Math 431, Quiz #2: Solutions

1. 20 points Assume that four out of five dentists recommend sugarless gum for their patients who chew gum. Let X be the number of dentists a pollster must ask until he or she finds the tenth dentist who recommends sugarless gum for gum-chewing patients.

(a) Find $E(X)$, the expected value of X. Give your answer as an exact fraction or decimal.

(b) What is the probability that X is less than $E(X)$?

Solution:

(a) X is a negative binomial random variable with $p = 4/5$ and $r = 10$, so $E(X) = r/p = 25/2 = 12.5$.

(b) $\binom{10-1}{10-1}(4/5)^{10}+\binom{11-1}{10-1}(4/5)^{10}(1/5)^1+\binom{12-1}{10-1}(4/5)^{10}(1/5)^2 = (1+10(1/5)+55(1/5)^2)(4/5)^{10} = 0.558....$

2. 20 points I have a coin that comes up heads 2 times out of 3 on average. If I toss the coin 6 times, what is the probability that it comes up heads an odd number of times?
Solution: The number of times heads comes up is a binomial random variable with $p = 2/3$ and $n = 6$. Call it X. Then $P(X = 1) = \binom{6}{1}(2/3)^1(1/3)^5 = 12/3^6$, $P(X = 3) = \binom{6}{3}(2/3)^3(1/3)^3 = 160/3^6$, and $P(X = 5) = \binom{6}{5}(2/3)^5(1/3)^1 = 192/3^6$, so the probability that X is odd is $(12 + 160 + 192)/3^5 = 364/729 = .499...$ (just a hair under 1/2).

3. 20 points A certain mutagen causes an average of five mutations in chromosome 17. Assume these mutations are governed by an approximate Poisson process. In what percent of the chromosomes will the mutagen cause exactly ten mutations? You must express your final answer as a percentage.
Solution: Let X be the number of mutations. X is governed by a Poisson distribution with parameter $\lambda = 5.0$. $P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}$, so $P(X = 10) = e^{-5}\frac{5^{10}}{10!} = (.0067)(\frac{9765625}{3628800}) = .018....$ Therefore we expect 1.8 percent of the chromosomes to have exactly ten mutations. (If you don't have a calculator, "$100e^{-5}\frac{5^{10}}{10!}$ percent" would also be an acceptable answer.)

It turns out that the probability that the mutagen will cause ten or more mutations is about twice as big: 3.2 percent, to two significant figures.

4. 20 points A large grab-bag contains 5 small bags of pretzels and 10 small bags of potato chips, all of which feel the same. Let X be the number of bags of pretzels you get if you reach into the large grab-bag and take 8 of the small bags at random. What specific type of probability distribution governs X? (That is, is it Bernoulli? binomial? geometric? etc.) Give the mean and standard deviation of X.

Solution: This is a hypergeometric distribution with $N_1 = 5$, $N_2 = 10$, $N = 15$, and $n = 8$, so $\mu = n(N_1/N) = 8(5/15) = 8/3 \approx 2.67$, $\sigma^2 = \frac{N-n}{N-1} n \frac{N_1}{N} \frac{N_2}{N} = (\frac{7}{14})(8)(\frac{5}{15})(\frac{10}{15}) = 8/9$, and $\sigma = \sqrt{8/9} \approx 0.943$.

5. 20 points Suppose the continuous variable X is uniformly distributed on the set $[1, 3] \cup [6, 10]$. We saw in class on Tuesday that the expected value of X is 6. What is the standard deviation of X?

Solution: $f_X(x)$ equals $\frac{1}{6}$ for x between 1 and 3 or between 6 and 10, and equals 0 for all other values of x. Hence

$$\begin{aligned} E(X^2) &= \int_{\mathbf{R}} x^2 f_X(x)\, dx \\ &= \int_1^3 x^2 \cdot \frac{1}{6}\, dx + \int_6^{10} x^2 \cdot \frac{1}{6}\, dx \\ &= \frac{x^3}{18}\Big|_1^3 + \frac{x^3}{18}\Big|_6^{10} \\ &= \frac{1}{18}(3^3 - 1^3 + 10^3 - 6^3) \\ &= 45, \end{aligned}$$

$\sigma^2(X) = E(X^2) - [E(X)]^2 = 45 - 6^2 = 9$, and $\sigma(X) = \sqrt{9} = 3$.