Enumeration of Tilings

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August 11, 2014

Note: This survey article will be a chapter in the Handbook of Enumerative Combinatorics, to be published by CRC Press in early 2015.

1. Introduction and overview

We begin with some examples.

Example 1: Figure 1 shows a region (a 2-by-6 rectangle) that has been tiled by **dominos** (1-by-2 rectangles) in one of 13 possible ways. More generally, the number of domino tilings of the 2-by-n rectangle (for $n \geq 1$) is the nth Fibonacci number (if one begins the sequence $1, 2, 3, 5, 8, \ldots$).

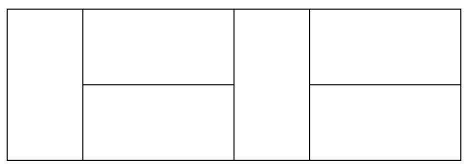

FIGURE 1. A domino tiling (one of 13).

Example 2: Figure 2 shows a region, composed of 24 equilateral triangles of side-length 1, that has been tiled by **lozenges** (unit rhombuses with internal angles of 60 and 120 degrees) in one of 14 possible ways. The region shown in Figure 2 is part of a one-parameter family of regions whose nth member has $\binom{2n}{n}/(n+1)$ tilings by lozenges. (See [**47**] and [**51**] for more on this manifestation of the Catalan numbers.)

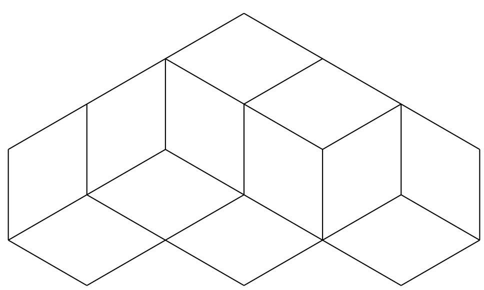

FIGURE 2. A lozenge tiling (one of 14).

Examples 1 and 2 are prototypes of the sorts of problems discussed in this chapter.

Specifically, we have some compact region R in the plane (in Example 1, R is the 2-by-6 rectangle $[0, 6] \times [0, 2]$) and a finite collection of other compact subsets called **prototiles** (in Example 1, the prototiles are $[0, 1] \times [0, 2]$ and $[0, 2] \times [0, 1]$), and we are looking at unordered collections $\{S_1, \ldots, S_m\}$ such that (a) each S_i is a translate of a prototile, (b) the interiors of the S_i's are pairwise disjoint, and (c) the union of the S_i's is R. We call such a collection a **tiling** of R. Given a region to be tiled and a set of prototiles, we want to know: *In how many different ways can the region be tiled?*

What concerns us here are not individual tiling problems but infinite families of tiling problems, with each family determining not just one number but an infinite sequence of them, indexed by a size parameter we will typically call n. The set of prototiles remains the same, while the size of R goes to infinity with n. In some cases, we obtain a sequence that is governed by a closed-form formula or a recurrence relation; in other cases, we settle for asymptotics. Often we find that the individual terms of the sequence exhibit number-theoretic patterns (e.g., they are perfect squares or satisfy certain congruences) that call out for a theoretical explanation, even in the absence of a general formula for the terms. The article [**137**] contains many such problems, as well as discussions of general issues in enumeration of tilings that overlap with the discussions contained in this chapter.

Here are two more examples of such one-parameter tiling problems:

Example 3 (Thurston and Lagarias-Romano): Figure 3 shows a "honeycomb triangle" of order 10 with one hexagon omitted (the central hexagon), tiled by 18 regions called **tribones**. W. Thurston [**165**] conjectured that for each way of packing $3n(n+1)/2$ tribones into a honeycomb triangle of order $3n+1$, the unique uncovered cell will be the center cell. Lagarias and Romano [**107**] proved this conjecture and also showed that the number of such packings is 2^n.

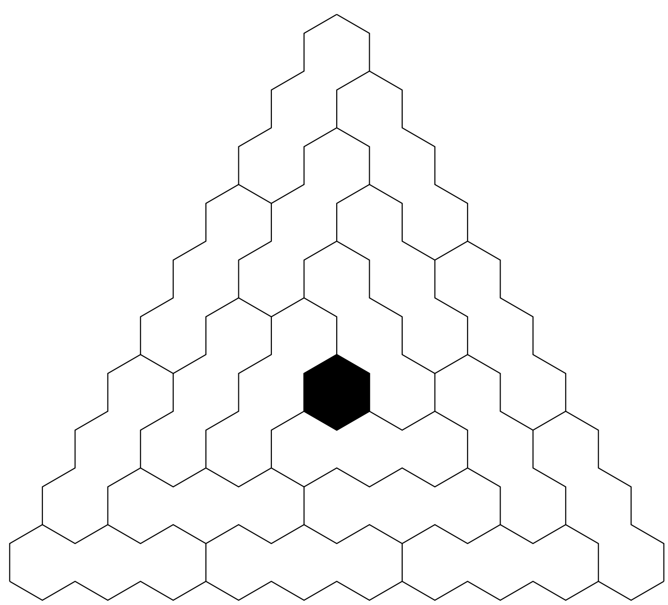

FIGURE 3. A tribone tiling (one of 8).

Example 4 (Moore): Figure 4 shows a 5-by-5 square tiled by ribbon tiles of order 5, where a **ribbon tile** of order n is a union of n successive unit squares, each of which is either the rightward neighbor or the upward neighbor of its predecessor.

It is an amusing exercise to show that the number of tilings of the n-by-n square by ribbon tiles of order n is $n!$. (The proof is not hard, but it is not obvious.) This unpublished example was communicated to me by Cris Moore.

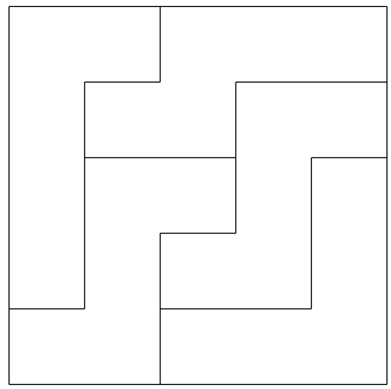

FIGURE 4. A ribbon tiling (one of 120).

These last two examples are pretty, but they are not typical of what you will learn about in this chapter (outside of the first part of Section 2). First, they do not serve as examples of a general method. Second, the sequences one gets grow too slowly to be typical of the state of the art in the modern theory of enumeration of tilings. Calling $n!$ "slow-growing" may seem surprising, since it grows faster than any exponential function. However, one thing that the nth Fibonacci number, the nth Catalan number, $n!$, and 2^n have in common is that their logarithms grow linearly in n (up to a factor of $\log n$ or smaller). In contrast, the theory of enumeration of tilings shows its true strength with exact enumerative results involving functions of n whose logarithms grow like n^2. A prototype of theorems of this kind concerns regions called Aztec diamonds [**53**]. An **Aztec diamond** of order n consists of $2n$ centered rows of unit squares, of respective lengths $2, 4, \ldots, 2n-2, 2n, 2n, 2n-2, \ldots, 4, 2$. Figure 5 shows an Aztec diamond of order 5 tiled by dominos in one of the 32,768 possible ways. More generally, an Aztec diamond of order n has exactly $2^{n(n+1)/2}$ tilings by dominos. This result has now been proved a dozen different ways, though some of the differences are more cosmetic than conceptual.

Highly analogous to domino tilings of Aztec diamonds are lozenge tilings of hexagons. Call a hexagon **semiregular** if its internal angles are 120 degrees and opposite sides are of equal length (more generally, call a polygon with an even number of sides semiregular if opposite sides are parallel and of equal length). A semiregular hexagon with side-lengths a, b, c, a, b, c can be tiled by lozenges in exactly

$$\frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)}$$

ways, where $H(0) = H(1) = 1$ and $H(n) = 1!2! \cdots (n-1)!$ for $n > 1$; an equivalent expression is

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

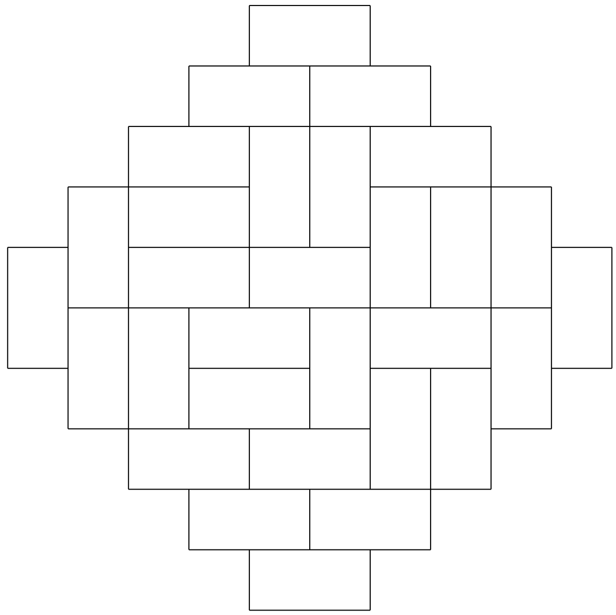

FIGURE 5. A domino tiling of an Aztec diamond (one of 32,768).

This result goes back to [**119**], although MacMahon was not studying tilings but plane partitions; plane partitions whose parts fit inside an a-by-b rectangle and with all parts less than or equal to c have three-dimensional Young diagrams that are easily seen to be in bijection with lozenge tilings of the a, b, c, a, b, c semiregular hexagon [**156**]. Figure 6 shows a regular hexagon of order 4, tiled by lozenges in one of the 232,848 possible ways.

Could there be higher-dimensional analogues of these tiling problems, involving sequences whose logarithms grow like n^3 (or faster)? Curiously, such analogues are almost entirely lacking (and I will briefly return to this issue later, in a speculative vein, at the close of the chapter). For now, I wish only to point out that any theory of exact enumeration of tilings that attempts to catch too many tiling problems in its net is likely to find that net ripped apart by the sea monsters of $\#P$-completeness. (For a definition of $\#P$-completeness, see [**125**].) Hardly anyone believes that $\#P$-complete problems can be solved efficiently. Beauquier, Nivat, Rémila, and Robson [**6**], Moore and Robson [**124**], and Pak and Yang [**129**][**130**] have given examples of classes of two-dimensional tiling problems exhibiting $\#P$-completeness. So there is little hope of solving the problem of counting tilings in its full generality, even in two dimensions. Still, there is much that can be done.

By far the most successful theory of enumeration of two-dimensional tilings is the theory of perfect matchings of a planar graph. A **perfect matching** of a graph $G = (V, E)$ is a subset E' of the edge-set E with the property that each vertex $v \in V$ is an endpoint of one and only one edge $e \in E'$. Problems involving perfect matchings of a planar graph G may not immediately appear to be tiling problems, but they can easily be recast in this framework: simply take the planar dual of G (the planar graph whose vertices, edges, and faces correspond respectively to the faces, edges, and vertices of G) and define a tile to be the union of any two

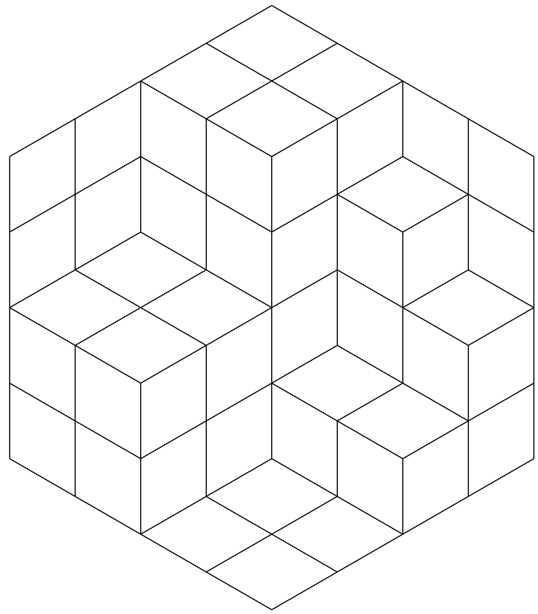

FIGURE 6. A lozenge tiling of a hexagon (one of 232,848).

regions in the dual graph that correspond to adjacent vertices in G. (To make this correspondence precise, one may need to limit the kinds of translation that can be applied to the prototiles.) In the other direction, some tilings correspond to perfect matchings. E.g., the domino tiling of Figure 5 corresponds to the perfect matching in Figure 7. For a more careful definition of Aztec diamonds and their dual graphs, see [**53**]. See also Figure 1 from [**135**], which shows the relationship between rhombus tilings of a hexagon and dual perfect matchings, and Figure 2 from [**135**], which shows the relationship between domino tilings of a square and dual perfect matchings.

(In the graph theory literature, a matching of a graph G is a subset E' of the edge-set E with the property that each vertex $v \in V$ is an endpoint of *at most one* edge $e \in E'$. I will comment later on the problem of counting matchings, or as physicists call them, monomer-dimer covers. Until then, I will use the term "matching" to refer exclusively to *perfect* matchings.)

The problem of counting matchings of a general graph is $\#P$-complete [**167**], but when the graph is planar (or not too far from being planar), the problem of counting matchings can be reduced to linear algebra, specifically, to the evaluation of determinants (and Pfaffians) of matrices. This technology was developed by the mathematical physicist Kasteleyn, who (using physicists' language) thought of his work as providing a way to evaluate "the partition function of the dimer model" (we will explain that language later). Thanks to Kasteleyn's work [**78**] [**79**] [**80**], much of the known body of theory relating to enumeration of tilings in the plane can be viewed as a subspecialty of the field of determinant evaluation, as developed by Krattenthaler and others (see for instance [**93**] and [**95**]). However, in many instances the matrices that arise do not belong to a general class of matrices to which established methods of determinant evaluation apply; in those instances, the only

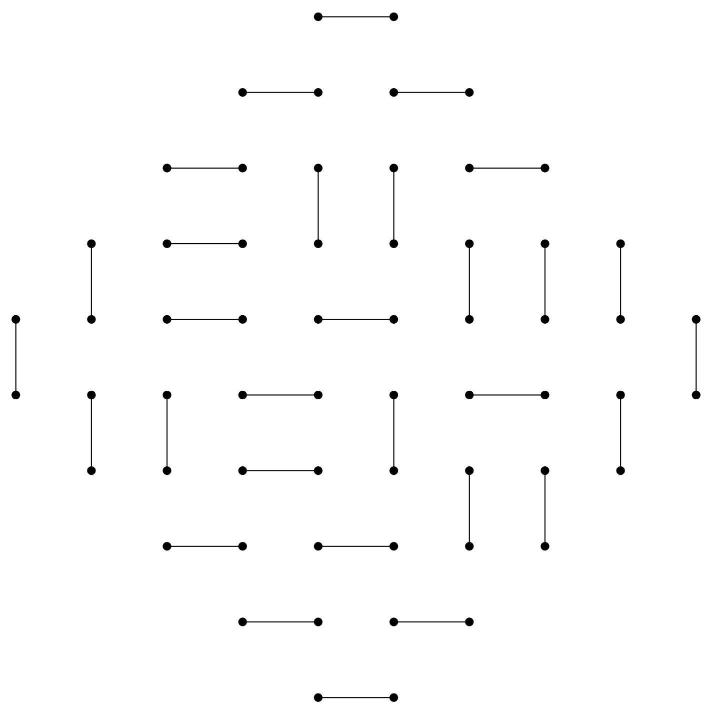

FIGURE 7. A matching of an Aztec diamond graph (one of 32,768).

known ways of evaluating the determinant make crucial use of the combinatorics and geometry of the tiling picture.

The main models I will treat are domino-tilings and lozenge-tilings, and the main methods of analysis I plan to discuss (in varying levels of detail) are: the transfer matrix method (Section 2); the Lindström-Gessel-Viennot method, Kasteleyn-Percus method, and spanning tree method (Section 3); representation-theory methods (Section 4); and Ciucu complementation and factorization, Kuo condensation, and domino shuffling (Section 5). In Section 6, I will try to give a sense of the broader context into which the enumeration of tilings fits, and convey some sort of historical sense of the way in which the field has evolved (or at least point the historically-minded reader toward earlier articles from which an accurate history of the subject could be assembled). It should be stressed that many of the tools discussed in Sections 2 through 5 were developed not by combinatorialists but probabilists, chemists, and physicists; a knowledge of the broader scientific literature has served combinatorialists well in the past and is likely to do so in the future.

In Section 7, I will discuss themes that have emerged from the study of matchings of planar graphs such as special patterns of edge-weights that, although nonperiodic, seem in some sense to be natural and in any case give rise to nice q-analogues of integer enumerations; three distinct ways in which the concept of symmetry plays a role; and certain sorts of patterns that often appear in the prime factorization of the integers that arise from enumeration of matchings.

In Section 8, I will describe some of the software that exists for exploring problems in enumeration of matchings (sadly in disuse and disrepair, but that surely will change if there is sufficient new activity in this field to justify investment in infrastructure). In Section 9, I will give an admittedly personal and biased list of issues on which progress is needed. Unlike [**137**], which focused on specific problems in enumeration, the current list is about larger themes.

My aim will be not to provide definitive and general statements of results, but to give attractive examples that will make combinatorially-inclined readers want to learn more about (and perhaps work in) this area, and will help researchers starting out in this area get a better sense of what they might want to read.

For a recent article on tilings by other authors, see [**2**].

2. The transfer matrix method

Looking down each of the six columns in Figure 1, we see either a vertical tile, the left halves of two horizontal tiles, or the right halves of two horizontal tiles. Moreover, if we know, for each of the six columns, which of these three pictures we see (Vertical, Left, or Right), then we know the whole tiling. For the tiling shown in Figure 1, the "code" is $VLRVLR$. However, not every length-six sequence consisting of V's, L's, and R's corresponds to an allowed tiling. The codes that correspond to tilings are precisely the ones that arise from paths of length six in the graph shown in Figure 8, starting at the circled node on the left and ending at the circled node on the right, where the symbols V, L, and R correspond respectively to edge of slope 0, -1, and $+1$. For instance, the path shown in bold has successive edges of type V, L, R, V, L, and R, and therefore corresponds to the tiling of Figure 1.

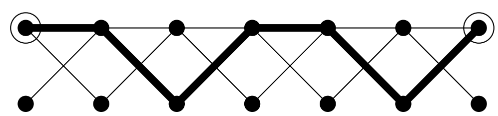

FIGURE 8. A trellis graph for counting domino tilings.

Figure 8 is an example of what I will call a **trellis graph** (though my use of the term is slightly different from the standard definition): a graph with vertex set $V_0 \cup V_2 \cup \cdots \cup V_n$ where each V_k $(0 \leq k \leq n)$ is finite and all edges connect a vertex in V_{k-1} to a vertex in V_k for some $1 \leq k \leq n$. For each $1 \leq k \leq n$, let M_k be the matrix with $|V_{k-1}|$ rows and $|V_k|$ columns whose i,jth entry counts the edges from the ith element of V_{k-1} to the jth element of V_k. Then it is easy to see that the number of paths from the ith element of V_0 to the jth element of V_n equals the i,jth entry in the matrix product $M_1 M_2 \cdots M_n$. We call the M_k's **transfer matrices**. In our case, all the matrices are the same matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, so we can readily compute the number of tilings as the upper-left entry of M^6, which is 13. More generally, we see that the number of domino-tilings of the 2-by-n rectangle is the upper-left entry of M^n, which is the nth Fibonacci number.

The same method can be applied to count domino-tiling of an m-by-n rectangle. The number of rows and columns in the matrix M grows exponentially with m, but Theorem 4.7.2 of [**158**] assures us that for fixed m, the number of domino-tilings of an m-by-n rectangle, viewed as a function of n, satisfies a linear recurrence relation with constant coefficients.

This method can be applied much more broadly to tiling problems in which the region being tiled is “essentially one-dimensional”, in the sense that only one dimension is growing as n increases. For example, if a_n denotes the number of ways to pack an 8-by-8-by-n box with 1-by-1-by-2 bricks that can be used in each of the three possible orientations, then the general theory tells us that a_n must satisfy a linear recurrence. However, the span of this recurrence could be extremely long; an off-the-cuff upper bound on the span is 10^{50}, which can probably be improved a bit but not by much without a good deal of work.

By using matrices whose entries are not just 0's and 1's, we can get more mileage out of the method. Suppose for instance we want to discriminate between tilings on the basis of how many horizontal or vertical dominos they have. Then we would look at powers of the matrix $M = \begin{pmatrix} y & x \\ x & 0 \end{pmatrix}$ (since each V corresponds to a vertical domino and each L or R corresponds to two horizontal half-dominos). The upper left entry of M^6 is $x^6 + 6x^4y^2 + 5x^2y^4 + y^6$, which tells us that the tiling of Figure 1 is one of exactly 6 tilings having 4 horizontal dominos and 2 vertical dominos.

Typically one exploits the transfer-matrix method by diagonalizing M (and if one is only interested in the growth rate of the entries of M^n, one merely computes the dominant eigenvalue of M, invoking the Perron-Frobenius theorem to conclude that the dominant eigenvalue is a simple eigenvalue of the matrix).

In rare cases, one can actually get formulas for the entries of the powers of M and skip the diagonalization step. One way of counting the lozenge tilings of semiregular hexagons provides an example of this: see Proposition 2.1 of [**41**] and its proof. One can argue that such a proof does not deserve to be called an example of the transfer-matrix method, since no linear algebra is involved. (Perhaps it should be called “enumeration by exact dynamic programming” or “the method of proving a result by finding a stronger result that can be proved by induction”.)

We conclude by returning to the x, y-enumeration of domino tilings discussed above, putting it in the framework of statistical mechanics, and indicating how the transfer matrix method can be applied to all one-dimensional lattice models. We already mentioned in Section 1 that domino tilings of a region are in bijection with matchings of the graph dual to the tessellation of the region by unit squares. In physics, a (perfect) matching is called a dimer cover or dimer configuration. We assign each dimer cover a weight equal to the product of the weights of its constituent edges, where a horizontal (resp. vertical) edge has weight x (resp. y). In physics, the weights have an interpretation in terms of energy (see [**5**]) but we will not concern ourselves with that here. The sum of the weights of the configurations is called the **partition function** , and is denoted by Z. The probability measure that assigns to each configuration probability equal to the weight of the configuration divided by the normalizing constant Z is called the **Gibbs measure** associated with that weighting. In the case where all configurations have the same weight, and the probability distribution is correspondingly uniform, we define the **entropy** of the system as the logarithm of the number of (equally likely) states. More generally, if a system has N states with respectively probabilities p_i, we define the entropy as $\sum_{i=1}^{N} p_i \log 1/p_i$, which coincides with the former definition if $p_i = 1/N$ for $1 \leq i \leq N$.

These concepts, with some work, can be extended to infinite systems as well, through appropriate rescalings and limit-procedures. Some care is helpful for dealing with the boundary conditions, but these turn out not to matter so much in one dimension. For instance, in the case of the 2-by-$2n$ dimer model with edge-weights x and y (which we take to be positive), we might take the logarithm of the number of tilings of the 2-by-n rectangle, divide by the area $2n$, and take the limit as $n \to \infty$, obtaining the normalized, or per-site, entropy of the model, a function of x and y. Or we might look at 2-by-n configurations in which dominos are allowed to protrude at the ends. The same asymptotics will prevail, since the entries of M^n must all grow at the same rate.

This argument is not applicable to higher-dimensional tilings, and indeed, the per-site entropy turns out to be extremely sensitive to the boundary conditions. For example, if one modifies the Aztec diamond of order n so that it has only one row of length $2n$ instead of two, the number of tilings drops from $2^{n(n+1)/2}$ down to 1.

In the case where all the states of a one-dimensional lattice model have the same probability, the transfer matrix method tells us that the entropy is just the logarithm of the dominant eigenvalue. In particular, for one-dimensional lattice models of this kind, if all the entries of the transfer matrix are 0's and 1's, the entropy must be the logarithm of an algebraic number. This, too, fails in higher dimensions (at least as far as is known); although the entropy for the square ice model introduced in section 4 is the logarithm of the algebraic number $8/(3\sqrt{3})$, the entropy of the dimer model on a square grid to be discussed in section 3 is G/π (where G is Catalan's constant), which is not believed to be the logarithm of an algebraic number.

3. Other determinant methods

3.1. The path method. Returning to Example 2 of Section 1, let us decorate each tile according to its orientation in the fashion shown in Figure 9, so that Figure 2, with all its tiles decorated, becomes Figure 10.

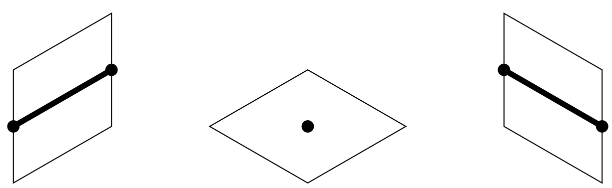

FIGURE 9. Decorated lozenges.

The decorations on the tiles yield a path from the leftmost node (marked P) to the rightmost node (marked Q), along with some nodes that are not visited by the path. In this way, one can see that the tilings of the region are in bijection with Dyck paths of length 6, and hence are enumerated by the Catalan number 14.

Figure 11 shows a domino tiling of a 3-by-4 rectangle decorated with a similar sort of path. This **Randall path** [**116**] is obtained by drawing a node at the midpoint of every other vertical edge (that is: when two vertical edges are neighbors in either the horizontal or vertical sense, exactly one of them has a node). Every tile then either has a single node in its interior or two nodes on its boundary, and

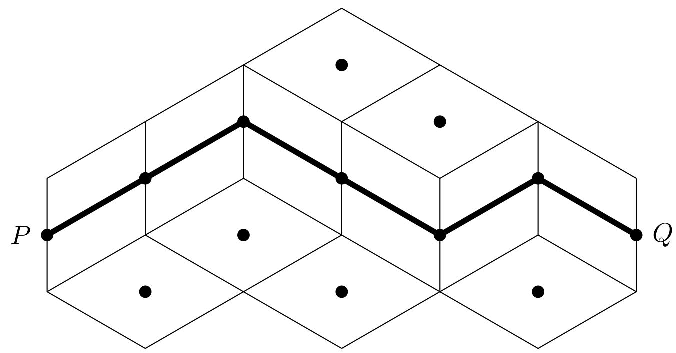

FIGURE 10. From tilings to paths.

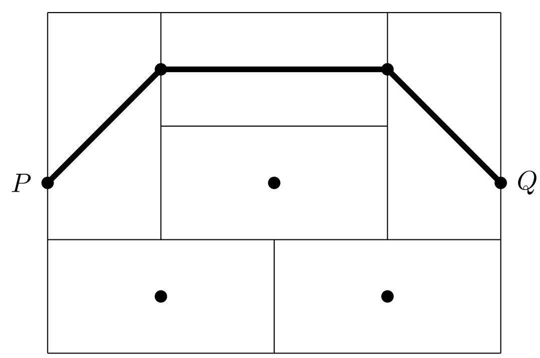

FIGURE 11. A tiling and a path.

in the latter case we join the nodes by a line segment. All the edges in the path go from left to right, with slope 0, +1, or −1. It is easy to use dynamic programming to show that there are 11 paths from P to Q, and hence 11 domino tilings of the rectangle, since there is a bijection between tilings and paths.

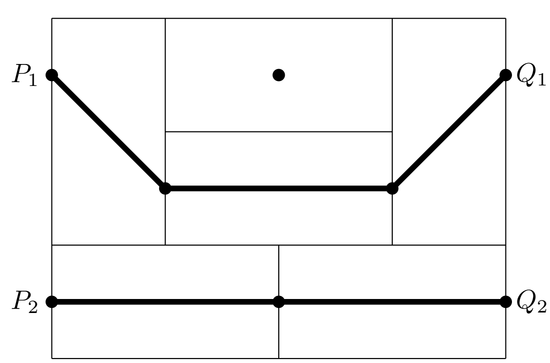

FIGURE 12. A tiling and two nonintersecting paths.

Figure 12 shows the other way to decorate the tiling of Figure 11 by drawing a node on every other vertical edge. Now we get not just one path but two non-intersecting paths, one joining P_1 to Q_1 and one joining P_2 to Q_2. There are 6 paths from P_i to Q_j when $i = j$ but only 5 when $i \neq j$. Lindström's lemma [**113**], rediscovered by Gessel and Viennot [**65**], shows that the number of pairs of nonintersecting paths from P_1 and P_2 to Q_1 and Q_2 respectively equals the determinant of the 2-by-2 matrix whose i, jth entry counts the paths from P_i to Q_j. Thus the number of tilings is

$$\begin{vmatrix} 6 & 5 \\ 5 & 6 \end{vmatrix} = 11 \text{ (again)}$$

since there is a bijection between tilings and pairs of nonintersecting paths.

This method generalizes to larger collections of paths in an acyclic directed graph. As long as the only way to join up the points $P_1, \ldots, P_m$ to the points $Q_1, \ldots, Q_m$ by nonintersecting paths is to join P_i to Q_i for all $1 \leq i \leq m$, the number of such families of nonintersecting paths equals the determinant of the matrix whose i, jth entry is the number of paths from P_i to Q_j. In many cases one can use this to count matchings of graphs. By employing this method, which was first used in [**67**] and was also discovered by chemists John and Sachs [**77**], one can write the number of lozenge tilings of a semiregular hexagon as the determinant of a matrix of binomial coefficients [**65**] (see also [**19**]); likewise one can write the number of domino tilings of an Aztec diamond as the determinant of a matrix of Schröder numbers [**17**][**55**]. A recent (and purely bijective) approach to counting nonintersecting Schröder paths is [**13**]. For historical background on Lindström's Lemma, see footnote 5 in [**96**] as well as the Notes to Chapter 2 of [**158**].

3.2. The permanent-determinant and Hafnian-Pfaffian method. Consider the 3-by-4 grid graph (dual to the 3-by-4 rectangle considered in the previous subsection) and two-color the vertices so that each black vertex has only white neighbors and vice versa. Number the white vertices 1 through 6 and the black vertices 1 through 6. The ordinary 12-by-12 adjacency matrix of this graph has determinant 0, as does the 6-by-6 bipartite adjacency matrix whose i, jth entry is 1 or 0 according to whether the ith white vertex and jth black vertex are connected or not. However, if we let K be the modified bipartite adjacency matrix whose i, jth entry is 1, $\mathbf{i}$ $(= \sqrt{-1})$, or 0 according to whether the ith white vertex and jth black vertex are connected by a horizontal edge, a vertical edge, or no edge, then we find that the modulus of the determinant of K equals the number of matchings of the graph. Likewise, if we replace the 1's in the original 12-by-12 adjacency matrix with suitable unit complex numbers (complex numbers of modulus 1), we obtain a matrix whose determinant has modulus equal to the square of the number of matchings of the graph.

This is a consequence of a general theorem of Kasteleyn. We discuss first the case of bipartite graphs (tacit in Kasteleyn's original papers [**78**] [**79**] [**80**] and made explicit by Percus [**131**]). If a bipartite graph G with $2N$ vertices is planar, then the non-zero entries of its bipartite adjacency matrix can be replaced by unit complex numbers so that the modulus of the determinant of the resulting modified matrix equals the permanent of the original matrix and hence equals the number of perfect matchings of G. More specifically, let us say (using terminology introduced by Kuperberg) that an edge-weighting of a bipartite planar graph that assigns each edge a unit complex number is **(Kasteleyn-)flat** if the alternating product of the edge-weights around a face $e_1^{+1}e_2^{-1}e_3^{+1}e_4^{-1}\cdots$ is -1 or $+1$ according to whether the number of edges is 0 or 2 mod 4 (this condition does not depend on the cyclic ordering of the faces). In our case, each face has 4 sides, so we need the alternating product to be -1, and indeed we have $(\mathbf{i})^{+1}(1)^{-1}(\mathbf{i})^{+1}(1)^{-1} = -1$. (The fact that flatness holds around the exterior face is a consequence of flatness around the interior faces; one can also check directly in the case of the 3-by-4 grid graph that the alternating product around the 14-sided exterior face is $\mathbf{i}^3/\mathbf{i}^3 = +1$.) Let K be the bipartite adjacency matrix of a bipartite planar graph G, with non-zero entries replaced by Kasteleyn-flat edge weights. It can be shown that if one expands the determinant of K into $N!$ terms then all the non-zero terms in the expansion are unit complex numbers with the same phase. Consequently, the matchings of the graph can be counted by taking the modulus of the determinant of the modified adjacency matrix obtained from the flat edge-weighting. This is the **permanent-determinant method**.

To see this method applied to general rectangular subgraphs of the square grid, see [**133**]. It can be shown that the per-site entropy of the dimer model on a square grid is $G/2\pi$, where $G = \sum_{n=0}^{n-1}(-1)^n/(2n+1)^2 = 1/1 - 1/9 + 1/25 - \ldots$ (Catalan's constant).

Now suppose G is a non-necessarily-bipartite planar graph with $2N$ vertices. Recall from [**101**] that the Pfaffian of an antisymmetric $2N$-by-$2N$ matrix A equals a signed sum of terms of the form $a_{i_1,j_1}a_{i_2,j_2}\cdots a_{i_N,j_N}$ where $i_1, j_1, \ldots, i_N, j_N$ ranges over all permutations of $1, 2, \ldots, 2N$ for which $i_1 < j_1$, $i_2 < j_2$, $\ldots$, $i_N < j_N$ and $i_1 < i_2 < \cdots < i_N$; such permutations are in one-to-one correspondence with the ways of dividing $1, 2, \ldots, 2N$ into pairs. Kasteleyn showed that if a graph G is planar, then there is an antisymmetric matrix A whose i,jth entry is a unit complex number when the ith and jth vertices of G are adjacent and 0 otherwise, such that the modulus of the Pfaffian of A equals the number of perfect matchings of G. Moreover, the determinant of A equals the square of the Pfaffian of A, and hence has modulus equal to the square of the number of perfect matchings of G. This is the **Hafnian-Pfaffian method** (where loosely speaking the Hafnian is to the Pfaffian as the permanent is to the determinant; see [**101**] for details).

Kasteleyn's method also works for graphs with (positive) edge-weights attached; one simply multiplies the weights in the weighted adjacency matrix by the required "Kasteleyn phases". Kasteleyn showed how to modify the method to cope with graphs on surfaces of small genus, but a discussion of this would take us too far afield.

We return to the bipartite case. It is known that the entries of the inverse Kasteleyn-Percus matrix K^{-1} have significance of their own; for instance, when the ith white vertex and the jth black vertex are adjacent, the absolute value of the j,ith entry of K^{-1} is the probability that the edge joining those two vertices will belong to a matching chosen uniformly at random. (For this and much else on probabilistic aspects of matchings, see [**83**].) A recent treatment of the inverse Kasteleyn matrix for Aztec diamond graphs is [**22**].

Kuperberg [**105**] has shown that the Gessel-Viennot matrix for a bipartite planar graph (typically of size $O(n)$) and the Kasteleyn-Percus matrix of that graph (typically of size $O(n^2)$) are not really different entities, and that it is possible to turn the latter into the former by means of combinatorially meaningful determinant-preserving matrix reductions. Extending this to the Pfaffian case for non-bipartite matching remains an open problem.

3.3. The spanning tree method. Figure 13 shows a domino tiling of a 5-by-5 square with a corner cell removed. The centers of certain cells have been marked, displaying a 3-by-3 grid within the 5-by-5 grid; when a domino contains a marked square, we draw an arrow from the center of its marked square to the center of its unmarked square. The dominos containing a marked square determine a spanning tree rooted at the missing corner, as shown in Figure 14.

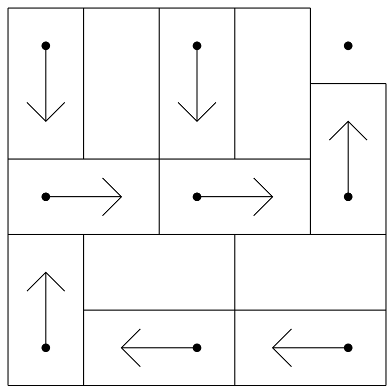

FIGURE 13. A tiling and a tree.

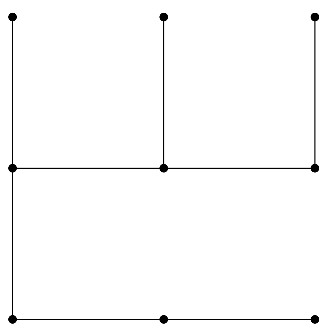

FIGURE 14. The tree made clearer.

Temperley [**163**] gave a method for turning certain dimer problems into spanning tree problems. This method was extended [**86**] and later was shown to apply to all bipartite planar graphs [**87**].

The strength of the method hinges on the fact that the Matrix Tree Theorem (see e.g. Theorem 5.6.8 of [**157**] or p. 57 of [**10**]) gives a formula for the number of spanning trees of a graph as the determinant of an associated matrix. For applications, see [**29**],[**31**],[**117**],[**118**]. To my knowledge, no one has as of yet exposed a link between the Kirchhoff matrix of a bipartite planar graph and either the Kasteleyn-Percus matrix or the Gessel-Viennot matrix.

The connection between tilings and trees yields a connection with the abelian sandpile model; see [**60**].

Matchings and spanning trees fit into a bigger picture involving specializations of the Tutte polynomial. See [**168**] as well as [**154**] and [**21**].

4. Representation-theoretic methods

(Note: Much of the information in this section was kindly provided by Greg Kuperberg.)

There are two broad classes of applications of representation theory to enumeration of matchings, which we might call "trace" and "determinant" methods. In the trace method, one interprets the number of tilings as the trace of some matrix and then uses representation theory to compute that trace. (In many cases, the matrix in question is the identity matrix so the trace is just the dimension of the vector space.) In the determinant method, one interprets the number of tilings as the determinant of some matrix and then uses representation theory to compute that determinant.

Larsen's proof of the formula for the number of domino tilings of an Aztec diamond [**53**] is an example of the trace method. Kuperberg [**100**] and Stembridge [**159**] (the latter building upon Proctor [**132**]) independently applied this approach to the study of lozenge tilings of hexagons under various symmetry constraints. In some of these results, the matrix one studies is the permutation matrix of an involution, so that the trace counts the fixed points of the involution.

Examples of the determinant method can be found in [**103**]. Here, one uses the representation theory of the Lie algebra sl(2). If e, f, and h form a basis for sl(2) in the usual way, a block of the ladder operator e goes from the $h = -1$ eigenspace to the $h = +1$ eigenspace, and the number of tilings equals the determinant of this block (up to a normalization factor). The representation theory of sl(2) lets one diagonalize this operator and thereby compute the determinant. This works for various symmetry classes of lozenge tilings of hexagons. It would be interesting to see trace methods applied to symmetry classes of domino tilings of Aztec diamonds.

Representation-theoretic arguments have also been successful in enumerative problems involving ASMs (Alternating Sign Matrices). Although the initial view of ASMs in [**122**] does not seem like it has anything to do with tilings, there is an attractive interpretation of ASMs as tilings that fill regions called "supergaskets" with curved tiles called "gaskets" and "baskets". See Figure 15 below for a picture of such a tiling, and the article [**138**] for an explanation of the link between tilings of this kind and ASMs viewed as n-by-n arrays of +1's, −1's, and 0's satisfying various constraints. It is also worth mentioning that domino tilings of Aztec diamonds were in a sense discovered by Robbins and Rumsey [**149**] as an outgrowth of their work on ASMs, even though the tilings picture was not made explicit. The picture of ASMs as tilings has not been helpful to researchers, but the dual picture of ASMs as states of the six-vertex model (see Figure 16) plays a key role in Kuperberg's proof [**102**] that the number of ASMs of order n is equal to the product $\prod_{k=0}^{n-1}(3k+1)!/(n+k)!$ (conjectured by Mills, Robbins and Rumsey [**122**] and first proved by Zeilberger [**175**]).

In the **six-vertex model** (also called the **square ice model**), edges of a square grid are assigned orientations in such a way that at each vertex there are two arrows pointing in and two arrows pointing out. (At each vertex, there are

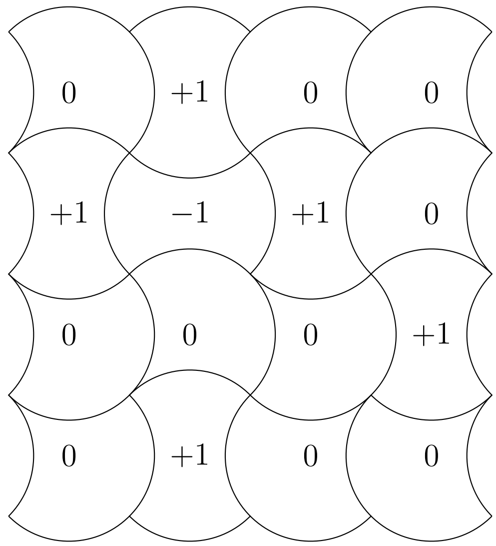

FIGURE 15. A tiling by baskets and gaskets (one of 42), with the associated alternating-sign matrix.

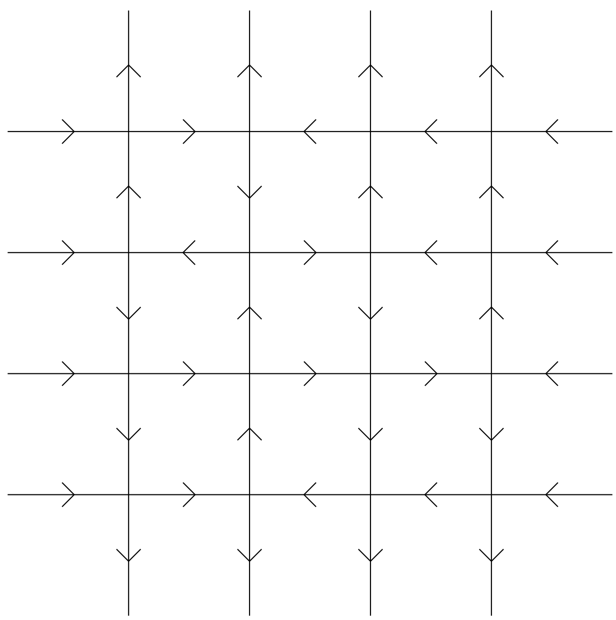

FIGURE 16. A state of the six-vertex model with domain-wall boundary conditions.

$\binom{4}{2} = 6$ possible ways to choose orientations for the incident edges: hence the name of the model.) As in the case of domino and lozenge tilings, we find that the boundary conditions for the six-vertex model that are most relevant to physics are not the ones most conducive to exact enumeration. Just as the logarithm of the number of domino tilings of the Aztec diamond falls short of the full entropy of the dimer model on the square grid, the logarithm of the number of alternating-sign matrices falls short of the full entropy of the square ice model. Curiously, although the former entropy is not believed to be the logarithm of an algebraic number, the latter number is the logarithm of the algebraic number $8/(3\sqrt{3})$ (traditionally written as $(4/3)^{3/2}$ in the physics literature, starting with its original appearance in [**112**]). Although the model with free boundary conditions is exactly solvable in the sense of physics (that is, in the sense that one can find a formula for the entropy), it is not known to be exactly enumerable (in the non-asymptotic sense). For exact enumerations, it appears one must impose the "domain-wall boundary conditions" (so dubbed by Korepin; see [**91**]) that prevail in the particular square ice configurations that correspond to ASMs. We will use this term in a more general way to denote boundary conditions in which different "frozen" phases of the system govern the behavior of different parts of the boundary, so that the interior is subject to competing influences (and can spontaneously divide itself into multiple macroscopic domains in which the tiling displays qualitatively different microscopic behavior).

For enumeration of ASMs of order n, the relationship with representation theory has been quite intimate, and some of the proofs in the literature make uses of different sorts of representation theory arguments at different stages in the proof. In the first place, when one relaxes the grid-structure underlying the square ice model and looks at ice-configurations of a more general kind, one finds Reidemeister-style moves that preserve the partition function or affect it in predictable ways; this is the theory of Yang-Baxter relations, and it is tied in with the q-deformed representation theory of sl(2). Against this background, Izergin and Korepin were able to write the partition function in terms of the determinant of an n-by-n matrix ([**73**],[**91**]) involving $2n$ variables associated with the rows and columns of the ice-grid. When one seeks to apply this formula to count ASMs, one obtains an indeterminate of type $0/0$, but by approaching the singularity along an artfully chosen path, Kuperberg was able to obtain the desired enumeration. For more details, see [**102**] and [**106**]. Okada [**127**] brought Lie group characters into the picture, with two distinct quantum parameters playing roles (one tending toward 1 as in Kuperberg's original proof, the other a cube root of unity). Okada showed that for all the proved or conjectural formulas in the second of Kuperberg's articles, the requisite determinant evaluations are associated with characters of Lie groups; for instance, the formula Kuperberg proved in [**102**] is associated with an irreducible representation of GL$(2n)$. Symmetry classes of alternating-sign matrices have also been enumerated by Razumov and Stroganov [**144**][**145**][**146**].

As in the case of other tiling problems with nice answers, we find that enumeration with the extra constraint of symmetry also gives rise to tiling problems with nice answers (which are usually harder to prove).

A central mystery of the subject is the remarkable fact that the number of fully symmetrical lozenge tilings of a regular hexagon of side $2n$ (which are in bijection with Totally Symmetric Self-Complementary Plane Partitions of order $2n$) are equinumerous with ASMs of order n. Both are governed by the product formula that appears above, but there is not at this time a conceptual explanation of why the two numbers should be the same. For instance, despite intensive work by combinatorialists over the past several decades, no bijection between TSSCCPs and ASMs is known.

5. Other combinatorial methods

A very fertile approach to enumeration of matchings, discovered by Eric Kuo while he was an undergraduate member of the Tilings Research Group at MIT, is a graphical analogue of Dodgson condensation. (For background on Dodgson condensation, see [**16**].) Figure 17 shows an Aztec diamond graph G of order 4 divided by two diagonals of slope $+1$ and two diagonals of slope -1. If one cuts the figure with

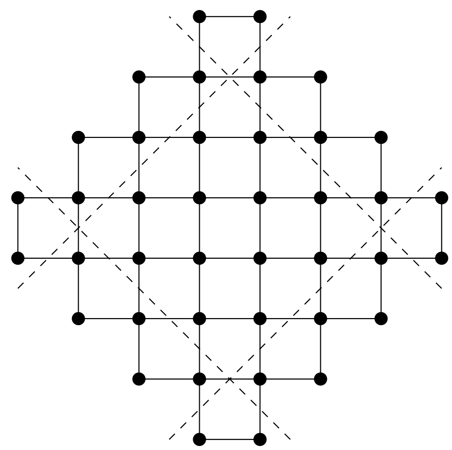

FIGURE 17. Kuo condensation for Aztec diamonds.

one of the two lines of slope $+1$ and one of the two lines of slope -1, one obtains an Aztec diamond graph of order 3 (and three smaller regions). Let G_N, G_S, G_E, and G_W respectively be the Northern, Southern, Eastern, and Western 3-by-3 Aztec diamond subgraphs obtained in this way, and let G_C be the central 2-by-2 Aztec diamond subgraph. Then the original version of **Kuo condensation** (Theorem 2.1 of [**99**]) tells us that $M(G)M(G_C) = M(G_N)M(G_S) + M(G_E)M(G_W)$, where $M(\cdot)$ denote the number of matchings of a graph. In this case, since the graphs G_N, G_S, G_E, and G_W are isomorphic, we obtain the relation $M(G_4)M(G_2) = 2M(G_3)^2$ (where G_n denote the Aztec diamond graph of order n), and more generally $M(G_{n+1})M(G_{n-1}) = 2M(G_n)^2$, which allows us to solve for $M(G_n)$ using the initial conditions $M(G_0) = 1$ and $M(G_1) = 2$. See [**99**] and [**98**]. For more details on Kuo condensation and numerous applications, see (in chronological order) [**171**] [**170**] [**90**] [**62**] [**35**] [**36**] [**34**] [**23**].

When one applies Kuo condensation repeatedly, keeping track of all the different subgraphs of the original graph that occur, one accumulates data that are mostly easily stored in a three-dimensional array in which the entries form an octahedral lattice (dual to a cubical lattice), in which the numbers $M(G)$, $M(G_C)$, $M(G_N)$, $M(G_S)$, $M(G_E)$, and $M(G_W)$ are associated with vertices that form an octahedron. That is to say, the enumerations of matchings satisfy the octahedron relation. This equation originated in the theory of discrete integrable systems, and has found numerous applications in combinatorics. Here it is worth pointing out that Kuo condensation can be applied with weightings of a fully general kind, where each edge is assigned its own weight independently of all the others.

A superficially different but probably related approach to enumerating matchings is (generalized) **domino-shuffling** [**140**], also known as **urban renewal**. There is some confusion in the literature regarding the intended meanings of these two terms. At least originally, urban renewal referred to a certain kind of local modification of an edge-weighted bipartite planar graph that has a predictable effect on the partition function. In this it drew much inspiration from the Temperley-Fisher method [**164**] of enumerating matchings, which used similar local mutations to turn states of the dimer model into states of an Ising model on a suitable graph. (See Chapter 13 of [**125**], particularly sections 13.6 and 13.7. Also see [**121**], in which the the dimer and Ising models are solved by use of Pfaffians in quite complete detail. Yet another worthwhile article on the subject is [**123**].) The inspiration for the terminology drew upon an image of small pieces of the graph as being "cities" that communicate with the outside world via certain ports (as well as being an obscure pun on a style of urban planning that had a brief vogue in the 1970s). Strictly speaking, urban renewal is an operation not on graphs but on edge-weighted graphs. By performing such operations throughout a graph in a systematic fashion (this is what was meant by "generalized domino-shuffling", on account of its relationship to the domino-shuffling algorithm introduced in [**53**]), one can often obtain a simpler graph of the same kind, giving rise to a recurrence relation for the number of matchings. In addition to having "local factors" associated with individual edges, it is often handy to collect together "global factors" that are associated not with individual edges but with the graph as a whole. The basic urban renewal operation is depicted in Figure 18. If an edge-weighted graph G having a city of the kind shown

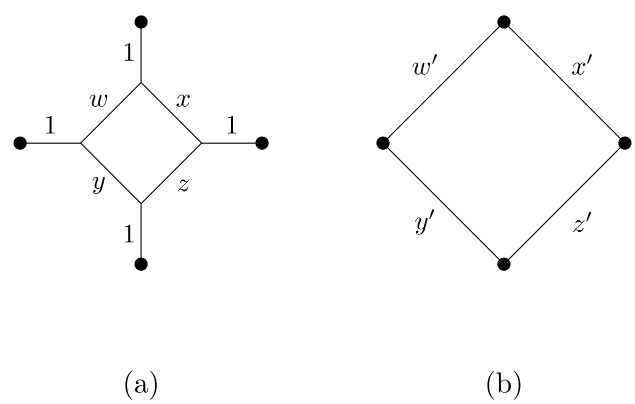

Figure 18. Urban renewal.

in panel (a) is modified by replacing that city by a city of the kind shown in panel (b) (remaining otherwise unchanged), with $w' = z/(wz + xy)$, $x' = y/(wz + xy)$, $y' = x/(wz+xy)$, and $z' = w/(wz+xy)$, then the sum of the weights of the matchings of the first graph equals $wz+xy$ times the sum of the weights of the matchings of the second graph. Like Kuo condensation, generalized domino-shuffling can be applied with weightings of a fully general kind, where each edge is assigned its own weight independently of all the others. Using urban renewal allows one to show fairly directly that the number of matchings of the Aztec diamond graph of order n equals 2^n times the number of matchings of the Aztec diamond of order $n-1$. Urban renewal was used in [**172**] to show that, for a family of finite graphs derived from the infinite square-octagon graph, the number of matchings is given by 5^{n^2} or $2 \times 5^{n^2}$ (depending on which member in the family is being discussed).

Di Francesco[**50**] gives a way to think about generalized domino-shuffling that unifies it with the octahedron recurrence, via certain sorts of discrete surfaces in the octahedron lattice.

The usefulness of generalized domino-shuffling is partly limited by the adherence to a square grid. Ciucu's complementation method [**26**] remedies this defect by introducing the idea of one weighted graph H' being the complement of another weighted graph H relative to some ambient graph G (the "cellular completion" of H), and establishing a formula relating the partition functions for matchings of H and matchings of H' (Theorem 2.1); the Δ-factors in that theorem can be recognized as a generalization of the face-factors in generalized domino-shuffling. A similarly flexible factorization theorem [**25**] gives a way to write the partition function of a weighted plane graph with reflection-symmetry as the product of the partition functions of two smaller graphs (up to a predictable multiplicative factor). For applications, see [**27**] and [**109**]. It is worth mentioning that Yan and Zhang [**171**] used Ciucu factorization to prove a version of Kuo condensation. Thus in a sense Ciucu factorization implies Kuo condensation.

Finally, a different sort of proof of the enumeration of lozenge-tilings of a hexagon comes from Krattenthaler's bijective proof [**94**] of Stanley's hook-content formula for the generating function for semistandard Young tableaux of a given shape. His proof yields, as a spin-off, a way of sampling from the uniform distribution on the set of lozenge tilings of a semiregular hexagon (by way of associated semistandard tableaux). It is perhaps too isolated an achievement to be considered a method, but it deserves study and imitation.

6. Related topics, and an attempt at history

The monomer-dimer model was first considered as a simplified model of a gas [**61**] and later as a model of adsorption of diatomic molecules on a surface. However, the mathematical difficulties posed by the monomer-dimer model in two or three dimensions, combined with a certain amount of mismatch between physical reality and the simplified model, did not lead to successful applications of the theory.

Fortunately for combinatorialists, physicists like Fisher, Kasteleyn and Temperley were not dissuaded and, emboldened by Onsager's success [**128**] in exactly solving the two-dimensional Ising model, invested considerable effort and ingenuity in finding exact solution methods (described in Sections 2–5) for the pure dimer model [**78**][**164**], even though that model is even more removed from the aforementioned target applications than the monomer-dimer model. Lieb's exact solution of the square ice model [**112**] — that is, his calculation of the per-site entropy — and the close agreement between Lieb's answer and the experimentally-determined residual entropy of actual (non-square!) three-dimensional ice gave further impetus to the study of two-dimensional lattice models in statistical mechanics.

The statistical mechanics literature (to which [**5**] serves as an excellent introduction) provides a unifying point of view for models that at a combinatorial level might seem to be unrelated. For instance, the dimer model on a square grid and the square ice model can both be seen as special cases of the eight-vertex model (an extension of the six-vertex model that permits vertices with all arrows pointing in or all arrows pointing out), with different sorts of weights attached to the edges of the graph; the dimer model is easier to solve than the square ice model because the former is part of the "free-fermion regime" of the eight-vertex model [**56**]. Statistical physicists Grensing, Carlsen, and Zapp [**68**] were the first researchers in any subject to look at domino tilings of Aztec diamonds, or rather (in their way of looking at things) dimer configurations in the dual graph. They went so far as to assert without proof the formula for the number of dimer configurations, but their published explorations stopped there, presumably because the model did not give rise to the known per-site entropy; they thereby missed out on discovering not just a beautiful combinatorial theorem but a rich body of theory that grew out of it, relating the dimer model to PDE and algebraic geometry [**83**]. This story bears out the observation that undiscovered mathematical treasures often lie in the trashbins of physicists.

A more harmonious match between theory and reality arose from work of theoretical chemists studying benzenoid hydrocarbons. These molecules, often depicted as having rings of alternating single and double bonds, actually have a more subtle quantum-mechanical nature, and can be viewed to a first approximation as being in a superposition of different valence-states or "Kekulé structures" [**67**]. The number of Kekulé structure on a graph has implications for its chemical properties, so chemists developed methods for counting these structures; indeed, a version of the Lindström-Gessel-Viennot method discussed above was found independently by Horst Sachs and his co-workers. (The class of regions mentioned in Example 2 of Section 1 was originally studied by chemists.) See [**77**], [**47**], and [**151**].

For a more recent application of tilings to the physical sciences, see [**9**], which like [**61**] uses the monomer-dimer model as a model of adsorption of diatomic molecules on a surface.

There has also been work on stepped surfaces (e.g., [**8**], [**110**], and [**59**]) of which tilings can be seen as two-dimensional projections (cf. our earlier discussion of Figure 6 and its interpretation as the three-dimensional Young diagram of a plane partition). Here the interests of physicists have a certain amount of overlap with the interests of probabilists and representation theorists. A phrase that has recently been associated with this line of work is "integrable probability theory" [**11**].

Probabilists are often interested in limit-shapes arising from tiling models when one takes the uniform distribution (or, more generally, distributions arising from a weighting) on the set of tilings of a large region and sends the size to infinity. For both random domino tilings of large Aztec diamonds and random lozenge tilings of large regular hexagons, one sees (with overwhelmingly high probability) a sharp boundary between a central circular region in which tiles of different orientations are mixed together and outer regions in which all tiles are aligned with their neighbors [**39**],[**41**]. There is a great deal of interest in the corresponding problem for alternating-sign matrices; there is a widely-believed conjecture for the asymptotic shape of the boundary between the central "liquid" domain and the four outer "solid" domains [**83**] of large random ASMs, but no rigorous proof is currently known. See [**42**], [**43**], and [**44**].

Probabilists are also interested in algorithms for randomly sampling from the uniform distribution (or, more generally, distributions arising from a weighting) on the set of tilings of a large region. In the case of matchings of planar graphs, a number of methods are available; see Section 8 for a discussion of this. Note that one method for random sampling is to iteratively make decisions in accordance with conditional probabilities (obtainable as ratios of tiling-enumerations); in the case of matchings of planar graphs, this is feasible because of Kasteleyn's method, and some tricks introduced by David Wilson [**169**] make this competitive with other methods. This method of exactly sampling from the uniform distribution cannot currently be applied to alternating-sign matrices because we lack exact formulas for the relevant conditional probabilities. It seems unlikely that the counting problems that arise are anywhere close to $\#P$-complete, but we cannot currently rule out this possibility.

The broad applicability of Kasteleyn's method has inspired a whole line of research in computer science called the theory of holographic algorithms, in which one tries to see how many other counting problems one can solve using variants of Kasteleyn's trick, via computational elements called matchgates. See [**70**] and [**18**].

The Aztec diamond and its relatives have also played a role in the study of random matrices, the connecting link being fluctuations in the boundaries between phase-domains; e.g., in the Aztec diamond of order n, fluctuations are of order $n^{1/3}$ and converge in distribution to the Tracy-Widom distribution. For more about boundary fluctuations, see [**76**]. For an account more focused on lozenge tilings, see [**58**].

Kuo's condensation lemma is related to the octahedron relation (also called the discrete Hirota equation; see section 12 of [**97**]). In this way, the study of enumeration of matchings leads to the theory of (discrete) integrable systems, and has connections to cluster algebras; [**66**] and [**50**] are good sources of more information.

Lastly, it should be mentioned that tilings, like other combinatorial objects, often arise in connection with algebraic objects. Standard examples of this phenomenon are matchings of complete bipartite graphs (which are implicit in the formula for the determinant of a matrix expanded as a sum of $n!$ monomials) and Young tableaux (which occur throughout the representation theory of the symmetric group). Matchings of Aztec diamond graphs, in disguised form, index the terms the expansion of the λ-determinant of Robbins and Rumsey [**149**] as a sum of Laurent monomials. Since lozenge tilings of a semi-regular hexagon can be converted to semistandard Young tableaux, they index terms in Schur polynomials associated with rectangular Young diagrams. More recently, square-triangle tilings of equilateral triangles have been used in a modern reformulation of the Littlewood-Richardson rule for expressing the product of two Schur functions as a linear combination of Schur functions [**143**]; see [**176**].

7. Some emergent themes

7.1. Recurrence relations. When have a sequence of tiling problems, the answer is the integer sequence $a_1, a_2, \ldots$ whose nth term counts the number of solutions to the nth tiling problem. What kinds of questions have nice answers? Niceness is subjective, so for present purposes we define a "nice answer" to be a sequence that appears in the mathematical literature and can be found via Sloane's On-line Encyclopedia of Integer Sequences (`http://oeis.org`) or has patterns that can be inferred empirically (e.g., satisfies a linear recurrence relation). When a_n grows exponential in n^2 (as is usually the case for interesting two-dimensional tiling enumerations), there is no chance of finding a linear recurrence. So what else can we try?

One answer is, rational recurrences. The sequence $a_n = 2^{n(n+1)/2}$ satisfies the recurrence $a_{n+1} = 2a_n^2/a_{n-1}$, and this is not accidental; writing the recurrence in the form $a_{n-1}a_{n+1} = a_n a_n + a_n a_n$, we see it as an instance of Kuo condensation. Applying this insight in reverse, one can start with a nonlinear recurrence like $a_{n-2}\,a_{n+2} = a_{n-1}\,a_{n+1} + a_n^2$ (satisfied by the Somos-4 sequence 1,1,1,1,2,3,7,23,59,... described in [**64**] and [**63**]) and reverse-engineer an appropriate family of bipartite planar graphs whose nth member has a_n matchings; see [**14**] and [**155**]. Just as the existence of a linear recurrence satisfied by the sequence $a_1, a_2, \dots$ is signaled by the existence of a value of m for which the Hankel matrix

$$H(a_{n+1}, a_{n+2}, \dots, a_{n+2m-1}) := \begin{pmatrix} a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ a_{n+2} & a_{n+3} & \cdots & a_{n+m+1} \\ \vdots & \vdots & \ddots & \\ a_{n+m} & a_{n+m+1} & \cdots & a_{n+2m-1} \end{pmatrix}$$

appears to be singular for all n in some range, the existence of a Somos-type recurrence satisfied by the sequence $a_1, a_2, \dots$ is signaled by the existence of values of k and m for which the Somos matrix

$$H(a_{n+1}, \dots, a_{n+2m-1}) * T(a_{n+k+1}, \dots, a_{n+k+2m-1})$$

appears to be singular for all n in some range, where $T(\dots)$ denotes the Toeplitz matrix obtained by writing each row of $H(\dots)$ backwards and $*$ denotes the Kronecker product of matrices (Somos, personal communication). E.g., the 3-by-3 Somos matrix

$$H(a_1, a_2, a_3, a_4, a_5) * T(a_3, a_4, a_5, a_6, a_7) = \begin{pmatrix} a_1a_5 & a_2a_4 & a_3a_3 \\ a_2a_6 & a_3a_5 & a_4a_4 \\ a_3a_7 & a_4a_6 & a_5a_5 \end{pmatrix}$$

is singular when $a_1, a_2, \dots$ satisfies the Somos-4 recurrence, and this remains true when all subscripts are shifted by adding a constant. It seems possible that some tiling-enumeration problems for which no exact formula has been discovered will turn out to governed by nonlinear recurrences, which might be discoverable by use of Somos matrices and related tools.

Inasmuch as rational recurrences like Somos-4 often arise as projections of a three-dimensional octahedron relation, an attractive research strategy for finding a nonlinear recurrence might be to imbed the sequence as a strand of a higher-dimensional array that satisfies the octahedron relation. This is one way to think about the approach to counting domino tilings of squares used in [**141**], although of course that construction had the benefit of hindsight (in the form of Kuo condensation).

7.2. Smoothness. In the absence of visible patterns linking the terms of a sequence, one sign of an underlying regularity is properties of the individual terms themselves, as integers. One sort of pattern can appear when one looks at how the integers factor into primes. Call a sequence $a_1, a_2, \dots,$ **smooth** if the largest prime factor of a_n is $O(n)$, and **ultra-smooth** if the largest prime factor of a_n is $O(1)$, that is, bounded. (Another word that is sometimes used instead of "smooth" is "round".) For example, $H(3n)H(n)^3/H(2n)^3$ (the number of lozenge tilings of a regular hexagon of side n) is smooth; $2^{n(n+1)/2}$ (the number of domino tilings of an Aztec diamond of order n) is ultra-smooth. Quite often we find that the answers to enumerative questions are smooth sequences, and the smoothness is often a key to finding an exact product formula.

Powers of 2 arise from looking at matchings of Aztec diamond graphs [**53**] associated with the regular tiling of the plane by squares; powers of 5 arise from looking at matchings of "fortress graphs" [**172**] associated with the semiregular tessellation of the plane by squares and octagons; and powers of 13 arise from looking at matchings of "dungeon" and "Aztec dungeon" graphs [**27**] associated with the semiregular tessellation of the plane by squares, hexagons, and dodecagons, respectively. In all three cases, there is a highly symmetrical tessellation of the plane in the background, and the tiling problem is best thought of as the problem of counting matchings of certain finite subgraphs of the highly symmetrical infinite bipartite planar graph that arises as the dual of the tessellation. We can sharpen our inquiry by asking: what sorts of tessellations, in combination with what sorts of subgraphs of the graph dual to the tessellation, give rise to "ultra-smooth enumerations", i.e., enumerations in which only finitely many prime factors arise?

Regarding the choice of tessellation, it was observed by David Wilson that if you deform the square-hexagon-dodecagon graph so that it becomes a subgraph of the square grid, turn that into a square grid with edge-weights 0 and 1, and then perform generalized domino-shuffling (hereafter, "shuffling") three times, you get back the original edge-weights (modulo translation). This is analogous to Yang's discovery [**172**] that if you use urban renewal to turn the square-octagon graph into a square grid with edge-weights and apply shuffling twice, you get back what you started with (thereby obtaining exact enumeration of "diabolo tilings of fortresses"), and also analogous to the original application of urban renewal to Aztec diamonds [**136**].

More generally, consider an infinite (but locally finite) bipartite planar graph G with two independent translation-symmetries. Suppose that it gives rise, via a suitable sort of deformation, to a doubly-periodic weighting of the infinite square grid with 0's and 1's (with a caveat discussed below). Call this weighting W_0. Applying shuffling to W_0, one gets some weighting W_1; applying shuffling again, one gets some weighting W_2; etc. Suppose that $W_m = W_0$ for some m. Each W_i is doubly-periodic, and hence has only finitely many distinct "face-factors", where the face-factor associated with a face is $ac+bd$ where a, b, c, d are the consecutive edge-weights around the face. Hence only finitely many primes occur in the numerators of face-factors arising from the weightings W_0, W_1, $\dots$, W_{m-1}. These are the primes that occur in ultra-smooth enumerations of matchings of finite subgraphs of G.

Therefore, one question that is likely to shed light on the subject of ultra-smooth enumeration is, what doubly-periodic graphs G give rise to weightings of the square grid that come back to themselves after a finite number of iterations of shuffling? This is probably not the most natural setting for the question, since it gives the square grid an unjustified privileged role, and not all ultra-smooth enumerations arise in this way, but it is a place to start. (For more on iterated shuffling as an integrable system, see [**66**].) A related project is to classify triply-periodic solutions to the octahedron recurrence (with periodicity in two "space directions" and one "time direction").

However, there is more to the story than the infinite graph G: one must also pick out the correct sorts of finite subgraphs. Here we face an irony that drastically augments the disappointment Grensing, Carlsen, and Zapp must have felt in their abortive study of Aztec diamond graphs: the sorts of subgraphs that are amenable to simple combinatorial methods never include the sorts that are relevant to computing the per-site entropy of the dimer model on the infinite graph. If for instance we apply shuffling to compute the number of domino tilings of a $2n$-by-$2n$ square, in the interior we pick up the face-factors equal to 2, but along the boundary we find other numbers that increase in complexity as we apply new rounds of shuffling, and we ultimately lose control over the analytical form of the answer.

In "designing" various families of bipartite planar graphs that turned out to have ultra-smooth enumerations, a heuristic that has proved to be very successful, but whose success is quite mysterious, is what one might (borrowing physicists' nomenclature) call the domain-wall heuristic. This is illustrated by the considerations that led me to (re-)invent Aztec diamonds in the 1980s. I was aware of MacMahon's beautiful product formula for the (smooth, though not ultra-smooth) enumeration of lozenge tilings of hexagons, and Kasteleyn's formula for the (unsmooth) enumeration of domino tilings of squares, and wondered what was responsible for the difference. A reading of W. Thurston's article [**165**], focusing on the concept of height-functions for these two sorts of tilings, led me to notice that the height-function along the boundary of the square is (modulo tiny fluctuations) flat, while the height function along the boundary of the hexagon has dramatic rises and dips. Might a region in the square grid whose boundary consisted of four zigzag paths, with dramatic rises and dips in the height function, be a better analogue of the hexagon, and have similar nice enumerative properties? The answer was a resounding "yes"; nothing is more ultrasmooth than a sequence of powers of 2.

Similar considerations, applied to other doubly-periodic graphs, yielded other conjectural smooth enumerations, such as the ones later proved by Yang and by Ciucu. In every case, the starting point was a two-colored doubly-periodic tessellation (often obtained from [**69**]) in which black cells are surrounded by white cells and vice versa. In such a tessellation one draws infinite paths that abut only white cells on one side and only black cells on the other side. A collection of such zigzags divides the plane into subregions. The finite regions that arise in this fashion turn out suspiciously often to be associated with ultra-smooth enumerations.

With these sorts of domain-walls boundaries, it is inevitable the tiling will be more tightly constrained near the boundary than in the interior, and so it is not surprising that segregation of the region into different phase-domains with interesting asymptotic shapes can occur. However, it still seems magical that domain-wall boundaries give rise to ultra-smooth enumerations.

Moreover, in the case of lozenge tilings of hexagons, the explanation in terms of face-factors does not apply, and we do not get an ultra-smooth enumeration, but we still get a smooth one, with prime factors that grow only linearly in size. The same observation applies to gaskets-and-baskets tilings: when one choose paths through the plane that lift to maximally steep paths in the height-function picture, the regions the paths cut out are supergaskets, whose tilings (which are in bijection with ASMs) yield smooth enumerations.

Now comes the caveat that I warned you about: Kuperberg's method of turning general bipartite planar graphs into subgraphs of the square grid gives rise to weightings with lots of 0's, and these are not always amenable to shuffling when expressions of the form 0/0 arise. The article [**74**] discusses some ways of circumventing this problem, but a complete solution awaits discovery. This issue has connections with a similar issue that arises when one carries out Dodgson condensation to evaluate determinants of matrices [**16**].

7.3. Non-periodic weights. With q a formal indeterminate, define the q-integer $[n] = 1 + q + \cdots + q^{n-1}$, the q-factorial $[n]! = [1][2]\cdots[n]$, and the q-hyperfactorial $H([n]) = [1]![2]!\cdots[n-1]!$. Many of the integer-valued enumerations associated with matchings of graphs can be obtained as the $q = 1$ specializations of more refined q-enumerations, in which one "q-counts" tilings by summing the quantity $q^{h(T)}$ as T varies over all the tilings, where h is some integer-valued "global height" associated with T. More specifically, one can write $h(T)$ as a sum of local heights associated with the vertices of the tiling (or with the faces of the dual graph), as in W. Thurston's exposition [**165**] of Conway's approach to tilings [**45**]; see also [**134**].

One can fit q-enumeration into the framework of edge-weighted enumeration, but at a cost: the pattern of weights will no longer be periodic. For instance, looking at a regular procession of edges, one may find that their edge weights follow the pattern $q^1, q^{-2}, q^3, q^{-4}, \ldots$.

Factorizations in $\mathbb{Z}[q]$ are more informative than factorizations in $\mathbb{Z}$; 12 could be 2 times 6 or 3 times 4, but [2] times [6] is different from [3] times [4], so in the case of smooth enumerations that give product formulas, empirical q-enumerations for small regions give more clues about what the general formula might be. In this way q-analogues can sometimes provide information that illuminates the $q = 1$ enumerations. A recent example of this (applied to a nonsmooth enumeration) came from Adam Kalman's work on Aztec pillows, discussed in the 2014 version of [**137**]. Indeed, I like to think of the step of replacing $\mathbb{Z}$ by $\mathbb{Z}[q]$ as just the first step down a road that, if you can reach the end, replaces what started out as a mere number by a multivariate polynomial (or Laurent polynomial) in which every coefficient is 1. In this way, the monomials themselves are seen to encode the combinatorial objects being counted. This "very many variables" point of view, which draws its inspiration from its success in the case of Aztec diamonds, was a key ingredient in the work of Carroll and Speyer on groves [**20**].

It should be mentioned (apropos of Kalman's work) that there really is not a hard dichotomy between smooth and nonsmooth enumerations; in particular, a single enumeration can exhibit both smooth and nonsmooth features, with certain small primes occurring with very high exponents and larger primes occurring with much smaller exponents.

A recent and interesting way to add variables, different from the height-functions approach, is the "elliptic" approach of Borodin, Gorin, and Rains [**12**].

7.4. Other numerical patterns. Sometimes in examining the prime factorization of an enumeration, one sees dominance not of a particular finite set of primes but of an infinite arithmetic progression of primes. This is the case for the sequence whose terms count domino tilings of squares: one sees numbers in whose prime factorizations primes congruent to 1 mod 4 predominate.

Sometimes the patterns one sees involve not the primes but their exponents. For instance, when all the exponents in an enumeration are even (that is, when the answer is a perfect square for all n), surely there is something going on that demands an explanation. A variant of this phenomenon is that certain answers to tilings problems are always twice a perfect square. Typically such patterns are explained by appealing to symmetries of the graph (e.g., [**75**] and [**25**]).

Sometimes one sees patterns in the congruence classes of the enumerations themselves (not their prime factors). The prototype for this behavior is seen in the Fibonacci numbers, which are periodic mod m for every m, and more generally any sequence whose terms satisfy a linear recurrence relation over the integers whose characteristic polynomial has first and last coefficients equal to ± 1. It is less obvious that this periodicity property should be true for the Somos-4 sequence 1,1,1,1,2,3,7,23,59,... (since the defining recurrence is not linear but rational) but periodicity still holds in this case [**120**] and in other similar cases. In the case of tiling problems, Cohn has shown that the sequence whose nth term counts domino tilings of the $2n$-by-$2n$ square is periodic modulo m whenever m is a power of 2 [**38**].

Divisibility properties constitute a special case of congruence properties. For one example of a result asserting divisibility, see [**72**], which shows that the number of perfect matchings of any member of a certain family of graphs is always divisible by 3.

Lastly, one sometimes sees that later terms of a sequence tend to be divisible by certain earlier terms. In the case of the Fibonacci sequence (which counts domino tilings of rectangles of width 2), this observation goes back centuries. A generalization of this classical fact that applies to domino tilings of rectangles of arbitrary fixed width was proved by Strehl [**161**] (see also problem 21 in [**137**]). Quite recently, MIT undergraduate Forest Tong has found a combinatorial explanation for a broad generalization of this phenomenon [**166**].

7.5. Symmetry. When a region has symmetries, one can count tiling of the region that are invariant under the full symmetry group of the region or some subgroup. Choosing a larger subgroup will of course reduce the number of tilings, but curiously, even though the numbers get smaller, the problems get harder! (Though sometimes when the subgroup is too large, the number of tilings drops to 0, and then the problem is trivial.) The original example of this sort of story is Stanley's article on symmetry classes of plane partitions [**156**]; the smaller subgroups were the first ones to be settled (starting with the trivial subgroup, which was settled by MacMahon), culminating in the case of Totally Symmetric Self-Complementary Plane Partitions (TSSCPPs). The final piece of this ten-symmetry-classes-of-plane-partitions story was not put into place until 2011 [**92**].

Matchings that are invariant under a symmetry can be thought of as matchings of a quotient graph, although some care is required when the action is not free: vertices or edges whose orbit is of anomalously small cardinality require special treatment in "orbigraph matchings" (see [**104**]).

More mysterious than the above is the $q = -1$ phenomenon discovered by Stembridge and significantly generalized by Reiner, Stanton, and White in their notion of the cyclic sieving phenomenon [**147**] (see also [**152**] and [**148**]). Consider the set S of lozenge tilings of some hexagon, either unconstrained or constrained to possess certain symmetries of the hexagon. Let $f(q)$ be the q-enumeration of the tilings. It is clear that $f(1)$ is the number of tilings in the set S we are looking at. What is surprising is that, in certain cases of this kind, $f(-1)$ turns out to have enumerative significance of its own: it is the number of tilings in S that are invariant under some involution on S arising from a symmetry of the hexagon. See [**160**].

A different symmetry concerns the size parameter n, rather than the region being tiled. The quantity 5^{n^2}, which counts matchings of fortress graphs, is invariant under the substitution that replaces n by $-n$. The quantity $2^{n(n+1)/2}$, which counts matchings of Aztec diamond graphs, is invariant under the substitution that replaces n by $-1-n$. The quantity $(\phi^{n+1} - (-1/\phi)^{n+1})/\sqrt{5}$, which counts matchings of the 2-by-$2n$ grid, is invariant under the substitution that replaces n by $-2-n$, up to a predictable factor of ± 1. This phenomenon is called **reciprocity**, and it can be seen in many exact enumerations. To cite a recent example, Ciucu and Lai [**108**] [**37**] have proved a conjecture of Matt Blum (problem 25 in [**137**]) and shown that, for positive integers a, b with $b \geq 2a$, the number of tilings of the hexagonal dungeon of type $a, 2a, b$ is $13^{2a^2} 14^{\lfloor a^2/2 \rfloor}$. This quantity is an even function of a; that is, it is invariant under the substitution that replaces a by $-a$. Likewise one notices that the polynomials that appears in formulas (7) through (10) in [**150**] (as well as many other formulas in that article, both proved and conjectural) are even functions of n.

In some situations, reciprocity can be seen as a special case of a more general phenomenon linking two enumeration problems that yield respective integer sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$; the two problems are "mutually reciprocal" if (upon defining $a_0, a_{-1}, \ldots$ and $b_0, b_{-1}, \ldots$ appropriately) one has $a_n = b_{c-n}$ and $b_n = a_{c-n}$ for some constant c.

In the case of the 2-by-$2n$ grid, and for other 1-dimensional matching problems, the beginnings of a satisfactory explanation for reciprocity can be found in [**139**] and [**1**]. However, for 2-dimensional tiling problems, nothing has been done. Since shuffling can be done in reverse, one might guess that the octahedron recurrence would provide the scaffolding for an explanation, via an extension of Kuo condensation to signed graphs like the ones considered by [**1**] (featuring various sorts of negative vertices and edges). Indeed, the rich octahedral symmetries of solutions to the octahedron recurrence call out for such a point of view, in which there would not be such a stark difference between "space" and "time", let alone between the positive and negative time-directions.

Although the introduction of signed graphs might seem like a conceptual extravagance, there is a very practical application they might have for the pattern-seeking combinatorialist. If one is looking for a recurrence relation that governs a one-sided sequence $a_1, a_2, \ldots$, and if one suspects that the unknown recurrence relation would actually be satisfied by a two-sided sequence $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ that extends the original one-sided sequence, and if moreover one has a principled guess for what the numbers $a_0, a_{-1}, a_{-2}, \ldots$ are, then one could apply methods like the Somos matrix singularity test to a finite sequence of the form $a_{-r}, a_{-r+1}, \ldots, a_0, \ldots, a_{s-1}, a_s$. The success of the method hinges on $r+s$ being large, so one wants to take r and s individually as large as possible; that is, the data will guide us toward the correct recurrence (assuming there is one!) if we give it as many terms as we can, in both the forward and reverse directions.

8. Software

The page http://jamespropp.org/software.html gives links to a number of resources that are likely to be useful to researchers conducting experimental work on matchings of bipartite planar graphs and associated tilings.

The program vaxmaple (developed by Greg Kuperberg, David Wilson and myself) allows one to count the matchings of a bipartite planar graph using the method of Kasteleyn. One inputs a graph in VAX-format (described in [**137**]) and the program outputs a Maple program which, if fed into Maple, returns the number of matchings of the graph.

David Wilson's vaxmacs package gives the user all the power of vaxmaple and more in an easy-to-use interactive environment based on the popular editor emacs. The documentation for this package is contained in the program itself. You will need emacs-19.30 or later in order to get full use of vaxmacs.

Harald Helfgott's program ren, written in C, implements the generalized domino-shuffling algorithm of [**140**].

If you want to count matchings of some non-bipartite planar graph, Matt Blum's TCL-program graph lets you define an arbitrary planar graph, and Ben Wieland's perl-program planemaple (which graph "knows about") lets you count the matchings of this graph efficiently, exploiting Kasteleyn's method. The program graph will also let you count matchings of non-planar graphs, but it does so by the brute-force method of finding them all, which will not be practicable when the graph is large. Both programs require the user to make some modifications to the program header.

Several of the above programs come equipped with compatible side-programs for printing attractive pictures of tilings, with various configurable options. Some can even be used to compute conditional probabilities of sub-configurations in a random matching, or to generate a random matching.

In addition to the above programs, which are intended as research tools, there are also programs of a more pedagogical nature.

In the 1990s, Jason Woolever created a Java applet available at http://jamespropp.org/cftp/ that allows one to generate random tilings of a variety of regions by means of Coupling From The Past (or CFTP). CFTP is a variant of Monte Carlo methods, and gives an approach to bias-free sampling from various sets of combinatorial objects [**142**]. The speed of CFTP correlates directly with the speed at which a certain Markov chain mixes, namely, the chain in which one makes random local changes in the tiling by increasing or decreasing the height at a point. It may be noticed that CFTP is considerably slower for tilings of fortresses and dungeons than for the other cases that are presented. That is because random diabolo tilings of fortresses, unlike random domino tilings of Aztec diamonds or random lozenge tilings of hexagons, exhibit not only frozen domains and a liquid domain, but also a gaseous domain (in the terminology of [**83**]). In height-function terms, the height function is (up to small local fluctuations) flat throughout a large expanse of the tiling far away from the boundary, so that the graph of the height-function exhibits a broad plateau. This plateau can be at any of several heights, and it is very hard for the Markov chain to change the plateau-height from one of its likely possible values to another, since this requires changing the local height at $O(n^2)$ locations, and any motion in this direction is likely to be reversed almost immediately by a kind of surface tension (entropic rather than energetic in nature). Thus there are bottlenecks in the Markov chain, and mixing is (or at least appears to be) exponentially slow. CFTP is therefore not an efficient way of generating random diabolo tilings of fortresses; other methods (such as generalized domino-shuffling or the conditional probability method) must be used instead. The same is true for dungeons.

In 2009 Ben Wieland modified the code to generate random alternating-sign matrices; see `http://nokedli.net/asm-frozen/`. It should be mentioned that for the specific case of alternating-sign matrices of large order, CFTP is the only known efficient method of generating unbiased samples from the uniform distribution.

For tilings of Aztec diamonds (or TOADs, as Hal Canary proposed calling them) there is an attractive Java applet `jamespropp.org/toadshuffle/` created by Canary that uses the original Kuperberg-Propp [**53**] version of domino-shuffling (with stages of annihilation, sliding, and creation) to iteratively generate uniformly random tilings of ever-larger Aztec diamonds.

9. Frontiers

Here are some directions in which further work is needed.

1. It was mentioned above that ASMs and TSSCPPs are equinumerous; specifically, both families of combinatorial objects are enumerated by the sequence $1, 1, 2, 7, 42, 429, \ldots$. This remains largely a mystery, as no bijective proof is known.

In a similar but more tractable vein, it has been noticed that the sequence $1, 1, 2, 8, 64, 1024, \ldots$ has arisen in several contexts [**53**] [**46**] [**126**] [**114**]. To the extent that all of these formulas have been given bijective proofs, one can provide bijections between the different models. However, some bijections are more natural than others, and one might hope to bring some precision to bear on the question of how similar, and how different, these various models are.

2. As I mentioned in subsection 7.1, as a step towards understanding the class of ultra-smooth enumerations of matchings, it would be good to classify the triply-periodic solutions of the octahedron recurrence. A related project is to better understand all doubly-periodic bipartite planar graphs. For any k, one can hope to classify all such graphs in which the fundamental domain contains k white vertices and k black vertices, up to deformations that preserve the combinatorial structure.

3. Subsection 7.5 mentions the role of symmetry. In that passage, we thought about starting with a finite graph and looking at matchings that possess invariance under some symmetry of the graph. One may take a different point of view and look at cofinite (or quasitransitive) group-actions on a finite graph, where we say an action on a graph is cofinite if there are only finitely many vertex-orbits and edge-orbits. If the infinite graph is for instance the the square grid, then quotient graphs include tori, Klein bottles, and cross-caps, as well as more arcane orbifold-type quotients. The case of the torus (that is, enumeration of matchings of a grid on a torus) is classical [**78**] and the case of the Klein bottle was settled by Lu and Wu [**115**] along with related problems in which the regions are given boundaries (ribbon graphs, Möbius strips, etc.). However, the case of the cross-cap remains unsettled.

4. Kenyon, inspired by earlier work on resistor networks and Ising models embedded in the complex plane, studied the bipartite planar dimer model on isoradial graphs [**81**]. In that article Kenyon showed (partly answering the final question

raised in [**40**]) that any periodic isoradial dimer model has entropy which is the volume of a certain 3D ideal polytope. Li recently proved a conformal invariance theorem for this class of models [**111**]. The connection between dimer models and hyperbolic geometry is still a mystery.

5. We have seen a host of exact enumerative results coming from the dimer model on a variety of graphs, as well as a few exact enumerative results coming from the six-vertex model on a single graph, namely, the square grid. Might there be exact enumerative results for the six-vertex model on other 4-regular graphs, such as the Kagome (or star-of-David) lattice, or other statistical mechanics models of a similar kind, such as the twenty-vertex model on the 6-regular triangular lattice?

We might try to move the theory of enumeration of tilings beyond the framework of dimer and ice-type models. An important step in this direction has been taken in the theory of square-triangle-rhombus tilings, mentioned at the end of Section 6.

6. It has been hard to get traction on the problem of counting monomer-dimer configurations in 2 (or more) dimensions, but other variants of dimers have been amenable to algebraic methods. Successful efforts in this direction include the double dimer model [**85**] and the monopole-dimer model [**3**]. Likewise, a variant of spanning trees that can be enumerated using a variant of the matrix tree theorem is cycle-rooted spanning forests [**84**].

If one is content with asymptotic results (as opposed to exact enumerations), and to work in the regime where the number of monomers is much smaller than the number of dimers, progress can be made; indeed, a great deal of interesting work has been done, much of it by Ciucu ([**28**], [**30**], [**32**], [**33**], [**24**]). A guiding theme here is that correlational behavior of the model, in the bulk limit, mimic features of electrostatics. See also [**57**] and [**52**].

7. Ribbon tilings (see Example 4 from Section 1) have not played much of a role in this chapter, but there has been some extremely interesting recent work on ribbon tilings of a special kind, namely "Dyck tilings". We will not define the term or delve into the subject here, but we point the reader toward [**88**] and [**89**] as articles in this rapidly growing literature, as well as the article [**153**] showing the relevance of Dyck tilings to the study of Kazhdan-Lusztig polynomials.

8. As was mentioned in Section 1, Pak, Yang, and others have shown that tiling problems in which the allowed tiles and the region being tiled are all rectangles can be $\#P$-complete. Even as we try to expand the domain of efficiently solvable tiling problems, we should try to find more and more constrained classes of tiling problems that, despite their strictures, still can be shown to be $\#P$-complete.

9. By taking the q-analogue of the formula $H(3n)H(n)^3/H(2n)^2$ for the number of rhombus tilings of a regular hexagon of side-length n and sending n to infinity, one recovers MacMahon's formula for the number of unconstrained plane partitions:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

One can think of this formula as q-enumerating a particular infinite set of rhombus-tilings of the plane. Let T_0 be a tiling of the plane that divides the plane into 120-degree sectors, with no tiles crossing between one sector and another. Suppose T is a tiling of the plane that agrees with T_0 outside some bounded region (call such a tiling cofinitely equivalent to T_0). Then, interpreting the two tilings as surfaces, we can define the rank of T as the volume of the portion of space bounded between the two surfaces (normalized so that the volume of a cube is 1). MacMahon's formula enumerates the tilings T by rank; that is, MacMahon's product is equal to $q^{\mathrm{rank(T)}}$, summed over all tilings T cofinitely equivalent to T_0.

A beautiful "domino-analogue" of this result is the formula

$$\prod_{n=1}^{\infty} \frac{(1+q^{2n-1})^{2n-1}}{(1-q^{2n})^{2n}}$$

conjectured independently by Szendrői [**162**] and Kenyon [**82**] and proved by Young [**173**][**174**]. One might ask whether other formulas of this kind await discovery.

Recent work of Bouttier, Chapuy, and Corteel [**15**] creates a unifying context for results of this sort and more standard results in the theory of enumeration of tilings. Especially noteworthy is the development of the theory of vertex operators, introduced in [**174**].

10. It is natural to try to extend the existing theory of enumeration of tilings into higher dimensions. However, this may not be possible. For instance, considering the fact that the double product

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \frac{i+j}{i+j-1}$$

counts lattice paths in a rectangle, and the fact that the triple product

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

counts plane partitions in a box (and their associated discrete spanning surfaces), one might hope that the quadruple product

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \prod_{l=1}^{d} \frac{i+j+k+l-2}{i+j+k+l-3}$$

might count some sort of solid partitions (associated with some sort of discrete 3-surfaces in 4-space). However, the quadruple product is not even a whole number when $a = b = c = d = 2$! (For recent work on enumerating solid partitions, see [**49**].)

Nonetheless, one can get some exact enumerative results relating to discrete 2-surfaces in n-space. (This should not be too surprising, when one considers that it is easy to count discrete 1-surfaces in n-space with fixed boundary using multinomial coefficients.) The first result along these lines was Elnitsky's enumeration of lozenge tilings of a semiregular octagon with side-lengths $a, b, 1, 1, a, b, 1, 1$ [**54**]: the number of tilings is

$$2(a+b+1)!(a+b+2)!/a!b!(a+2)!(b+2)!$$

In the case of a semiregular octagon with side-lengths $a, 1, b, 1, a, 1, b, 1$, the enumerations have much larger prime factors, so no nice product formula is possible, but there is still a double sum formula found by Elnitsky; for more details, see [**48**]. It should be mentioned that for both of these two-parameter families of semiregular octagons, Stembridge's $q = -1$ phenomenon has been proved to occur; in the

$a, b, 1, 1, a, b, 1, 1$ case the result is due to Elnitsky [**54**]: in the $a, 1, b, 1, a, 1, b, 1$ case it is due to Bailey [**4**] (Corollary 3.3 on page 25). See also [**71**].

One might also seek an exact formula for the number T_n of lozenge tilings of a $2n$-gon with all side-lengths equal to 1, but this appears to be a hard problem. We know from [**7**] that $\lim(\log T_n)/n^2 < \frac{1}{2}$, but little is known about this sequence aside from asymptotics.

Jumping off from this example into a general consideration of the growth rates of sequences that arise in enumerative combinatorics, let us say that a combinatorial sequence $a_1, a_2, \ldots$ (along with the combinatorial problem that gave rise to it) is of grade k if the logarithm of a_n grows like n^k; that is, $(\log \log a_n)/\log n$ converges to k. (For problems with more than one parameter, assume that all parameters are of the same order as n.) Combinatorics of grade 0 concerns sequences like the sequence of triangle numbers $1, 3, 6, 10, \ldots$. Combinatorics of grade 1 concerns sequences like the Fibonacci and Catalan sequences; Examples 1 through 4 of Section 1 are all at grade 1. Combinatorics of grade 2 largely concerns genuinely two-dimensional tiling problems, many of which (like rhombus tilings of the a, b, c hexagon) can be recast as problems involving discrete 2-surfaces in 3 dimensions; more generally, rhombus tilings of a semiregular polygon with $2n$ pairs of parallel sides can be recast as discrete 2-surfaces in n dimensions.

In the past century, combinatorics moved from grade 1 to grade 2. However, we seem to be stuck there. Perhaps there is no such subject as grade 3 combinatorics (and perhaps, relatedly, there are no non-trivial exactly solvable 3-dimensional lattice models). Or perhaps there is such a subject, which we will begin to glimpse once we have gotten through more of the grade 2 curriculum and the teacher deems us ready for higher things.

Acknowledgments: This chapter would not have been possible without the assistance and encouragement of dozens of people, especially Federico Ardila, Arvind Ayyer, Art Benjamin, Jin-Yi Cai, Clara Chan, Sunil Chhita, Timothy Chow, Mihai Ciucu, Henry Cohn, Filippo Colomo, David desJardins, Nicolas Destainville, Julien Dubedat, Noam Elkies, Patrik Ferrari, Philippe di Francesco, David Gamarnik, Ira Gessel, Jan de Gier, Vadim Gorin, Tony Guttmann, Rick Kenyon, Christian Krattenthaler, Eric Kuo, Greg Kuperberg, Tri Lai, Lionel Levine, Russell Lyons, John Mangual, Barry McCoy, Roderich Moessner, Cris Moore, Igor Pak, Kyle Petersen, Dana Randall, Vic Reiner, Dan Romik, Bruce Sagan, Nicolau Saldanha, Robert Shrock, Richard Stanley, John Stembridge, Jessica Striker, Hugh Thomas, Dylan Thurston, Carlos Tomei, David Wilson, Fred Wu, and Benjamin Young. Thanks also to my wife Sandi Gubin for her encouragement and patience.

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