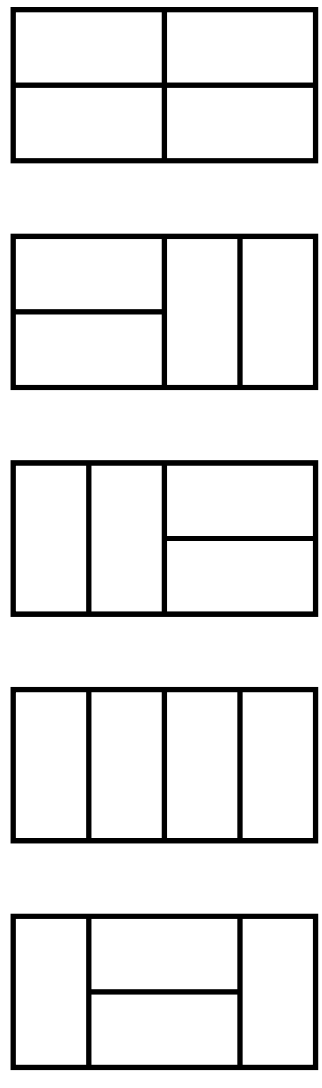
148. **Greedy** Codes (with V. Pless), J. Combin. Theory (A), 64 (1993), 10-30.

84. On minimal regular digraphs with **girth** 4 (with Li Qiao). Czech. Math. J. 33 (108) (1983), 439-447.

The article whose main point I'll be illustrating in this talk is

138. The symbiotic relationship of Combinatorics and Matrix Theory, 43 MS pages, Linear Alg. and its Applic., 162-164 (1992), 65-105

along with passages from two books: Introductory Combinatorics (first published in 1977) and Combinatorial Matrix Theory (with H.J. Ryser, published in 1991).



The five perfect covers of a
2-by-4 chessboard by dominoes

If $f(n)$ denotes the number of different perfect covers of a a 2-by-n chessboard by dominoes (1-by-2 rectangles), then we have

$$f(1) = 1,$$

$$f(2) = 2,$$

and

$$f(n) = f(n-1) + f(n-2) \text{ for all } n \geq 3.$$

Hence the sequence

$$f(1), f(2), f(3), f(4), f(5), f(6), \ldots$$

is the Fibonacci sequence

$$1, 2, 3, 5, 8, 13, \ldots$$

Re-indexing the recurrence relation as

$$f(n+2) = f(n+1) + f(n)$$

and rewriting it as

$$f(n) = f(n+2) - f(n-1)$$

we see that there is a unique natural way of extending the sequence backwards:

$$\ldots, \mathbf{5}, -\mathbf{3}, \mathbf{2}, -\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{1}, 1, 2, 3, 5, 8, 13, \ldots$$

These extrapolated terms are just the ordinary Fibonacci numbers, up to sign (this is easy to prove by induction).

If you think of the 2-by-0 board as the empty board, then it makes sense that $f(0) = 1$, but it's hard to see what to make of $f(-1)$, $f(-2)$, etc.

Something similar happens if we define $g(n)$ as the number of different perfect covers of a 3-by-n chessboard by dominoes, and extend the sequence backwards by making use of the recurrence relation

$$g(n) = 4g(n-2) - g(n-4)$$

in reverse:

$$\ldots, \mathbf{41}, \mathbf{0}, \mathbf{11}, \mathbf{0}, \mathbf{3}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, 0, 3, 0, 11, 0, 41, 0, 153, \ldots$$

This phenomenon (called **combinatorial reciprocity**) also holds for perfect covers of a m-by-n board, for any fixed value of m.

What's going on?

The extrapolated quantities

$$\ldots, f(-3), f(-2), f(-1)$$

$$\ldots, g(-3), g(-2), g(-1)$$

etc., some of which are negative, are actually "counting" something (up to sign)!

To see what they're counting,

USE LINEAR ALGEBRA

and

ADD MORE VARIABLES.

LINEAR ALGEBRA:

The Fibonacci number 5 is equal to the upper-left entry of the two-by-two matrix given by the matrix product

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

MORE VARIABLES:

The upper-left entry of the two-by-two matrix

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & 0 \end{pmatrix} \begin{pmatrix} a_4 & b_4 \\ c_4 & 0 \end{pmatrix}$$

is

$$b_1c_2b_3c_4 + b_1c_2a_3a_4 + a_1a_2b_3c_4 + a_1a_2a_3a_4 + a_1b_2c_3a_4,$$

a polynomial with five terms whose coefficients are all equal to 1.

(Check what happens when all variables are set equal to 1.)

These terms can be put into one-to-one correspondence with the five perfect covers of the 2-by-4 board.

More generally, if we put

$$M(n) = \begin{pmatrix} a_n & b_n \\ c_n & 0 \end{pmatrix}$$

then the upper-left entry of the matrix product

$$P(n) = M(1)M(2)M(3)\cdots M(n)$$

is a polynomial whose coefficients are all equal to 1.

Setting all the variables $a_1, b_1, \ldots, c_n$ in this polynomial equal to 1 yields the Fibonacci number $f(n)$.

The terms of this polynomial (of which there are $f(n)$) can be put into one-to-one correspondence with the $f(n)$ perfect covers of the 2-by-n board in a systematic way.

Two points of view:

1. The objects we care about are the (perfect domino) covers of a 2-by-n board, and each monomial of degree n **encodes** one such cover. We view the polynomial as the "generating function" for the set of covers.

2. The objects we care about are the monomials **themselves**, and the perfect domino covers are merely pictorial representations of them. Only certain monomials are "legal" (or "grammatical"). The grammar can be expressed in purely algebraic terms, but the grammar is easier to think about in terms of pictures (namely domino covers).

Recall that we defined

$$P(n) = M(1)M(2)M(3)\cdots M(n-1)M(n)$$

for all $n \geq 1$. It's clear how to define $P(n)$ for all $n < 1$: take the relation $P(n) = P(n-1)M(n)$, re-index it as $P(n+1) = P(n)M(n+1)$, and rewrite it as $P(n) = P(n+1)M(n+1)^{-1}$.

$$\begin{aligned}
P(3) &= M(1)M(2)M(3) \\
P(2) &= M(1)M(2) \\
P(1) &= M(1) \\
P(0) &= I \\
P(-1) &= M(0)^{-1} \\
P(-2) &= M(0)^{-1}M(-1)^{-1} \\
P(-3) &= M(0)^{-1}M(-1)^{-1}M(-2)^{-1} \\
&\vdots
\end{aligned}$$

It turns out that each of the matrices $P(-1)$, $P(-2)$, $P(-3)$, …defined in this way is a matrix whose entires are *Laurent polynomials*.

(A *Laurent monomial* is some coefficient times a product of finitely many variables each raised to some (positive or negative) integer power, such as $5x^1y^{-2}$. A *Laurent polynomial* is a sum of Laurent monomials.)

E.g., the upper-left entry of $P(-4)$ is

$$c_0 b_{-1}^{-1} c_{-2}^{-1} b_{-3}^{-1} + c_0 a_{-1} b_{-1}^{-1} c_{-1}^{-1} a_{-2} b_{-2}^{-1} c_{-2}^{-1} b_{-3}^{-1}$$

(a 2-term Laurent polynomial in which each coefficient is $+1$) and the upper-left entry of $P(-5)$ is a 3-term Laurent polynomial in which each coefficient is -1.

When n is negative, all the coefficients in $P(-n)$ are equal to $(-1)^n$.

Hence, setting all variables equal to 1, we see that for $n < 0$, $f(n)$ equals the number of monomials in the upper-left corner of $P(n)$, with a plus-sign if n is even and a minus-sign if n is odd.

For a more satisfying combinatorial interpretation of $f(n)$ with $n < 0$, interpret those monomials pictorially, in analogy with what works for n positive.

The big difference is that exponents can now be -1 as well as 0 or 1, so the interpretation via pictures initially seems more complicated.

Surprise: we get domino covers again, but with a different encoding!

Main result for rectangular boards:

Fix $m \geq 1$, and let $h(n)$ denote the number of perfect domino covers of a m-by-n board ($n \geq 1$).

The numbers $h(n)$ satisfy a linear recurrence relation with constant coefficients, and so may be extrapolated in a unique natural fashion to non-positive values of n.

These extrapolated values satisfy the reciprocity relation

$$h(-2-n) = \varepsilon_{m,n} h(n)$$

where $\varepsilon_{m,n} = -1$ if $m \equiv 2 \pmod 4$ and n is odd, and $\varepsilon_{m,n} = +1$ otherwise.

See `www.combinatorics.org/Volume_8/Abstracts/v8i1r18.html`

Why is $h(-n)$ equal to $\pm h(n-2)$ instead of $\pm h(n)$?

David Speyer used the transfer matrix method to prove a more general result that explains this:

Fix m. Let A^* (resp. B^*) be the set consisting of the m leftmost (resp. rightmost) squares in an m-by-n rectangle. For any $A \subseteq A^*$ and $B \subseteq B^*$, let $h_{A,B}(n)$ be the number of partial domino covers of the rectangle that cover every square in the m-by-n board except the squares in $A \cup B$ (well-defined combinatorially when $n \geq 2$, and defined for all other integers n by extrapolation). Then

$$h_{A,B}(-n) = \pm h_{B^* \setminus B, A^* \setminus A}(n).$$

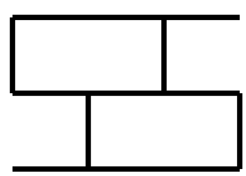
In particular, with $A = B =$ the empty set ϕ,

$$\begin{aligned} h_{\phi,\phi}(-n) &= \pm h_{B^*,A^*}(n) \\ &= \pm h_{\phi,\phi}(n-2). \end{aligned}$$

Does this approach only work for rectangular boards?

It works for lots of boards, including boards in higher dimensions (e.g., k-by-m-by-n boards, with k, m fixed and n varying, covered by 1-by-1-by-2 bricks).

Some work has also been done on other two-dimensional surfaces (including Möbius strips and Klein bottles; see `cond-mat/0110035` by Lu and Wu).



A perfect cover of a 2-by-3
Möbius strip by dominoes

In every case that's been tried, there seems to be a reciprocity phenomenon.

(The projective plane hasn't been tried yet.)

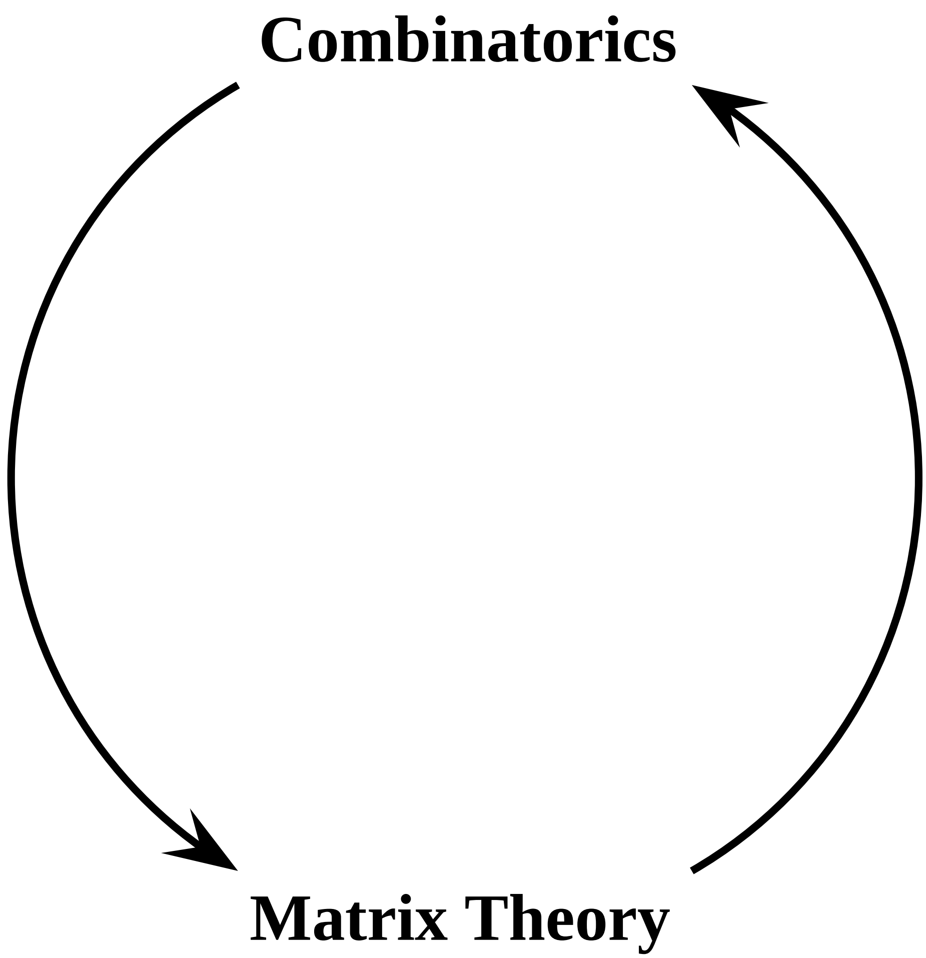
Does this approach require that the tiles be dominoes, or more generally "diform tiles" (unions of two adjacent cells)?

Other tiles can be used (though reciprocity results do not always apply; the phenomenon is somewhat sensitive to the tile-set chosen).

For example, if the region to be tiled is a rectangle and the allowed tiles are dominoes and monominoes (1-by-1 squares), then a version of reciprocity still applies. One can interpret these results via signed enumeration, but now signs can interfere destructively as well as constructively.

See `math.CO/0304359` by Anzalone, Baldwin, Bronshtein, and Petersen.

Summary



Concluding testimonial

rab (Heb.), noun: teacher (lit. “great one”).

RAB is both!