# Dynamical Algebraic Combinatorics in the Combinatorial and Piecewise-Linear Realms

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Based on collaborations with David Einstein,
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and conversations with Arkady Berenstein, Shahrzad Haddadan,
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Slides for this talk are available at
`http://jamespropp.org/dac.pdf`

# **Three realms**, six phenomena

Realms:

**combinatorial**,

**piecewise linear** = PL = cpl = tropical, and

**birational** = geometric.

# Three realms, **six phenomena**

Phenomena:

**periodicity**, **orbit-equivalence**, **cyclic sieving**;

**invariance**, **homomesy**, **reciprocity**.

Background:

$X$, a set;
$T : X \to X$, an invertible transformation; and
$F : X \to \mathbb{R}$, a statistic.

# Phenomena for $(X, T)$

**Periodicity**: For all $x \in X$, $T^n x = x$ (with $n \ll |X|$).

**Orbit-equivalence**: For all $k \geq 0$,

$$\#\{x : T^k x = x\} = \#\{x' : (T')^k x' = x'\}$$

(proof-strategy: find an equivariant bijection, i.e., a bijection $\phi : X \to X'$ with $\phi \circ T = T' \circ \phi$).

**Cyclic sieving**: For all $k \geq 0$,

$$\#\{x : T^k x = x\} = |p(\zeta^k)|$$

where $T$ is of period $n$, $\zeta$ is a primitive $n$th root of 1, and $p(\cdot)$ is some polynomial.

# Phenomena for $(X, T, F)$

**Invariance**: For all $x$, $F(Tx) = F(x)$.

**Homomesy**: There exists $c$ such that for every orbit $\mathcal{O}$ in $X$, the average of $F(x)$ for $x$ in $\mathcal{O}$ equals $c$.

**Reciprocity**: For all $x$, $F(x) = -G(T^k x)$ for some special combinations of $F$, $G$, and $k$ (implies that $F + G$ is "0-mesic")

(Side note: reciprocity gets its name from its manifestations in the birational realm, where it takes the form $F(x) = 1/G(T^k x)$.)

# An example (?) of homomesy

Conjecture 5.8 from the problems list:

$X =$ the set of noncrossing partitions of $\{1, 2, \ldots, n\}$;

$T =$ a product of "toggles";

$F =$ the number of blocks.

To fit this neatly into Striker's generalized toggle framework, replace each noncrossing partition $\pi$ by the set of pairs $(i, j)$ such that $i, j$ are in the same block $B$ of $\pi$, $i < j$, and there is no intermediate element $k$ in $B$ with $i < k < j$.

Note that the set $S$ of such pairs uniquely determines $\pi$. Call these pairs "arcs". The splitting/merging operation is tantamount to removing/adding an arc.

# Conjecture 5.8 as an example of generalized toggling

$X$ = the set of all subsets $S$ of

$$\{(1,2),(1,3),\ldots,(1,n),(2,3),\ldots,(n-1,n)\}$$

that correspond to noncrossing partitions $\pi$ of $\{1,2,\ldots,n\}$;

$T$ = the composition of the toggles in the order $\tau_{1,2}, \tau_{2,3}, \ldots,$ $\tau_{n-1,n}, \tau_{1,3}, \tau_{2,4}, \ldots, \tau_{1,n}$ (applying $\tau_{1,2}$ first and $\tau_{1,n}$ last).

$F$ = the number of blocks of $\pi$
= $n$ minus the number of arcs in $S$.

Note that

$$\tau_{i,j}(S) = \begin{cases} S' := S \triangle \{(i,j)\} & \text{if } S' \in X, \\ S & \text{otherwise.} \end{cases}$$

# Sage takes the case

With 24 lines of my code about noncrossing partitions,
sitting on top of 60 lines of code written by Jessica,
sitting on top of thousands of lines of code written by hundreds of people (some of them in this room),
I was able to get quick answers (the most time-consuming part was debugging my code).

E.g., for $n = 7$ (with $|X| = 429$), the statistic $F$ takes the average $(7 + 1)/2 = 4$ on each of the sixteen orbits.

The homomesy conjecture has been verified up to $n = 8$.

# REU-problem-generating engine #1

Iterate products of bijections.

Example:
For $P$ a finite poset, let
$J(P) =$ the set of order ideals (down-sets) of $P$,
$F(P) =$ the set of filter (up-sets) of $P$.
$A(P) =$ the set of antichains of $P$, and

Cameron-Fon-Der-Flaass operation:

$$J(P) \to F(P) \to A(P) \to J(P)$$

Brouwer-Schrijver operation:

$$A(P) \to J(P) \to F(P) \to A(P)$$

# REU-problem-generating engine #2

Compose involutions.

Example:

Rowmotion: Toggle from top to bottom

Promotion: Toggle from left to right

Locomotion: Toggle in a crazy order

# REU-problem-generating engine #3

Combine non-homomesies to obtain homomesies.

Example:

$X$ = the set of $k$-element subsets of $\{1, \ldots, n\}$

$T$ = the function that adds 1 (mod $n$) to each element of $S$

$F_1(S) = \min(S)$

$F_2(S) = \max(S)$

$F_1$ and $F_2$ aren't homomesic, but $F_1 + F_2$ is!

# Feature spaces and their homomesic subspaces

Any linear combination of homomesies is a homomesy.

Paradigm: Given

$X$ a set,

$T : X \to X$ an invertible transformation, and

$V$ a vector space of functions $F : X \to \mathbb{R}$ (a "feature space"),

find the subspace of $V$ consisting of the functions $F \in V$ that are homomesic under $T$.

# What you see depends on what you look at

The Cameron-Fon-der-Flaass transformation on order ideals and the Brouwer-Schrijver transformation on antichains have the same orbit structure, but very different homomesy stories.
Let $P = [a]x[b]$, so that each transformation has period $a + b$.
For any subset $S \subseteq P$, let

$$1_x(S) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

If $F : J(P) \to \mathbb{R}$ is the sum of $1_x$ for all $x$ in some **file**/column of $P$, $F$ is homomesic under Cameron-Fon-der-Flaass.
If $F : A(P) \to \mathbb{R}$ is the sum of $1_x$ for all $x$ in some **fiber** of $P = [a] \times [b]$, $F$ is homomesic under Brouwer-Schrijver.
(Each has other homomesies too; see Propp-Roby and Einstein-Propp for details.)

# What's special about $[a] \times [b]$?

$P = [a] \times [b]$ is worth studying because (as we'll see) it exists in all three realms.

The most natural directions for generalizing are to minuscule posets and root posets.

$[a] \times [b] \times [c]$ is more challenging, but worth trying as well.

The example of a product of two chains has some conceptual centrality, because it has strong links to Schützenberger promotion of SSYT's, which we'll discuss after a necessary detour.

# Stanley's $P$-partitions

For $P$ a finite poset, a $P$-partition is a weakly order-reversing map $f$ from $P$ to the nonnegative integers.

A framed $P$-partition with ceiling $n$ is a weakly order-reversing map $f$ from $P$ to $\{0, 1, \ldots, n\}$.

E.g., the indicator function of an order ideal in $P$ is a framed $P$-partition with ceiling 1.

# Toggling $P$-partitions

Given a $P$-partition $f$, and given $x \in P$, define $\tau_x f = f'$ as follows: for all $y \in P$, let

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ a + b - f(x) & \text{if } y = x \end{cases}$$

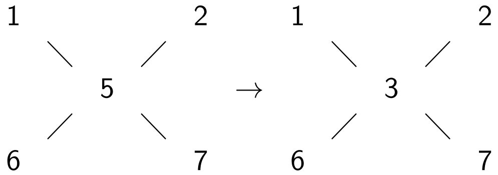
where

$$a = \max\{f(y) : y > x\},$$

$$b = \min\{f(y) : y < x\}.$$

# An example

Example with $f(x) = 5$, $a = 2$, $b = 6$, $f'(x) = 3$:



# A technicality

We need to make sure that all entries stay in $[0, n]$.

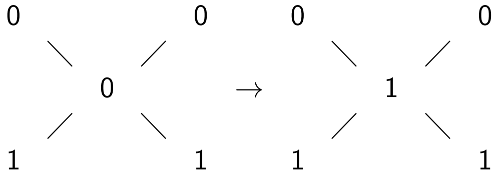
It's convenient to do this by adjoining to the poset new elements $\hat{1}$ and $\hat{0}$ at the top and bottom respectively, and to require $f(\hat{1}) = 0$ and $f(\hat{0}) = n$.

We never toggle $f$ at $\hat{0}$ or $\hat{1}$.

# Toggling order ideals is a special case

When $n = 1$, toggling framed $P$-partitions is just Striker-Williams toggling.

E.g.:




Here $a = 0$, $b = 1$, and $f'(x) = 0 + 1 - f(x)$.

# Other things to do to framed $P$-partitions

Let $\mathrm{FPP}(P, n)$ be the set of framed $P$-partitions with ceiling $n$.

The toggling operation $\tau_x : \mathrm{FPP}(P, n) \to \mathrm{FPP}(P, n)$ is an involution.

Define rowmotion on framed $P$-partitions by toggling from top to bottom; define promotion on framed $P$-partitions by toggling from left to right.

Note that Striker-Williams promotion is just the case $n = 1$.

# Upshot

Let $\mathrm{SSYT}(A, B, N)$ be the set of semistandard Young tableaux with $A$ rows, $B$ columns, and all entries between 1 and $N$.

There's an equivariant bijection from $\mathrm{SSYT}(A, B, N)$ under Schützenberger promotion to $\mathrm{FPP}(P, n)$ under framed $P$-partition promotion, with $P = [A]x[N - A]$ and $n = B$.

(The bijection is not an "algorithm": no tricky iterative processes are required!

Two steps: Convert a SSYT to a Gelfand-Tsetlin triangle and then extract appropriate entries to form the framed $P$-partition.

See `http://jamespropp.org/gtt-promotion.txt` for details.)

# An infinite toggle group

Let $\mathrm{PP}(P)$ be the disjoint union of $\mathrm{FPP}(P, n)$ with $n \geq 0$.

We can define $\tau_x$ on $\mathrm{PP}(P)$ in the obvious way, so that each $\tau_x$ acts as a permutation on $\mathrm{FPP}(P, n)$ for each $n$.

Define the $P$-partition toggle group as the group generated by the maps $\tau_x$.

Claim: This group is infinite, even for $P = [2]x[2]$!

In particular, Einstein showed that a particular simple element of the toggle group ("locomotion": the four primitive toggles composed in a particular sequence) is of infinite order.

Even though the action of Einstein's element is of finite order on each individual $\mathrm{FPP}(P, n)$, the lcm of these orders is infinite.

# Cutting $P$-partitions down to size

To get a handle on what's going on, it's useful to scale down $\mathrm{FPP}(P, n)$ by $n$, so that we're looking at order-reversing maps from $P$ to $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}$.

In fact, it's useful to relax the rationality constraint, so that we're looking at order-reversing maps from $P$ to $[0, 1]$.

In fact, it's useful to turn things upside down, so that we're looking at order-preserving maps from $P$ to $[0, 1]$: that is, we're looking at points in Stanley's $\mathcal{O}(P)$, the order polytope of $P$.

# Fiber-flipping

Fiber-flipping: Given a dissection of a polytope into parallel line-segments, map each line segment to itself so as to exchange the endpoints.

Toggling a framed $P$-partition is equivalent to applying a fiber-flipping map to the order polytope.

(In fact, there's a way to extend the fiber-flipping map from $\mathcal{O}(P)$ to itself to a map from all of $\mathbb{R}^{|P|}$ to itself, so that most of the nice things that happen inside $\mathcal{O}(P)$, like periodicity and homomesy, happen outside too, but that's part of the story I won't have time to go into.)

# Things to do to the order polytope

Fiber-flipping is a (continuous) piecewise linear map from the polytope to itself. We call it PL-toggling.

PL-toggling is an involution on $\mathcal{O}(P)$.

The restriction of PL-toggling to the vertices of $\mathcal{O}(P)$ corresponds to combinatorial toggling (the vertices, viewed as functions from $P$ to $[0,1]$, are the indicator functions of the complements in $P$ of the order ideals).

PL-rowmotion, PL-promotion, and PL-gyration are defined in obvious ways.

# What if you do them over and over?

When you iterate such maps, the maximal domains of linearity (pieces? grains? granules? slivers?) typically get smaller.

When the map is of finite order, this process is self-limiting and ultimately reverses itself.

When the map is of infinite order, what happens is much more mysterious.

Not all the pieces/grains/granules/slivers shrink!

# It's complicated

Non-trivial fact: When the map is of infinite order, there must be at least one infinite orbit. However, the points of infinite order need not be dense in the polytope. There can be subpolytopes that come back to themselves via the identity map in $n$ steps, for certain values of $n$.

We don't know whether these subpolytopes jointly have full measure.

Define the spectrum of the map as the set of $n$ such that the points whose orbit has size $n$ is of positive measure.

Example: Let $P$ be the tetrahedral poset associated with ASMs of order 4.

Under the action of PL-gyration, the spectrum apparently begins $\{8, 24, ...\}$.

This is probably related to resonance (whatever that is!).

# Examples of resonance

Back to the combinatorial realm:

(Rectangular tableaux, promotion) $\rightarrow$ (Multisets, rotation):
resonant frequency is the ceiling

(ASMs, gyration) $\rightarrow$ (Link patterns, rotation):
resonant frequency is $2n$

(ASMs, superpromotion) $\rightarrow$ ?
resonant frequency is $3n - 2$

(Order ideals in $[a] \times [b] \times [c]$, rowmotion) $\rightarrow$ ?
resonant frequency is $a + b + c - 1$

But in some ways, the concept of a single resonant frequency isn't rich enough to capture the phenomena we're starting to see.

# Resonance for tableaux

The 2772 SSYT's with shape 3,1,1 and entries between 1 and 9 belong to 56 orbits of size 9, 14 orbits of size 18, 56 orbits of size 27, and 14 orbits of size 36.

Compare:

The 4752 SSYT's with shape 3,1,1 and entries between 1 and 10 belong to 72 orbits of size 10, 25 orbits of size 20, 84 orbits of size 30, and 25 orbits of size 40, and (oops!) 1 orbit of size 4 and 1 orbit of size 8.

Note the oddball orbits of size 4 and 8.

Why do we call 10 (and maybe 20, 30, and 40 as well) resonances, but not 4?

One principled ground for discounting the 4 is looking at asymptotics, as the ceiling approaches infinity.

This brings us back into the world of PL maps.

# Our first example, revisited

Recall the case $n = 7$ of Conjecture 5.8, in which there were sixteen orbits; those orbit sizes are

133, 109, 39, 39, 31, 15, 13, 11, 9, 9, 6, 3, 3, 3, 3, and 3.

Note the oddball orbit of size 6.

So one might say that our "fundamental period" here is 1, and that "odd subharmonics" are favored and "even subharmonics" are suppressed — but only mostly!

It's not clear how to phrase a conjecture about what might happen for larger $n$, but something is going on.

# Even infiniter toggle groups

When $P$ isn't too small, the usual toggle group, a subgroup of $\text{Sym}(J(P))$ that one might call the "permutation toggle group", is a proper subgroup of the (infinite) $P$-**partition toggle group**.

This is a subgroup of the **piecewise linear toggle group** (one might also call it the "real toggle group") acting on $\mathcal{O}(P)$.

This is a quotient group of the **birational toggle group** (which Tom will discuss momentarily).

This is a quotient of the **free toggle group** on generators $\tau_x$ $(x \in P)$ of order 2 that commute when the two associated elements of $P$ are not adjacent in the Hasse diagram and satisfy no relations otherwise.

Which of these inclusions/quotients are strict?