On the Cookie Game

James Propp[1]
The Massachusetts Institute of Technology

Daniel Ullman[2]
The George Washington University

May 30, 1990

Abstract

On alternate days, each of two players eats a cookie from a cookie jar. Every cookie has a spoilage date after which it can not be eaten. The object of the game is to eat the last edible cookie. In this paper, we produce a strategy for winning this game when the cookies have distinct spoilage dates. One interesting feature of the game is that the player with the winning strategy can control not only the parity of the duration of the game but the duration itself.

[1]Supported by an NSF postdoctoral fellowship at the University of California at Berkeley during the writing of this article.

[2]Supported by a Junior Scholar Incentive Award from George Washington University.

1 Introduction

In a certain cookie jar are found a number of individually-wrapped cookies, each of which has its expiration (spoilage) date stamped on the wrapper. Two competitive siblings raid the jar on alternate days, each time stealing one not-yet-spoiled cookie. Eventually one of the siblings will encounter an empty jar or a jar containing only spoiled cookies. This gives the other sibling the spiteful pleasure of having eaten the last (unspoiled) cookie, which for the children in question is incomparably greater than the inherent pleasure of cookie-eating. The gloating sibling is deemed the winner in what we call the **cookie game** (first studied in [**2**]).

To formalize the game, we represent the state of the jar by a multiset of non-negative integers (called **components**), where a component of size k corresponds to a cookie with k remaining days of edibility. Thus, a spoiled cookie corresponds to a component of size 0; such a component is said to have **expired**. For convenience, we order the components from smallest to largest, say as $(a_1, a_2, ..., a_n)$ with $a_1 \leq a_2 \leq ... \leq a_n$. We call this n-tuple a **position** of the cookie game, though we sometimes call it a **game** when we associate it with the specific cookie game whose opening position it is.

Given a non-negative integer i, let i^- denote $\max(i-1, 0)$ and let A_k denote the position $(a_1^-, ..., a_{k-1}^-, a_{k+1}^-, ..., a_n^-)$ with $1 \leq k \leq n$. An **option** of a position $A = (a_1, a_2, ..., a_n)$ is a position of the form A_k with $a_k > 0$; we call A_k the kth option of A. At each turn, a player selects an option of the current position; this new position will have strictly fewer components than the old. An n-component cookie-game must therefore terminate in a position with no legal options in at most n moves, resulting in the victory of the player who just moved. It follows that one of the players has a winning strategy; we call this player the **favored player**, and say that a position (game) is of type $\mathcal{N}$ or type $\mathcal{P}$ according to whether the Next (first) or Previous (second) player is favored. If the favored player (call her Winnie) knows an algorithm for determining whether arbitrary positions are of type $\mathcal{N}$ or type $\mathcal{P}$, she can use this algorithm to implement a winning strategy against her adversary (Lucy), since she is guaranteed to win the game if she always moves to a type $\mathcal{P}$ position. (See [**1**] for the fundamental facts about two-player impartial games with complete information.)

This paper has two main results. The first is that the problem of determining whether an arbitrary position in the cookie game is of type $\mathcal{N}$ or type $\mathcal{P}$ can be reduced to the case in which the position $(a_1, a_2, ..., a_n)$ satisfies $1 \leq a_1 \leq a_2 \leq ... \leq a_n = n$ and $a_k \geq k$ for all $1 \leq k \leq n$. The second result gives a fairly quick way of determining the $\mathcal{N}$-versus-$\mathcal{P}$ type of all positions $(a_1, a_2, ..., a_n)$ that satisfy the aforementioned constraints as well as the additional constraint that no integer in the range $\{1, ..., n-1\}$ occurs more than once as a component. (That is, $1 \leq a_1 < a_2 < ... < a_k < a_{k+1} = a_{k+2} = ... = a_n = n$ for some $k < n$.) Taken together, these results give a complete theory of the cookie game in the special case that all the cookies have distinct expiration dates.

A striking feature of the second result is that it assigns to each position A within its purview a non-negative integer $D(A)$, satisfying the following properties:

(a) $D(A) = 0$ if and only if A has no options;

(b) when $D(A)$ is even, every option B of A satisfies $D(B) = D(A) - 1$;

(c) when $D(A)$ is odd, A has at least one option B satisfying $D(B) = D(A) - 1$, and the remaining options of A have odd D-value.

An easy induction shows that at most one such function can exist, and that it can be abstractly characterized as the **strategic duration** of the game – that is, the number of moves that the game will last, assuming that Winnie exercises one of her winning strategies. Note that most of the games studied by combinatorial game theorists do not have a well-defined strategic duration in this strong sense.

Readers familiar with the Steinhaus remoteness function and suspense number of an impartial game (see [**1**]) may check, by straightforward induction, that a position with a well-defined strategic duration has both remoteness and suspense equal to its strategic duration. The converse is not true, but it is the case that if a position has remoteness equal to suspense, *and* all positions accessible from it share this property, then the position has a well-defined strategic duration equal to its remoteness and suspense.

We now survey the results of this paper in greater detail. Say two positions A and B are **tantamount** (in symbols, $A \sim B$) if they have the same type, that is if both are $\mathcal{P}$-positions or both are $\mathcal{N}$-positions. Given a position $A = (a_1, a_2, ..., a_n)$, let $A' = (a'_1, a'_2, ..., a'_n)$ where $a'_k = \min(a_k, n)$ for $1 \leq k \leq n$; we call A' the **depletion** of A.

Depletion Theorem: If A' is the depletion of A, then $A' \sim A$. (Proof deferred to Section 2.)

Accordingly, in a position with n components, all components of size $> n$ may be regarded as being of size n.

Define the **surplus** of a position $A = (a_1, ..., a_n)$ (also to be called the **spoilage** of A, for reasons that will be clear later) by

$$s(A) = \max_{0 \leq k \leq n} k - a_k,$$

where $a_0 = 0$ by convention (thus forcing $s(A) \geq 0$). The **deletion** of A is the position $(a_{s+1}, a_{s+2}, ..., a_n)$, where $s = s(A)$.

Deletion Theorem: If A' is the deletion of A, then $A' \sim A$. (Proof deferred to Section 2.)

If one performs deletion followed by depletion, the resulting position $(a_1, ..., a_n)$ has surplus 0 and satisfies $1 \leq a_1 \leq ... \leq a_n \leq n$. We call such a position **reduced**. Note that since a reduced position has surplus 0, it satisfies $a_k \geq k$ for $1 \leq k \leq n$, and in particular $a_n \geq n$; since we also have $a_n \leq n$, it follows that $a_n = n$.

By the Depletion and Deletion Theorems, every position is tantamount to its reduced form. Hence, it suffices to determine, for each n, the respective types of the $\binom{2n}{n}/(n+1)$ reduced positions with n components. (To see why the Catalan number $\binom{2n}{n}/(n+1)$ occurs, observe that a sequence $1 \leq a_1 \leq a_2 \leq ... \leq a_n = n$ satisfying $a_k \geq k$ for all k corresponds to a lattice path from $(0,0)$ to (n,n) that never goes below the line $y = x$. Such paths are well-known to be enumerated by Catalan numbers.)

Call a reduced position $A = (a_1, ..., a_n)$ **strict** if no component of size $< n$ is repeated. Such positions are precisely those that arise as the reduced forms of positions in which all components are of distinct sizes. For instance, $(1,2,4,4)$ is the reduced form of $(1,2,4,5)$. There are 2^{n-1} strict positions of size $n \geq 1$, since each corresponds to a subset of $\{1,2,...,n-1\}$ under the correspondence $(a_1, a_2, ..., a_k, n, n, ..., n) \leftrightarrow \{a_1, a_2, ..., a_k\}$. Every strict position has surplus 0. Also, every option of a strict position, when put in reduced form, is itself strict.

Given two positions A and B, we write $A \succeq B$ ("A dominates B") if there is an injection from the components b_j of B to the components a_i of A, such that $a_i \geq b_j$ for all paired i, j. More concretely, if $A = (a_1, ..., a_m)$ and $B = (b_1, ..., b_n)$ with $m \geq n$, then $A \succeq B$ if and only if $a_{m-n+1} \geq b_1$, $a_{m-n+2} \geq b_2$, ..., $a_m \geq b_n$. (If $m - n$ leading 0's are added to B, this ordering coincides with componentwise domination.) If we restrict $\succeq$ to a partial ordering on the set of all strict positions with n components, we get a lattice $\mathcal{L}_n$ with 2^{n-1} elements. (In fact, $\mathcal{L}_n$ is isomorphic to Young's lattice on the set of number partitions, restricted to those partitions with all parts distinct and $\leq n-1$.)

Fix n. For all integers d satisfying $(2n-1)/3 \leq d \leq n$, we define the strict n-component position $C^{n,d}$ as follows: Define the parity function

$$p(d) = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 1 & \text{if } d \text{ is odd.} \end{cases}$$

Put $r = 2(n-d) + p(d)$ and $m = (d+r)/2$ (an integer $\leq d+1$). Let $C^{n,d} = (c_1, c_2, ..., c_n)$

with

$$\begin{aligned} c_i &= i && \text{for } 1 \le i \le r, \\ c_{r+i} &= r + 2i && \text{for } 1 \le i \le (d-r)/2, \\ c_{m+i} &= d + i && \text{for } 1 \le i \le n - d, \text{ and} \\ c_{m+n-d+i} &= n && \text{for } 1 \le i \le d - m, \end{aligned}$$

except in the extreme case $d = (2n-1)/3$, in which we define $c_i = i$ for $1 \le i \le n$. That is, we put

$$C^{n,d} = (1, 2, 3, ..., r-1, r, r+2, r+4, ..., d-2, d, d+1, d+2, ..., n-1, n, n, ..., n).$$

We call these positions $C^{n,d}$ the **critical positions** of the cookie game. It is easy to check that for $d = \lceil (2n-1)/3 \rceil$, one has $C^{n,d} = (1, 2, ..., n)$, the minimal element of $\mathcal{L}_n$. It can also be shown (see Section 3) that for each fixed n, the positions $C^{n,d}$ form a chain in $\mathcal{L}_n$, i.e. $C^{n,d-1} \prec C^{n,d}$ for $(2n-1)/3 \le d-1$, $d \le n$. Accordingly, for A in $\mathcal{L}_n$ we define

$$D(A) = \max\{d : A \succeq C^{n,d}\};$$

if A is not strict, but its reduced form A' is, we define $D(A) = D(A')$. We will show that $D(\cdot)$ satisfies the properties (a), (b), and (c) stated earlier, from which it easily follows that A is of type $\mathcal{N}$ or type $\mathcal{P}$ according to whether $D(A)$ is odd or even. As remarked above, it also follows that if Winnie uses one of her winning strategies in a game with initial position A, the game will last exactly $D(A)$ moves.

The following table gives the critical positions $C^{n,d}$ for all $n \le 9$.

n	d	$C^{n,d}$								
1	1	1								
2	1	1	2							
2	2	2	2							
3	2	1	2	3						
3	3	1	3	3						
4	3	1	2	3	4					
4	4	2	4	4	4					
5	3	1	2	3	4	5				
5	4	1	2	4	5	5				
5	5	1	3	5	5	5				
6	4	1	2	3	4	5	6			
6	5	1	2	3	5	6	6			
6	6	2	4	6	6	6	6			
7	5	1	2	3	4	5	6	7		
7	6	1	2	4	6	7	7	7		
7	7	1	3	5	7	7	7	7		
8	5	1	2	3	4	5	6	7	8	
8	6	1	2	3	4	6	7	8	8	
8	7	1	2	3	5	7	8	8	8	
8	8	2	4	6	8	8	8	8	8	
9	6	1	2	3	4	5	6	7	8	9
9	7	1	2	3	4	5	7	8	9	9
9	8	1	2	4	6	8	9	9	9	9
9	9	1	3	5	7	9	9	9	9	9

As an example, consider the position $A = (0, 1, 1, 3, 4, 5, 8, 8)$. Its deletion is the position

$(1,3,4,5,8,8)$, whose depletion in turn is the strict position $(1,3,4,5,6,6)$. Since

$$(1,3,4,5,6,6) \succeq (1,2,3,5,6,6) = C^{6,5}$$

but

$$(1,3,4,5,6,6) \not\succeq (2,4,6,6,6,6) = C^{6,6},$$

we have $D(A) = 5$, so that in particular A is an $\mathcal{N}$-position.

2 Reduction Theorems

Claim 1 *Let A be the position $(a_1, ..., a_n)$. Any component of size $> n$ may be depleted down to size n without affecting the type of the position.*

Proof: Let B be the position obtained by performing the depletion. Since a component of size $> n$ cannot expire within n moves, every legal course of play for A corresponds to a legal course of play for B, and vice versa. In particular, any winning strategy for one player in either of these games carries over to the other game. □

This verifies the Depletion Theorem.

Claim 2 *At least $s(A)$ spoiled cookies must remain at the end of the game A, and $s(A)$ is the smallest number with this property (hence the term "spoilage").*

Proof: Let $s = s(A)$; say $s = k - a_k$ (k may be 0). For all $1 \le i \le k$, $a_i \le a_k = k - s$; hence after $k - s$ moves, all k of the components $a_1, ..., a_k$ will be unavailable, even though only $k - s$ of them could have been removed by the players. Therefore at least $k - (k - s) = s$ of the components must be 0's at the end of $k - s$ moves. (If the game ended before $k-s$ moves were made, all the better: more than $n-(k-s) \geq s$ components must be 0's.)

On the other hand, suppose the players sign a treaty that obliges them to select $a_{s+1}, a_{s+2}, \ldots$ in succession. This is feasible, since on the ith turn the protocol involves choosing a component of size $a_{s+i} - (i-1)$, which is ≥ 1 since $(s+i) - a_{s+i} \leq s$. Under the treaty, only the s smallest components of A will actually expire. $\square$

Claim 3 *If A' is an option of A, $s(A') \geq s(A)$.*

Proof: Claim 2 says that $s(A)$ is the infimum, over all ways of playing out A to a terminal position, of the number of 0-components in that terminal position. Since every terminal position accessible from A' is accessible from A, $s(A') \geq s(A)$. $\square$

Claim 4

(a) *If $s(A) = s$ and $1 \leq i \leq s$ with $a_i > 0$, then the ith option of A is tantamount to the $(s+1)$st option of A.*

(b) *If $t \leq s(A)$ and B is the position obtained from A by deleting the t smallest components of A, then A is tantamount to B.*

Proof: By induction on A. When A is a terminal position (all components equal to 0), the first claim is vacuously true (A has no legal options) and the second is trivial. Now assume that (a) and (b) apply to all the options of A; we will verify that (a) and (b) hold for A as well. Since by Claim 3 $s(A_i), s(A_{s+1}) \geq s(A) = s$, we may apply (b) to A_i and A_{s+1}: by deleting the s smallest components of each, we see that both A_i and A_{s+1} are tantamount to the game C obtained by removing the $s+1$ smallest components of A and reducing the others by 1. Hence $A_i \sim C \sim A_{s+1}$, which gives (a). The just-proved fact that $A_i \sim A_{s+1}$ for all $i \leq s$ now tells us that if A has a winning option, it must have a winning option A_k with $k \geq s+1 > t$. Also, applying (b) to A_k (and using the fact that $s(A_k) \geq s \geq t$), we get $A_k \sim B_{k-t}$ for all $k > t$. Hence A has an option of type $\mathcal{P}$ if and only if B does. This implies $A \sim B$, which is (b). □

This verifies the Deletion Theorem (which is Claim 4(b) in the case $t = s(A)$). Phrased differently, the Deletion Theorem says that if a game is guaranteed to last at least k $(= n - s)$ moves, all but the k largest components may be ignored.

3 Strict Games

Let $A = (a_1, a_2, ..., a_n)$ be a strict position, with options $A_1, A_2, ..., A_n$; note that $A \succeq A_1 \succeq A_2 \succeq ... \succeq A_n$. Fix d with $(2n-1)/3 \leq d \leq n$, let $r = 2(n-d) + p(d)$ and $m = (d+r)/2$, as in Section 1.

Claim 5 *For A strict, the inequality $D(A) \geq d$ is equivalent to the system of inequalities*

$$\begin{aligned} a_{r+1} &\geq r+2, \\ a_{r+2} &\geq r+4, \\ &\vdots \\ a_m &\geq d. \end{aligned}$$

Proof: The fact that A is strict, in combination the inequality $a_m \geq d$, implies

$$\begin{aligned} a_1 &\geq 1, \\ a_2 &\geq 2, \\ &\vdots \end{aligned}$$

$$\begin{aligned} a_r &\geq r; \\ a_{m+1} &\geq d+1, \\ a_{m+2} &\geq d+2, \\ &\vdots \\ a_{m+n-d} &\geq n. \end{aligned}$$

These inequalities, together with those given in the statement of the Claim, yield $A \succeq C^{n,d}$. The reverse implication is obvious. □

Claim 6 $C^{n,d} \succ C^{n,d-1}$.

Proof: Put $n' = n$, $d' = d - 1$. If d is even, then d' is odd and $r' = 2(n' - d') + 1 = 2(n - d + 1) + 1 = 2(n - d) + 3 = r + 3$, so that

$$\begin{aligned} C^{n,d} &= (1, 2, ..., r, r+2, r+4, r+6, r+8, ..., d, d+1, d+2, ...) \\ &\succ (1, 2, ..., r, r+1, r+2, r+3, r+5, ..., d-3, d-1, d, ...) \\ &= C^{n,d-1}. \end{aligned}$$

If d is odd, then d' is even and $r' = 2(n' - d') = 2(n - d + 1) = 2(n - d) + 2 = r + 1$, so that

$$\begin{aligned} C^{n,d} &= (1, 2, ..., r, r+2, r+4, ..., d, d+1, ...) \\ &\succ (1, 2, ..., r, r+1, r+3, ..., d-1, d, ...) \\ &= C^{n,d-1}. \end{aligned}$$

□

Claim 7 $C^{n,d} \succ C^{n-1,d}$.

Proof: Put $n' = n-1$, $d' = d$. Regardless of whether d is even or odd, we have $r' = r-2$, so that

$$\begin{aligned} C^{n,d} &= (1,2,...,r-1,r,r+2,...,d,d+1,...) \\ &\succeq \quad (1,...,r-2,r,r+2,...,d,d+1,...) \\ &= C^{n-1,d}. \end{aligned}$$

□

Claim 8 *If A is strict and A' is A with its first component removed, then $D(A)-D(A') =$ 0 or 1.*

Proof: Put $A = (a_1, a_2, ..., a_n)$, $A' = (a_2, ..., a_n)$. The reduced form of A' is $B = (b_1, ..., b_{n-1})$, with $b_i = \min(a_{i+1}, n-1)$. Let $d = D(A)$, so that $A \succeq C^{n,d}$. In the case $d = (2n-1)/3$, the mere fact that A' has $n-1$ components implies that $D(A') \geq d$. Otherwise, let C' be the position of length $n-1$ obtained by removing the first component of $C^{n,d}$, so that $A' \succeq C'$. Since $C^{n,d} \succeq C^{n-1,d-1}$ (Claims 6 and 7), we have $C' \succeq C^{n-1,d-1}$, from which it follows that $A' \succeq C^{n-1,d-1}$. This implies $B \succeq C^{n-1,d-1}$, so that $D(A') = D(B) \geq d-1$. In both cases, we get $D(A') \geq d-1 = D(A)-1$.

On the other hand, put $d = D(A)+1$. $C^{n,d}$ is not meaningful if $D(A) = n$, but in this case it is easy to see directly that $D(A') \leq D(A)$. Otherwise, we have $A \not\succeq C^{n,d}$, which by Claim 5 implies the existence of an i ($1 \leq i \leq (d-r)/2$) with $a_{r+i} < r+2i$. Let $n' = n-1$, $d' = d$, $r' = r-2$. Putting $i' = i+1$, we see that $a'_{r'+i'} = a'_{r+i-1} = a_{r+i} < r+2i = r'+2i'$ and $1 \leq i' \leq (d-r)/2 + 1 = (d'-r')/2$. By Claim 5, this implies $A' \not\succeq C^{n',d'} = C^{n-1,d}$. Therefore $B \not\succeq C^{n-1,d}$, so that $D(A') = D(B) \leq d-1 = D(A)$. □

Claim 9 *If A and A' are consecutive options of some strict position, then $D(A) - D(A') =$ 0 or 1.*

Proof: Say A and A' are the kth and $k+1$st options of B, respectively. Putting A and A' in reduced form, write $A = (a_1, ..., a_n)$ and $A' = (a'_1, ..., a'_{n'})$; normally we have $n' = n$, the sole exception being the case in which B has a 1-component, k is 1, and $n' = n - 1$. This exceptional case is covered by Claim 8, so we may assume $n' = n$.

Let $d = D(A)$. We will use the relation $A \succeq C^{n,d}$ to prove $A' \succeq C^{n,d-1}$. Let $d' = d - 1$ and $r' = 2(n' - d') + p(d')$, which equals $r + 1$ or $r + 3$ according to the parity of d. Note that $a'_i = a_i$ for all $i \neq k$ and $a'_k < a_k$. Since $A \succeq C^{n,d} \succeq C^{n,d-1}$ (Claim 6), and since A agrees with A' except in the kth position, we need only check that a'_k is greater than or equal to the kth component of $C^{n,d}$. Indeed, by virtue of Claim 5, we need only concern ourselves with the case in which k is of the form $r' + i'$ with $1 \leq i' \leq (d' - r')/2$. We need to prove that $a_{r'+i'} \geq r' + 2i'$. Let $i = k - 4$; it is easy to check that $1 \leq i - 1 \leq (d - r)/2$. (In detail: $i = i' + 1$ or $i' + 3$, so $i - 1 \geq i' \geq 1$. Also, $r + 2i = (r + i) + i = k + i \leq k + i' + 3 = (r' + i') + i' + 3 = r' + 2i' + 3 \leq d' + 3 = d + 2$, implying $i - 1 \leq (d - r)/2$.) Hence, making use of the fact that A' is strict, together with the fact that $A \succeq C^{n,d}$, we have $a'_k > a'_{k-1} = a_{k-1} = a_{r+(i-1)} \geq r + 2(i - 1)$, so that $a'_k \geq r + 2i - 1$. But it is easily checked that $r + 2i - 1 \geq r' + 2i'$. (In detail: $r + 2i = k + i \geq k + i' + 1 = r' + 2i' + 1$.) Hence $a'_k \geq r' + 2i'$, as was to be shown. This verifies $D(A') \geq d - 1 = D(A) - 1$.

On the other hand, put $d = D(A) + 1$, so that $A \not\succeq C^{n,d}$. Since $A \succeq A'$, we have $A' \not\succeq C^{n,d}$, whence $D(A') \leq d - 1 = D(A)$. □

Claim 10 *Suppose $A = (a_1, .., a_n)$ is a strict reduced position with A' the reduced form of A_n. Suppose d is even. Then $D(A) \geq d$ if and only if $D(A') \geq d - 1$.*

Proof: Case 1: $a_1 = 1$, $n > 1$. Then A' has $n - 2$ components. Set $n' = n - 2$, $d' = d - 1$, $r' = r - 1$. Then $(d' - r')/2 = (d - r)/2$, and it is easily checked that the conditions $a'_{r'+i} \geq r' + 2i$ (of Claim 5 applied to A') are respectively equivalent to the conditions $a_{r+i} \geq r + 2i$ for $1 \leq i \leq (d-r)/2$. (In detail: $a'_{r'+i} = a'_{r-1+i} = a_{r+i} - 1$ and $r' + 2i = r + 2i - 1$.)

Case 2: $a_1 > 1$ or $n = 1$. Then A' has $n-1$ components. Set $n' = n-1$, $d' = d-1$, $r' = r+1$. Then $(d'-r')/2 = (d-r)/2 - 1$, and it is easily checked that the conditions $a'_{r'+i} \geq r' + 2i$ are respectively equivalent to the conditions $a_{r+(i+1)} \geq r + 2(i+1)$ for $1 \leq i \leq (d-r)/2 - 1$. (In detail: $a'_{r'+i} = a'_{r+1+i} = a_{r+1+i} - 1 = a_{r+(i+1)} - 1$ and $r' + 2i = r + 1 + 2i = r + 2(i+1) - 1$.) □

Claim 11 *Suppose $A = (a_1, .., a_n)$ is a strict reduced position with A' the reduced form of A_1. Suppose d is odd. Then $D(A) \geq d$ if and only if $D(A') \geq d - 1$.*

Proof: A' has $n-1$ components. Set $n' = n-1$, $d' = d-1$, $r' = r-1$. Then $(d'-r')/2 = (d-r)/2$, and it is easily checked that the conditions $a'_{r'+i} \geq r' + 2i$ are respectively equivalent to the conditions $a_{r+i} \geq r + 2i$ for $1 \leq i \leq (d-r)/2$. (In detail: $a'_{r'+i} = a'_{r-1+i} = a_{r+i} - 1$ and $r' + 2i = r + 2i - 1$.) □

Theorem 12 *Let $A = (a_1, ..., a_n)$ be a strict reduced position, with $d = D(A)$.*

(a) *$d = 0$ if and only if A is the empty position;*

(b) *if d is even, every option A_k of A satisfies $D(A_k) = d - 1$; and*

(c) *if d is odd, every option A_k of A satisfies $d - 2 \leq D(A_k) \leq d$, and at least one option satisfies $D(A_k) = d - 1$.*

Proof: (a): This part is clear, since every n-component strict reduced position with $n \geq 1$ dominates $(1, 2, ..., n)$ and thus has D-value at least $\lceil (2n-1)/3 \rceil \geq 1$.

(b): Since $D(A) \geq d$, it follows from Claim 10 that $D(A_n) \geq d - 1$, and since $D(A) < d + 1$, it follows from Claim 11 that $D(A_1) < d$. For all options A_k of A we have

$A_n \preceq A_k \preceq A_1$, which (by Claim 9) implies $d-1 \leq D(A_n) \leq D(A_k) \leq D(A_1) < d$. It follows that $D(A_k) = d-1$.

(c): Since $D(A) \geq d-1$, it follows from Claim 10 that $D(A_n) \geq d-2$, and since $D(A) < d+2$, it follows from Claim 11 that $D(A_1) < d+1$. For all options A_k we have $A_n \preceq A_k \preceq A_1$, implying $d-2 \leq D(A_n) \leq D(A_k) \leq D(A_1) \leq d$, which is the first part of the claim. To prove the other, we offer a non-constructive argument for the existence of A_k satisfying $D(A_k) = d-1$, as follows: Since $D(A) < d+1$, it follows from Claim 10 that $D(A_n) < d$, and since $D(A) \geq d$, it follows from Claim 11 that $D(A_1) \geq d-1$. Consider now the sequence of options $A_1, A_2, ..., A_n$; by Claim 9 their D-values can only decrease by 0 or 1 at each stage. Since $D(A_1) \geq d-1$ and $D(A_n) \leq d-1$, there must exist some intermediate option A_k with $D(A_k) = d-1$. □

Corollary 13 *The position A is of type $\mathcal{N}$ or type $\mathcal{P}$ according to whether $D(A)$ is odd or even.*

This completes the formal analysis of the cookie game in the case of strict positions. Our theory of D-values gives Winnie a winning strategy; when it is her turn to move the current position will have odd D-value, and she need only examine all her options and select one with even D-value. However, when the number of components n is large it may be tedious for her to examine all n options individually. Of course, since the options are arranged in order of descending D-values, she can use a bisection algorithm to narrow in on an option of the desired D-value by examining at most $\log_2 n$ options, but clearly she would be happier with a procedure which, given an $\mathcal{N}$-position A, determines a k such that A_k is a $\mathcal{P}$-position. Therefore we offer, without proof, the following procedure for determining a winning option A_k of a position A with odd D-value d:

1. If $a_{r+j} = r + 2j$ for some j with $0 \leq j \leq (d-r)/2$, and if j is the smallest such integer, then take any $k < r + j$.

2. If $a_{r+i} \geq r + 2i + 1$ for every i with $0 \leq i \leq (d-r)/2$ and $a_{r+j} = r + 2j + 1$ for some j with $-1 \leq j \leq (d-r)/2$ and if j is the smallest such integer, then take $k = r + j$.

3. If $a_{r+i} \geq r + 2i + 2$ for every i with $-1 \leq i \leq (d-r)/2$ and $a_{r+j} = r + 2j + 2$ for some j with $-2 \leq j \leq (d-r)/2 - 1$ and if j is the smallest such integer, then take any $k > r + j$.

It is easily checked that the alternatives are exclusive. To show that they are exhaustive as well, suppose that none of them hold, so that either

$$a_{r+j} < r + 2j \text{ for some } j \text{ with } 0 \leq j \leq (d-r)/2$$

or

$$a_{r+j} \geq r + 2j + 3 \text{ for all } j \text{ with } -2 \leq j \leq (d-r)/2 - 1.$$

In the first case, $D(A) < d$ by Claim 5, and in the second case, putting $d' = d + 1$, $r' = r - 3$, and $j' = j + 3$, we get

$$a_{r'+j'} \geq r' + 2j' \text{ for all } j' \text{ with } 1 \leq j' \leq (d' - r')/2,$$

so that $D(A) \geq d' = d + 1$.

It is left to the reader to verify that the above prescription for k gives a winning strategy.

4 Conclusion

If a position A is of type $\mathcal{N}$, it follows from part (b) of the preceding theorem (together with Claim 9) that the winning options of A form a **winning interval**: that is, there exist $1 \leq j \leq k \leq n$ such that A_i is a winning option of A if and only if $i \in [j, k]$. This verifies the Winning Interval Conjecture made in [**2**].

If $A = (1, 2, ..., n)$, then $D(A) = \lceil (2n-1)/3 \rceil$, which is even precisely when n is a multiple of 3. This verifies the observation, made in [**2**], that $(1, 2, ..., n)$ is a $\mathcal{P}$-position precisely when n is a multiple of 3. One can also use our analysis of strict positions in the cookie game to show that all positions of the form $(k, k+1, k+2, ...)$ do indeed satisfy the pattern suggested by Table 3 in the earlier article.

One remaining mystery concerning the strict positions is the apparent fact that they all have nim-value 3 or less. $(2, 3, 4, 4)$ has nim-value 3, but no strict position with 20 or fewer heaps has nim-value 4 or larger. That is, just as the options of a particular position A can have only three different D-values (namely $D(A)$, $D(A) - 1$, and $D(A) - 2$), it seems that they also can have only three different nim-values. This phenomenon awaits explanation.

We suspect that many of the proofs given above become more intelligible if positions in the cookie game are represented in some fashion other than the one we use here. A cryptic hint concerning the form such a representation might take comes from the following table, which shows, for various values of n and d, how many strict n-component positions there are with D-value d.

n \ d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1																
1		1															
2		1	1														
3			1	3													
4				5	3												
5				2	4	10											
6					2	20	10										
7						14	15	35									
8						4	12	77	35								
9							4	70	56	126							
10								36	56	294	126						
11								8	32	312	210	462					
12									8	216	240	1122	462				
13										88	180	1320	792	1716			
14										16	80	1100	990	4290	1716		
15											16	616	880	5434	3003	6435	
16												208	528	5148	4004	16445	6435

(Rows are indexed by $n = 0, 1, 2, ...$; columns are indexed by $d = 0, 1, 2, ...$) We will leave to the reader the pastime of finding patterns in this table; armed with a copy of Sloane [**3**], he or she should have no trouble discovering appearances of central binomial coefficients, coefficients of Chebyshev polynomials, and numbers associated with dissections of polygons. We have proved that some of these patterns do indeed hold, but we have no real idea why.

Of course, the set of strict positions is only a fairly small subset of the set of all cookie positions, and a full solution still seems far off. Most positions in the unrestricted case have no strategic duration (e.g., Lucy can force the game $(2,2,4,4)$ to last either 2 or 4 moves), so a new idea will be needed. One surprising feature of the theory of strict positions is that the parities of the individual components (which to the novice player seem extremely important) do not play a crucial role in the analysis; in the general theory, one expects that these parities will take on more importance.

REFERENCES

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways for Your Mathematical Plays,* Academic Press, 1982.

[2] Jim Propp, A New Take-Away Game; to appear in *Proceedings of the Strens Conference on Recreational and Intuitive Mathematics,*, R. K. Guy et. al., eds.

[3] Neil J. Sloane, *A Handbook of Integer Sequences,* Academic Press, 1973.