Math 431, Assignment #7: Solutions

(due 4/12/01)

1. Chapter 4, problem 17:

   (a) $P(X=i) = \lim_{t\to i^+} F(t) - \lim_{t\to i^-} F(t)$, so $P(X=1) = (\frac{1}{2}+\frac{1-1}{4}) - \frac{1}{4} = \frac{1}{4}$, $P(X=2) = \frac{11}{12} - (\frac{1}{2}+\frac{2-1}{4}) - \frac{1}{4} = \frac{1}{6}$, and $P(X=3) = 1-\frac{11}{12} = \frac{1}{12}$.

   (b) Since $P(X=\frac{3}{2})=0$, $P(\frac{1}{2} < X < \frac{3}{2}) = P(\frac{1}{2} < X \leq \frac{3}{2}) = P(X \leq \frac{3}{2}) - P(X \leq \frac{1}{2}) = F(3/2) - F(1/2) = (\frac{1}{2} + \frac{3/2-1}{4}) - (\frac{1/2}{4}) = \frac{1}{2}$.

2. Chapter 5, problem 4:

   (a) $P(X > 20) = \int_{20}^{\infty} f(x)dx = \int_{20}^{\infty} 10x^{-2}dx = -10x^{-1}|_{20}^{\infty} = 10(20)^{-1} = \frac{1}{2}$.

   (b) For $x < 10$, we have $F(x) = 0$, and for $x \geq 10$, we have $F(x) = \int_{-\infty}^{10} 0\,dt + \int_{10}^{x} 10t^{-2}\,dt = -10t^{-1}|_{10}^{x} = 10/10 - 10/x = 1 - 10/x$. Check this with part (a): $P(X > 20) = 1 - P(X \leq 20) = 1 - F(20) = 1-(1-10/20) = \frac{1}{2}$.

   (c) The probability that an individual unit functions for at least 15 hours is $P(X \geq 15) = P(X > 15) = 1 - P(X \leq 15) = 1 - F(15) = 1-(1-10/15) = 2/3$. Assuming that all six devices are independent of one another, the number of devices that last at least 15 hours (call it X) will be a binomial random variable with $n = 6$, $p = 2/3$. Hence the probability that at least 3 will survive this long is $P(X=3)+P(X=4)+P(X=5)+P(X=6) = \binom{6}{3}(2/3)^3(1/3)^3 + \binom{6}{4}(2/3)^4(1/3)^2 + \binom{6}{5}(2/3)^5(1/3)^1 + \binom{6}{6}(2/3)^6(1/3)^0 = (20\cdot 8 + 15\cdot 16 + 6\cdot 32 + 1\cdot 64)/3^6 = 656/729 \approx .900$. (Note: One could also compute $P(X=0)+P(X=1)+P(X=2)$ and subtract the sum from 1.)

3. Chapter 5, problem 7: In order for f to be a p.d.f, we must have $1 = \int_{\mathbf{R}} f(x)\,dx = \int_0^1 a + bx^2\,dx = (ax+\frac{1}{3}bx^3)|_0^1 = a + \frac{1}{3}b$. Also, we have $\frac{3}{5} = E(X) = \int_{\mathbf{R}} x\,f(x)\,dx = \int_0^1 x(a+bx^2)\,dx = \int_0^1 ax + bx^3)\,dx = (\frac{1}{2}ax^2 + \frac{1}{4}bx^4)|_0^1 = \frac{1}{2}a + \frac{1}{4}b$. These two linear equations involving a and b tell us that $a = 3/5$ and $b = 6/5$.

4. Chapter 5, problem 8: $E(X) = \int_0^\infty x^2 e^{-x} dx = (-x^2 - 2x - 2)e^{-x}|_0^\infty = 2$.

5. Chapter 5, problem 13: X is uniform on $[0, 30]$.

   (a) $P(X > 10) = \frac{2}{3}$.

   (b) $P(X > 25 \mid X > 15) = \frac{P(X>25)}{P(X>15)} = \frac{5/30}{15/30} = \frac{1}{3}$.

6. Chapter 5, theoretical exercise 7: Let μ denote the expected value of X, so that (by linearity of expectation) the expected value of $Y = aX + b$ is $\mu' = a\mu + b$. Then the variance of Y is $E([Y - \mu']^2) = E([(aX + b) - (a\mu + b)]^2) = E([a(X - \mu)]^2) = a^2 E([X - \mu]^2) = a^2\sigma^2$, so the standard deviation of Y is $\sqrt{a^2}\sigma$. or $|a|\sigma$. (Note: We are not told that $a \geq 0$, $a\sigma$ is not a fully correct answer.)

7. Chapter 5, theoretical exercise 8: Since $0 \leq X \leq c$, it follows that $X^2 \leq cX$. Taking the expected value of the random variables on both sides of the inequality, we get $E(X^2) \leq E(cX) = cE(X)$. Hence $\text{Var}(X) = E(X^2) - [E(X)]^2 \leq cE(X) - [E(X)]^2 = E(X)[c - E(X)] = c\alpha[c - c\alpha] = c^2\alpha(1 - \alpha)$. On the other hand, one can use calculus to show that for all real numbers α, $\alpha(1 - \alpha) \leq \frac{1}{4}$ (hint: the graph of $f(t) = t(1 - t)$ is a parabola pointing downward, and it must achieve its maximum value at the point where $f'(t) = 0$). So $\text{Var}(X) = c^2\alpha(1 - \alpha) \leq c^2\frac{1}{4}$, as was to be shown.

8. Chapter 5, theoretical exercise 12: Deferred.

9. Chapter 5, theoretical exercise 28: For all x between 0 and 1, $P(Y \leq x) = P(F(X) \leq x) = P(X \leq F^{-1}(x)) = F(F^{-1}(x)) = x$. Clearly $P(Y \leq x)$ equals 0 for $x < 0$ and equals 1 for $x > 1$. Hence Y has the same c.d.f. as a uniform random variable on $[0, 1]$.

   Note that this result has an important converse: If Y is uniformly distributed on $[0, 1]$, and we define X by $X = F^{-1}(Y)$ for some monotone increasing function F, then the random variable X will have F as its c.d.f. That is: If you want to generate a random variable X that is governed by a specific c.d.f. (denote it by F), just take a uniform random variable in $[0, 1]$ and apply F^{-1} to it! If this seems confusing to you, try a simple example, like an exponential random variable.