# Math 431, Assignment #9: Solutions

*(due 4/26/01)*

1. (a) Write $Y = e^X = g(X)$ where $g(x) = e^x$. Then $g^{-1}(y) = \ln y$ and $\frac{d}{dy}g^{-1}(y) = 1/y$ whenever $y > 0$. So (by Theorem 7.1) $f_Y(y)$ takes the value $(f_X(\ln y))/y$ if $y > 0$ and vanishes otherwise.

   (b) Write $Y = \ln X = g(X)$ where $g(x) = \ln x$. Then $g^{-1}(y) = e^y$ and $\frac{d}{dy}g^{-1}(y) = e^y$. So $f_Y(y)$ takes the value $(f_X(e^y))e^y$ for all $y$.

2. (a)

$$\begin{aligned} F_{XY}(a) &= P(XY \le a) \\ &= \iint_{xy \le a} f_X(x) f_Y(y)\, dx\, dy \\ &= \int_0^\infty \int_0^{a/y} f_X(x) f_Y(y)\, dx\, dy \\ &= \int_0^\infty F_X(a/y) f_Y(y)\, dy. \end{aligned}$$

   Therefore

$$\begin{aligned} f_{XY}(a) &= \frac{d}{da}\int_0^\infty F_X(a/y) f_Y(y)\, dy \\ &= \int_0^\infty (\frac{d}{da}F_X(a/y)) f_Y(y)\, dy \\ &= \int_0^\infty ((F_X'(a/y))\frac{d}{da}(a/y)) f_Y(y)\, dy \\ &= \int_0^\infty (f_X(a/y))\frac{1}{y} f_Y(y)\, dy. \end{aligned}$$

   (b) $f_{\ln X}(t) = e^t f_X(e^t)$ and $f_{\ln Y}(t) = e^t f_Y(e^t)$, so

$$\begin{aligned} f_{\ln(XY)}(a) &= f_{\ln X + \ln Y}(a) \\ &= \int_{-\infty}^{\infty} f_{\ln X}(a-t) f_{\ln Y}(t)\, dt \end{aligned}$$

$$= \int_{-\infty}^{\infty} e^{a-t} f_X(e^{a-t})\, e^t f_Y(e^t)\, dt$$
$$= e^a \int_{-\infty}^{\infty} f_X(e^{a-t}) f_Y(e^t)\, dt.$$

Finally,

$$\begin{aligned} f_{XY}(a) &= f_{\exp(\ln(XY))}(a) \\ &= f_{\ln(XY)}(\ln a)/a \\ &= e^{\ln a}\left(\int_{-\infty}^{\infty} f_X(e^{(\ln a)-t}) f_Y(e^t)\, dt\right)/a \\ &= a\left(\int_{-\infty}^{\infty} f_X(ae^{-t}) f_Y(e^t)\, dt\right)/a \\ &= \int_{-\infty}^{\infty} f_X(ae^{-t}) f_Y(e^t)\, dt. \end{aligned}$$

To see that the answer we got in part (a) agrees with the answer we got in part (b), put $y = e^t$: then we have $dy = e^t\, dt$, and

$$f_{XY}(a) = \int_{-\infty}^{\infty} f_X(ae^{-t})\, \frac{1}{e^t}\, f_Y(y)\, e^t\, dt;$$

the $e^t$ and $1/e^t$ cancel, giving

$$\int_{-\infty}^{\infty} f_X(ae^{-t}) f_Y(e^t)\, dt.$$

3. Chapter 6, problem 6: The wording of the problem is slightly ambiguous. The use of the word "spotted" suggests that we have to actually test the defective ones, even though the rest of the problem says we only have to "identify" the defectives (which in particular would permit us to deduce their identity without testing them). I'll solve the problem both ways. Let $N_1$ and $N_2$ denote the number of tests required until you identify the first and second defectives, respectively, under the assumption that you can use deduction, and let $M_1$ and $M_2$ denote the number of tests required until you actually *test* the first and second defectives, respectively.

   It is convenient to model this problem with a sample space with $\binom{5}{2} = 10$ equally likely outcomes, according to the locations of the two defective transistors: $DDGGG$, $DGDGG$, etc., where $D$ stands for a defective transistor and $G$ stands for a good transistor. Here the order of the $D$'s and $G$'s corresponds to the order in which the units are tested. (Technical point: under the first interpretation, there will be one or two units that don't get tested, because their status is deduced, so we can just list their status at the end. They'll have the same status as one another, so we don't have to worry about which is fourth and which is fifth.)

$DDGGG$: $N_1 = 1$, $N_2 = 1$; $M_1 = 1$, $M_2 = 1$.

$DGDGG$: $N_1 = 1$, $N_2 = 2$; $M_1 = 1$, $M_2 = 2$.

$DGGDG$: $N_1 = 1$, $N_2 = 3$; $M_1 = 1$, $M_2 = 3$.

$DGGGD$: $N_1 = 1$, $N_2 = 3$; $M_1 = 1$, $M_2 = 4$.

$GDDGG$: $N_1 = 2$, $N_2 = 1$; $M_1 = 2$, $M_2 = 1$.

$GDGDG$: $N_1 = 2$, $N_2 = 2$; $M_1 = 2$, $M_2 = 2$.

$GDGGD$: $N_1 = 2$, $N_2 = 2$; $M_1 = 2$, $M_2 = 3$.

$GGDDG$: $N_1 = 3$, $N_2 = 1$; $M_1 = 3$, $M_2 = 1$.

$GGDGD$: $N_1 = 3$, $N_2 = 1$; $M_1 = 3$, $M_2 = 2$.

$GGGDD$: $N_1 = 3$, $N_2 = 0$; $M_1 = 4$, $M_2 = 1$.

So the joint pmf of $N_1$ and $N_2$ assigns probability $\frac{1}{10} = 0.1$ to each of the pairs $(1,1), (1,2), (2,1), (3,0)$ and probability $\frac{2}{10} = 0.2$ to each of the pairs $(1,3), (2,2), (3,1)$, while the joint pmf of $M_1$ and $M_2$ assigns probability $\frac{1}{10} = 0.1$ to all of the pairs $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)$.

4. Chapter 6, problem 7: The probability that $X_1$ equals $i$ and $X_2$ equals $j$ is the probability of getting $i$ failures, then a success, then $j$ more failures, and then another success. So $p_{X_1,X_2}(i,j) = q^i p q^j p = p^2 q^{i+j}$, where $q = 1 - p$.

5. Chapter 6, problem 11: The probability that, of the 5 customers, 2 will buy an ordinary set, 1 will buy a color set, and 2 will leave empty-handed is governed by a multinomial distribution, and is therefore equal to

$$\binom{5}{2,1,2}(.45)^2(.15)^1(.40)^2 = \frac{5!}{2!1!2!}(.45)^2(.15)^1(.40)^2 = .1458.$$

(By the way: since when is a black and white considered "ordinary"? I'll bet Ross must be a fair bit older than I am!)

6. Chapter 6, problem 18: Geometrical solution: The pair $(X, Y)$ is uniformly distributed over the rectangle $\{(x, y) : 0 \le x \le L/2,\ L/2 \le y \le L\}$. The portion of this rectangle that satisfies the complementary condition $y - x \le L/3$ is the triangle with vertices $(L/6, L/2), (L/2, L/2), (L/2, 5L/6)$. This is a right triangle with base $L/3$ and height $L/3$, so its area is $(1/2)(L/3)(L/3) = L^2/18$. The full rectangle has area $(L/2)(L/2) = L^2/4$. Thus the probability that $Y - X \le L/3$ is $(L^2/18)/(L^2/4) = 4/18 = 2/9$, and the probability that $Y - X > L/3$ is $1 - 2/9 = 7/9$.

   Calculus solution: Clearly the answer doesn't depend on $L$, so let's take $L = 6$ for simplicity (to avoid fractions throughout most of the calculation). We can write the desired probability as $\iint_R f_{X,Y}(x, y)dxdy$, where $R$ is the region $\{(x, y) : 0 \le x \le 3 \le y \le 6,\ y - x \ge 2\}$. (Note that I've changed "greater than" to "greater than or equal to", but since we're dealing with jointly continuous random variables, this doesn't matter.) The integrand can be written as $f_X(x)f_Y(y) = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$. We'll just replace this by 1, and divide the integral by 9 when we're done. For each value of $x$, the point $(x, y)$ lies in $R$ if $y$ ranges between two values, but these values depend on what $x$ is. For $x$ between 0 and 1, the range for $y$ is from 3 to 6; for $x$ between 1 and 3, the range for $y$ is from $x + 2$ to 6. So our area integral splits up as two iterated integrals:

$$\iint_R f_{X,Y}(x, y)\, dx\, dy = \int_0^1 \int_3^6 1\, dy\, dx + \int_1^3 \int_{x+2}^6 1\, dy\, dx = 7,$$

   giving the final answer 7/9 as before.

7. Chapter 6, problem 22:

   (a) No, since the joint pdf does not factor.

   (b) $f_X(x) = \int_0^1 (x + y)\, dy = x + \frac{1}{2}$ for $0 < x < 1$ (and vanishes otherwise).

   (c) $\iint_{x+y<1} (x+y)\, dx\, dy = \int_0^1 \int_0^{1-x} (x+y)\, dy\, dx = \int_0^1 (xy + \frac{1}{2}y^2)|_0^{1-x}\, dx = \int_0^1 (x(1-x) + \frac{1}{2}(1-x)^2)\, dx = \int_0^1 (\frac{1}{2} - \frac{1}{2}x^2)\, dx = (\frac{1}{2}x - \frac{1}{6}x^3)|_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$.

8. Chapter 6, problem 28:

$$\begin{aligned} P(X_1/X_2 < a) &= \int_0^\infty \int_0^{ay} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \, dx \, dy \\ &= \int_0^\infty \left(1 - e^{-\lambda_1 a y}\right) \lambda_2 e^{-\lambda_2 y} \, dy \\ &= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 a} \\ &= \frac{\lambda_1 a}{\lambda_1 a + \lambda_2}. \end{aligned}$$

9. Chapter 6, problem 30: The number of typographical errors on each page should approximately be Poisson distributed and the sum of independent Poisson random variables is also a Poisson random variable. So the parameter $\lambda$ for errors-per-ten-pages is $10 \times 0.2 = 2$. So for (a) we get $\frac{e^{-2}2^0}{0!} = e^{-2} \approx .1353$ while for (b) we get $1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2} \approx .5940$.

10. Chapter 6, theoretical exercise 9: $P(\min(X_1, \ldots, X_n) > t) = P(X_1 > t, \ldots, X_n > t) = P(X_1 > t) \cdots P(X_n > t) = e^{-\lambda t} \cdots e^{-\lambda t} = e^{-n\lambda t}$, so $F_{\min(X_1,\ldots,X_n)}(t) = 1 - e^{-n\lambda t}$, thus showing that the minimum of the $n$ exponential random variables with rate $\lambda$ is exponential with rate $n\lambda$.

11. Treating $X+Y+Z$ as $(X+Y)+Z$, we get $f_{X+Y+Z}(a) = \int_0^1 f_{X+Y}(a-y)\,dy$. There are two ways to proceed.

    First method: Change variables to $t = a - y$, obtaining $f_{X+Y+Z}(a) = \int_{a-1}^a f_{X+Y}(t)\,dt$. If we sketch $f_{X+Y}$ (using the formula calculated in Example 3a in Chapter 6), we find that there are three non-trivial cases to consider. If $0 \le a \le 1$, then the area under the graph of $f_{X+Y}(t)$ between $t = a-1$ and $t = a$ is just a triangle with vertices $(0,0)$, $(a,0)$, and $(a,a)$, with area $a^2/2$. If $1 \le a \le 2$, then the area under the graph of $f_{X+Y}(t)$ between $t = a-1$ and $t = a$ is equal to the full area under the graph between $t = 0$ and $t = 2$ (which is 1) minus the areas of two isosceles right triangles, one with legs of length $a-1$ and the other with legs of length $2-a$. So this area is $1 - (a-1)^2/2 - (2-a)^2/2 = -a^2 + 3a - \frac{3}{2}$. If $2 \le a \le 3$, then the area under the graph of $f_{X+Y}(t)$ between $t = a-1$ and $t = a$ is an isosceles triangle with legs of length $2 - (a-1) = 3 - a$, with area $(3-a)^2/2$.

(If $a$ is less than 0 or greater than 3, the integrand vanishes between $a-1$ and $a$, and so the integral vanishes.) Therefore

$$f_{X+Y+Z}(a) = \begin{cases} \frac{1}{2}a^2 & \text{if } 0 \le a \le 1, \\ -a^2 + 3a - \frac{3}{2} & \text{if } 1 \le a \le 2, \\ \frac{1}{2}a^2 - 3a + \frac{9}{2} & \text{if } 2 \le a \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

As a way of checking your calculations, you should note that the functions agree at the juncture-points $a = 0, 1, 2, 3$ and that our formulas yield $f_{X+Y+Z}(a) = f_{X+Y+Z}(3-a)$, as predicted by symmetry considerations. (If $X, Y, Z$ are each distributed symmetrically around $\frac{1}{2}$, then $X+Y+Z$ must be distributed symmetrically around $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$.)

Second method: Write $f_{X+Y}$ as $g$ for short. There are three cases to consider. If $0 < a < 1$, $\int_0^1 g(a-y)\,dy = \int_0^a g(a-y)\,dy + \int_a^1 g(a-y)\,dy = \int_0^a a - y\,dy + \int_a^1 0\,dy = \ldots = \frac{1}{2}a^2$. If $1 < a < 2$, $\int_0^1 g(a-y)\,dy = \int_0^{a-1} g(a-y)\,dy + \int_{a-1}^1 g(a-y)\,dy = \int_0^{a-1} 2-(a-y)\,dy + \int_{a-1}^1 a-y\,dy = \ldots = -a^2 + 3a - \frac{3}{2}$. If $2 < a < 3$, $\int_0^1 g(a-y)\,dy = \int_0^{a-2} g(a-y)\,dy + \int_{a-2}^1 g(a-y)\,dy = \int_0^{a-2} 0\,dy + \int_{a-2}^1 2-(a-y)\,dy = \ldots = \frac{1}{2}a^2 - 3a + \frac{9}{2}$.