# Math 431, Assignment #10: Solutions

*(due 5/3/01)*

1. Chapter 6, problem 34: Let $X$ and $Y$ denote, respectively, the number of males and females in the sample that never eat breakfast. Since

$$E(X) = 200 \times .252 = 50.4,$$

$$\mathrm{Var}(X) = 200 \times .252 \times (1 - .252) = 37.6992,$$

$$E(Y) = 200 \times .236 = 47.2,$$

$$\mathrm{Var}(Y) = 200 \times .236 \times (1 - .236) = 36.0608,$$

it follows from the normal approximation to the binomial that $X$ is approximately distributed as a normal random variable with mean 50.4 and variance 37.6992, and that $Y$ is approximately distributed as a normal random variable with mean 47.2 and variance 36.0608. By Proposition 3.2, $X + Y$ is approximately distributed as a normal random variable with mean 97.6 and variance 73.7600 and $Y - X$ is approximately distributed as a normal random variable with mean $-3.2$ and variance 73.7600. Let $Z$ be a standard normal random variable.

(a)

$$\begin{aligned} P(X + Y \geq 110) &= P(X + Y \geq 109.5) \\ &= P\left(\frac{X + Y - 97.6}{\sqrt{73.76}} \geq \frac{109.5 - 97.6}{\sqrt{73.76}}\right) \\ &= P(Z > 1.3856) \approx .083. \end{aligned}$$

(b)

$$\begin{aligned} P(Y \geq X) &= P(Y - X \geq -.5) \\ &= P\left(\frac{Y - X - (-3.2)}{\sqrt{73.76}} \geq \frac{-.5 - (-3.2)}{\sqrt{73.76}}\right) \\ &= P(Z > .3144) \approx .377. \end{aligned}$$

2. Chapter 6, problem 42:

   (a)

$$f_{X|Y}(x|y) = \frac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dx} = (y+1)^2 xe^{-x(y+1)} \text{ for } x > 0;$$

$$f_{Y|X}(y|x) = \frac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dy} = xe^{-xy} \text{ for } y > 0.$$

   (b)

$$\begin{aligned} P(XY < a) &= \int_0^\infty \int_0^{a/x} xe^{-x(y+1)}\, dy\, dx \\ &= \int_0^\infty (1 - e^{-a})e^{-x}\, dx \\ &= 1 - e^{-a} \end{aligned}$$

   so $f_{XY}(a) = e^{-a}$ for $0 < a$. That is, $XY$ is an exponential r.v. of rate 1.

3. Chapter 6, problem 48: Let $X_1$, $X_2$, $X_3$, $X_4$, $X_5$ be the 5 numbers chosen. With probability 1, they are all distinct. There are 5 equally likely possibilities for which of them is the largest, then 4 remaining equally likely possibilities for which of them is the next largest, etc., for a total of $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ different situations, each of which has the same probability. In each of the 120 situations, the probability of having the median lie between 1/4 and 3/4 is the same as for each of the others. For simplicity, let's focus on the case in which $X_1 < X_2 < X_3 < X_4 < X_5$. The event $\{X_1 < X_2 < X_3 < X_4 < X_5 \text{ and } 1/4 < X_3 < 3/4\}$ can be broken down into events of the form $\{X_1 < X_2 < X_3 < X_4 < X_5 \text{ and } x < X_3 < x + dx\}$ where $x$ lies between 1/4 and 3/4, so its probability can be written as the integral

$$\int_{1/4}^{3/4} P(X_1 < X_2 < x < X_4 < X_5 \text{ and } x < X_3 < x + dx).$$

   Since $X_1, \ldots, X_5$ are independent,

$$P(X_1 < X_2 < x < X_4 < X_5 \text{ and } x < X_3 < x + dx)$$

splits up as the product

$$P(X_1 < X_2 < x)P(x < X_4 < X_5)P(x < X_3 < x + dx).$$

$P(X_1 < X_2 < x) = \int_0^x \int_0^t 1\ ds\ dt = \int_0^x t\ dt = x^2/2$. Likewise, $P(x < X_4 < X_5) = (1-x)^2/2$. Also, $P(x < X_3 < x+dx) = dx$. So the integral is $\int_{1/4}^{3/4} \frac{x^2(1-x)^2}{4}\ dx$, and the desired probability is $120\int_{1/4}^{3/4} \frac{x^2(1-x)^2}{4}\ dx$. (Section 6.6 contains a formula that gives you the equivalent answer $\frac{5}{2!2!}\int_{1/4}^{3/4} \frac{x^2(1-x)^2}{4}\ dx$.) The integral evaluates to approximately .79297.

4. Chapter 6, theoretical exercise 18: For $a < s < 1$, $P(U > s \mid U > a) = P(U > s)/P(U > a) = \frac{1-s}{1-a}$, whence $U \mid U > a$ is uniform on $(a, 1)$. For $0 < s < a$, $P(U < s \mid U < a) = P(U < s)/P(U < a) = \frac{s}{a}$, whence $U \mid U < a$ is uniform on $(0, a)$.

5. Chapter 7, problem 6 (also find the variance):

$$E(\sum_{i=1}^{10} X_i) = \sum_{i=1}^{10} E(X_i) = 10(7/2) = 35$$

$$\mathrm{Var}(\sum_{i=1}^{10} X_i) = \sum_{i=1}^{10} \mathrm{Var}(X_i) = 10(35/12) = 175/6$$

6. Chapter 7, problem 11 (also find the variance when $p = \frac{1}{2}$): For $i$ between 2 and $n$, let $X_i$ equal 1 if a changeover occurs on the $i$th flip and 0 otherwise. Then $E(X_i) = P(i-1 \text{ is } H,\ i \text{ is } T) + P(i-1 \text{ is } T,\ i \text{ is } H) = 2p(1-p)$. Hence the expected number of changeovers is $E(\sum_{i=2}^n X_i) = \sum_{i=2}^n E(X_i) = 2(n-1)p(1-p)$.

   In general, the events $X_i$ are not independent of each other. For instance, take $n = 3$. The expected value of $X_2X_3$ is the probability that $X_2$ and $X_3$ both equal 1, which is $P(1 \text{ is } H, 2 \text{ is } T, 3 \text{ is } H) + P(1 \text{ is } T, 2 \text{ is } H, 3 \text{ is } T) = p^2(1-p) + p(1-p)^2 = p - p^2$, which in general is not equal to $E(X_2)E(X_3) = 4p^2(1-p)^2$.

   However, when $p = \frac{1}{2}$, the probability of a changeover occurring at any stage is $\frac{1}{2}$ independently of everything that's happened before, up to and including the preceding toss. So in this case the $X_i$'s are indeed independent. Each $X_i$ has variance 1/4, and $\mathrm{Var}(\sum_{i=2}^n X_i) = \sum_{i=2}^n \mathrm{Var}(X_i) = (n-1)/4$.

7. Chapter 7, problem 15 (also find the variance): Let $X_i$ denote the number of white balls taken from urn $i$, and $X$ the total number of white balls taken. Then $E(X) = \sum E(X_i) = \frac{1}{6} + \frac{3}{6} + \frac{6}{10} + \frac{2}{8} + \frac{3}{10} = 109/60$. Also, the $X_i$'s are independent of each other, so $\text{Var}(X) = \sum \text{Var}(X_i) = \frac{1}{6}(1-\frac{1}{6}) + \frac{3}{6}(1-\frac{3}{6}) + \frac{6}{10}(1-\frac{6}{10}) + \frac{2}{8}(1-\frac{2}{8}) + \frac{3}{10}(1-\frac{3}{10}) = 739/720$.

8. Chapter 7, problem 16:

$$E(X) = \int_{y>x} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = e^{-x^2/2}/\sqrt{2\pi}.$$

9. Chapter 7, problem 22 (also find the variance): For $i = 1$ to 6, let $X_i$ denote the number of rolls after we've seen $i-1$ distinct numbers until we've seen $i$ distinct numbers. The $X_i$'s are independent geometric random variables with probability of success $p_i = (7-i)/6$, expected value $E(X_i) = 1/p_i = 6/(7-i)$, and variance $\text{Var}(X_i) = (1-p_i)/p_i^2 = 6(i-1)/(7-i)^2$. Hence $E(\sum X_i) = \sum E(X_i) = 6/6+6/5+6/4+6/3+6/2+6/1 = 14.7$ and $\text{Var}(\sum X_i) = \sum \text{Var}(X_i) = 0/6^2 + 6/5^2 + 12/4^2 + 18/3^2 + 24/2^2 + 30/1^2 = 23.99$.

10. Chapter 7, problem 25: $P(N \geq n) = P(X_1 \geq X_2 \geq \cdots \geq X_n) = 1/n!$ so $E(N) = \sum_{n=1}^{\infty} P(N \geq n) = \sum_{n=1}^{\infty} 1/n! = e$.