Piecewise-linear and birational toggling

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UMass Lowell

July 2, 2014

Special thanks to Tom Roby for help with preparing this talk.

Slides at http://jamespropp.org/fpsac14.pdf

Main concepts

- homomesy
- toggling
- rowmotion
- promotion
- reciprocity

All of these concepts apply to dynamical systems in three realms:

- the combinatorial realm
- the continuous piecewise-linear (cpl) realm
- the birational realm

Recent example: Armstrong-Stump-Thomas

Theorem (conjectured by Panyushev): Let W be a finite Weyl group of rank r and **Pan** the Panyushev complement on antichains in the root poset $\Phi^+(W)$. Then for any orbit $\mathcal{O}$ of **Pan** we have

$$\frac{1}{|\mathcal{O}|}\sum_{A\in\mathcal{O}}|A| = r/2.$$

DREW ARMSTRONG, CHRISTIAN STUMP, AND HUGH THOMAS

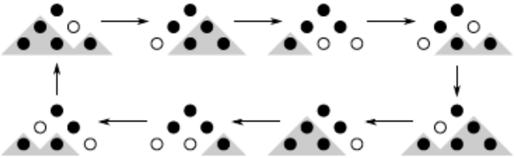
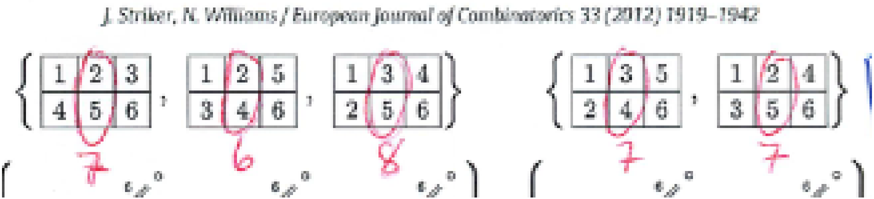


FIGURE 1. An orbit of the Panyushev complement

See http://arxiv.org/pdf/1101.1277.pdf.

More recent example: Bloom-Pechenik-Saracino

Theorem (conjectured by Propp): Let $\mathrm{SSYT}(m, n, k)$ be the set of semistandard Young tableaux t of shape $m \times n$ with entries bounded by k.
Let a, b be boxes in the $m \times n$ diagram related by 180° rotation.
For $t \in \mathrm{SSYT}(m, n, k)$, let $F(t)$ be the sum of the entries of t in boxes a and b.
Then $F(t)$ has the same average in each promotion-orbit in $\mathrm{SSYT}(m, n, k)$.



See http://arxiv.org/abs/1308.0546.

Homomesy

Given

- a set X,
- an operation $T : X \to X$ with $T^n = id$, and
- a function F from X to a field $\mathbb{K}$ of characteristic 0,

we say that F is *homomesic* under the action of T, or that the triple (X, T, F) exhibits *homomesy*, if for all $x \in X$ the average

$$\frac{1}{n} \sum_{k=0}^{n-1} F(T^k(x))$$

equals some c independent of x.

Examples abound: partitions, tableaux, colorings of graphs, independent sets in graphs, ASMs (abelian sandpiles models), ASMs (alternating sign matrices), ...

Homomesy and invariance

Given a vector space V of functions from X to $\mathbb{K}$,
let V_h be the linear subspace of homomesic functions
and V_i be the linear subspace of invariant functions
(functions $F : X \to \mathbb{K}$ with $F(Tx) = F(x)$ for all $x \in X$).

Easy fact: $V_h \cap V_i$ is the subspace of constant functions.

Equivalently, if we define V_h^0 as the subspace of "0-mesic" functions (functions $F : X \to \mathbb{K}$ with $F(x) + F(Tx) + \cdots + F(T^{n-1}x) = 0$ for all $x \in X$), then $V_h^0 \cap V_i = \{0\}$.

In some cases we have $V = V_h^0 \oplus V_i$ (e.g. see section 2.4 and 2.5 of the July 1, 2014 version of http://arxiv.org/abs/1310.5201), but even when this doesn't happen, we typically find (for "naturally occuring" X, T, V) that dim V_h is surprisingly large.

Order ideals

Let P be a poset.

Let X be the set of order-reversing maps f from P to $\{0,1\}$ (naturally identified with the set $J(P)$ of order ideals I of P).

Let V be the vector space of functions expressible as linear combinations of the indicator functions 1_x $(x \in P)$, where

$$1_x(I) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if } x \notin I, \end{cases}$$

i.e., the set of maps F of the form $F(f) = \sum_{x \in P} a_x f(x)$ for fixed coefficients $a_x \in \mathbb{K}$.

E.g., with $a_x = 1$ for all x, we get the cardinality function $F(I) = \sum_{x \in P} f(x) = |I|$.

Toggling

Given an order ideal $I \in J(P)$ and an $x \in P$, define

$$\tau_x(I) = \begin{cases} I \triangle \{x\} & \text{if } I \triangle \{x\} \in J(P), \\ I & \text{otherwise,} \end{cases}$$

where $\triangle$ denotes the symmetric difference.

Following Striker and Williams (http://arxiv.org/abs/1108.1172) we call τ_x "toggling at x".

τ_x is an involution on $J(P)$.

τ_x and τ_y commute unless $x \gtrdot y$ (x covers y) or $x \lessdot y$ (x is covered by y).

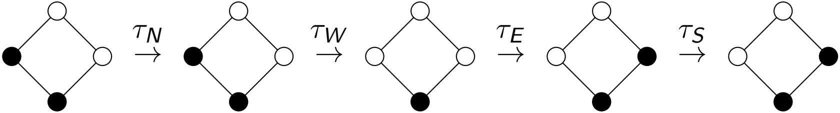
Rowmotion

Hereafter we focus on $P = [a] \times [b]$ (extensions to other posets are in progress).

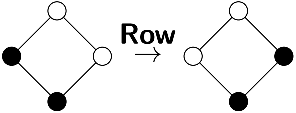
When $a = b = 2$, we label the 4 elements of the poset as N, S, E, and W in the Hasse diagram in the obvious way.

Following Striker and Williams: define $\mathbf{Row}(I)$ to be the result of successively toggling at all the elements of P from top to bottom; this is well-defined because of the commutativity property.

An example of rowmotion in $[2] \times [2]$



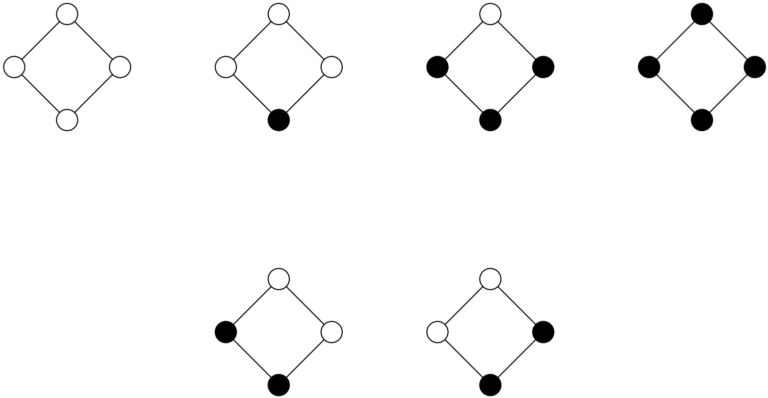
So



Periodicity

Theorem (Fon-der-Flaass 1993): **Row** on $P = [a] \times [b]$ is of order $a + b$.

$[2] \times [2]$: periodicity for rowmotion



We have an orbit of size 4 and an orbit of size 2.
Both orbits have size dividing $a + b = 2 + 2 = 4$.

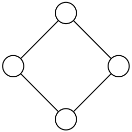
Homomesy for cardinality

Theorem (Propp and Roby): $F(I) = |I|$ is homomesic under rowmotion with average $c = ab/2$ in each orbit.

That is, for any orbit $\mathcal{O}$ of **Row** we have

$$\frac{1}{|\mathcal{O}|} \sum_{I \in \mathcal{O}} |I| = ab/2.$$

[2] × [2]: homomesy for cardinality under rowmotion



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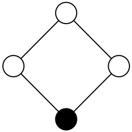

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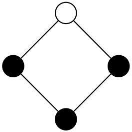

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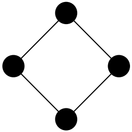

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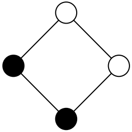

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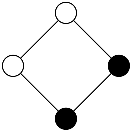


2

$$(0+1+3+4)/4 = (2)(2)/2 = (2+2)/2$$

Determining V_h (the subspace of homomesies)

Propp and Roby found other homomesies for rowmotion.

Einstein showed that Propp and Roby's list is complete;
that is, he determined V_h.

Side note: Rowmotion can also be defined for antichains as in the Armstrong-Stump-Thomas paper (in this context it is called the Panyushev complement); we get a different V (that is, the bijection between order ideals and antichains does not give a linear map between $V^{\text{order ideals}}$ and $V^{\text{antichains}}$), and V_h is quite different in the two cases.

From $J(P)$ to the order polytope of P

$J(P)$ is naturally identified with the set of order-reversing maps from P to $\{0, 1\}$.

We could just as well define toggling for the set of order-preserving maps from P to $\{0, 1\}$ (just exchange the roles of 0 and 1).

There is a natural way to lift toggling from the set of order-preserving maps from P to $\{0, 1\}$ to the set of order-preserving maps from P to $[0, 1]$.

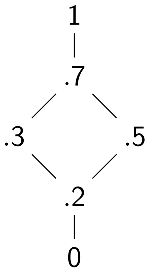
The set of such maps $f : P \to \mathbb{R}$, viewed as a subset of $\mathbb{R}^{|P|}$, is Stanley's order polytope $\mathcal{O}(P)$, whose vertices correspond to the order ideals of P.

The order polytope

Let $\hat{P}$ denote the augmented poset obtained from P by adjoining $\hat{0}$ and $\hat{1}$ satisfying $\hat{0} < x < \hat{1}$ for all $x \in P$.

$\mathcal{O}(P) \subset \mathbb{R}^{|P|}$ is the set of vectors associated with functions $\hat{f} : \hat{P} \to \mathbb{R}$ that satisfy $\hat{f}(\hat{0}) = 0$ and $\hat{f}(\hat{1}) = 1$ and are *order-preserving* ($x \le y$ in P implies $\hat{f}(x) \le \hat{f}(y)$ in $\mathbb{R}$).

E.g., for $P = [2] \times [2]$:



Toggling in the order polytope

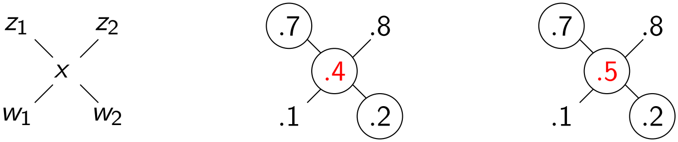
For each $x \in P$, define $\tau_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f (an order-preserving function from P to $[0,1]$) to the unique f' satisfying

$$\hat{f}'(y) = \begin{cases} \hat{f}(y) & \text{if } y \neq x, \\ \min_{z \gtrdot x} \hat{f}(z) + \max_{w \lessdot x} \hat{f}(w) - \hat{f}(x) & \text{if } y = x, \end{cases}$$

The involution τ_x is a cpl (continuous piecewise linear) map.

This definition is implicit in work of Kirillov and Berenstein; see also Pak (and probably others as well).

Example of toggling at a vertex



$$\min_{z \gtrdot x} \hat{f}(z) + \max_{w \lessdot x} \hat{f}(w) = .7 + .2 = .9$$

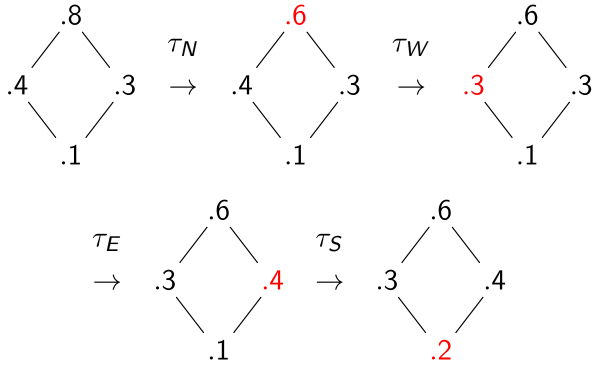
$$\hat{f}(x) + \hat{f}'(x) = .4 + .5 = .9$$

Rowmotion in the order polytope

Define rowmotion on $\mathcal{O}(P)$ ("cpl rowmotion") in analogy with rowmotion on $J(P)$ ("combinatorial rowmotion") as the result of performing cpl toggling at the vertices of P from top to bottom.

Combinatorial rowmtion is cpl rowmotion restricted to the vertices of $\mathcal{O}(P)$.

An example of cpl rowmotion



Promotion (an aside)

One can define an operation on $\mathcal{O}([a] \times [b])$ by toggling from left to right in the Hasse diagram instead of top to bottom.

We call this "cpl promotion", and denote it by **Pro**, since it is the cpl version of Striker and Williams' promotion operation.

It deserves this name: it can be shown that Schützenberger promotion on the set of semistandard Young tableaux of rectangular shape with A rows and B columns having entries between 1 and n is naturally equivariant with the action of **Pro** on the lattice points in the polytope obtained by dilating the order polytope of $[A] \times [n - A]$ by a factor of B.

Rowmotion and promotion on $\mathcal{O}(P)$ have the same orbit structure and homomesies, so henceforth we just discuss rowmotion.

Periodicity and homomesy for cpl rowmotion

Einstein-Propp: cpl rowmotion is of order $a + b$.

Periodicity in the cpl setting doesn't follow from periodicity in the combinatorial setting.

Question: Is there a self-contained proof of periodicity for cpl rowmotion?

We have also classified the homomesies of cpl rowmotion, and they are the same as the homomesies for combinatorial rowmotion; e.g., the function that maps f to $\sum_{x \in [a] \times [b]} f(x)$ is homomesic.

Detropicalizing toggling

The way we prove periodicity for cpl rowmotion (and with it our homomesy results) is by deriving it from a result in the birational setting.

Recall: A birational map from $\mathbb{C}^n$ to itself is a rational map $f : \mathbb{C}^n \to \mathbb{C}^n$ for which there exists a rational map $g : \mathbb{C}^n \to \mathbb{C}^n$ such that $f \circ g$ and $g \circ f$ are the identity map (off of a proper subvariety).

To lift toggling to the birational setting, we replace $+$, $-$, max, and min by $\times$, $\div$, $+$, and $\parallel$, where the "parallel sum" $x \parallel y$ is defined as $xy/(x + y) = 1/(1/x + 1/y)$.

Let $\sum^{+}$ denote the ordinary sum and $\sum^{\parallel}$ denote the parallel sum.

Toggling in the birational realm

Look at maps $f : P \to \mathbb{C}$ and the associated maps $\hat{f} : \hat{P} \to \mathbb{C}$ that send both $\hat{0}$ and $\hat{1}$ to 1 (this condition on $\hat{f}$ can be relaxed but it complicates things).

Ignoring the subvariety on which things blow up:

For each $x \in P$, define $\tau_x(f) = f'$ where

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ (\sum^{\parallel}_{z \gtrdot x} f(z))(\sum^{+}_{w \lessdot x} f(w))/f(x) & \text{if } y = x. \end{cases}$$

Rowmotion in the birational realm

Define birational rowmotion as doing birational toggling from top to bottom.

Periodicity Theorem (Grinberg-Roby): Birational rowmotion on $[a] \times [b]$ is of order $a + b$.

The homomesies for cpl rowmotion lift to homomesies for birational rowmotion: e.g., the function that maps f to $\sum_{x \in [a] \times [b]} \log |f(x)|$ is homomesic.

The three realms

The birational realm

$\downarrow$

The cpl realm

$\downarrow$

The combinatorial realm

Reciprocity

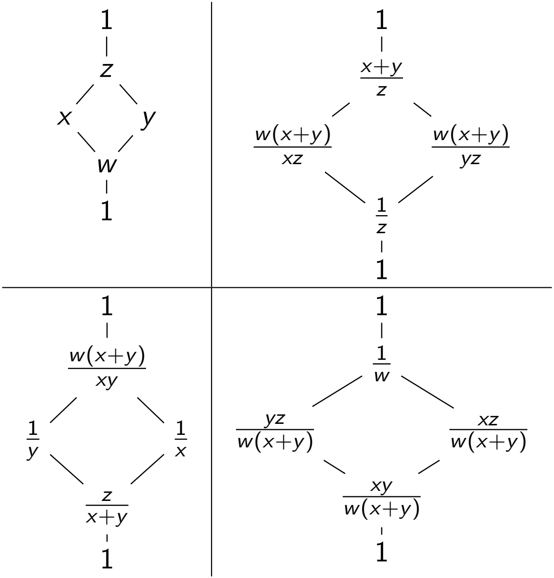
The Periodicity Theorem can be derived from:

Reciprocity Lemma (conjectured independently by Propp and Roby; proved by Grinberg-Roby): For $x = (i, j)$ in $[a] \times [b]$, we have

$$g(y) = 1/f(x),$$

where $y = (a + 1 - i, b + 1 - j)$ (the element related to x by 180° rotation) and $g = \mathbf{Row}^{a+b+1-i-j} f$ (here **Row** denotes birational rowmotion).

Reciprocity in $[2] \times [2]$



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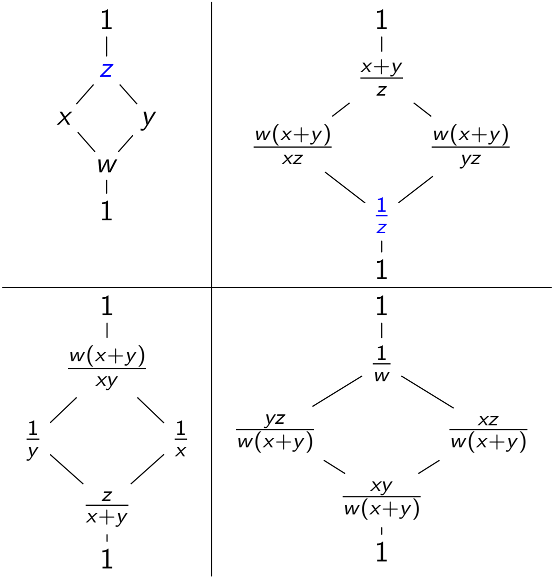

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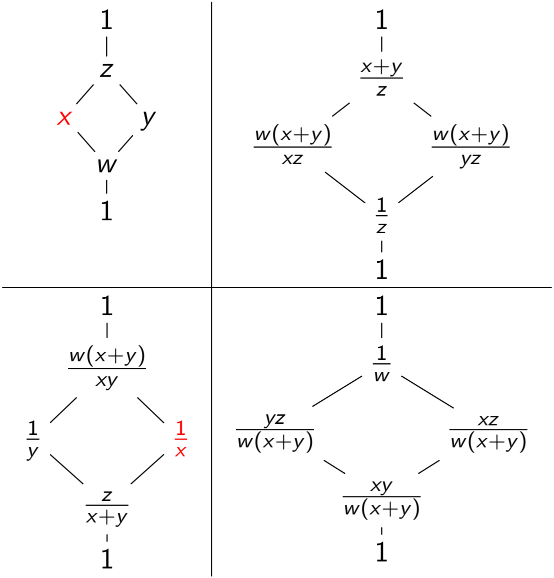

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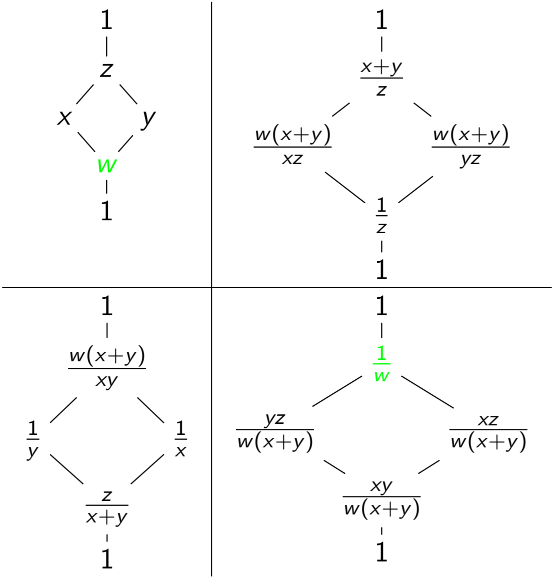


In quest of a simpler proof

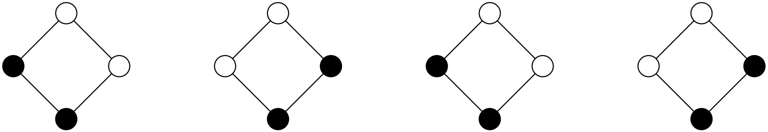
Question: Is there a self-contained combinatorial proof of the combinatorial version of birational reciprocity?

Combinatorial reciprocity: $x \in I$ if and only if $y \notin J$, where x, y are as above and $J = \textbf{Row}^{a+b+1-i-j}(I)$ (here **Row** denotes combinatorial rowmotion).

Such an argument might "lift" to the birational realm, yielding a simpler proof of the reciprocity theorem, from which everything else follows.

Note added after the talk: Hugh Thomas found such proof. Now we need to figure out how to "birationalize" it.

Example of combinatorial reciprocity



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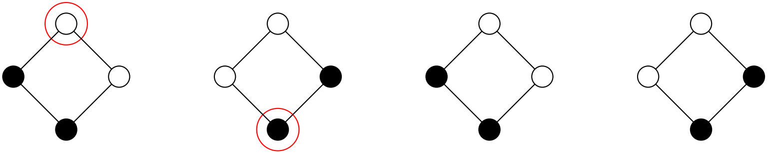

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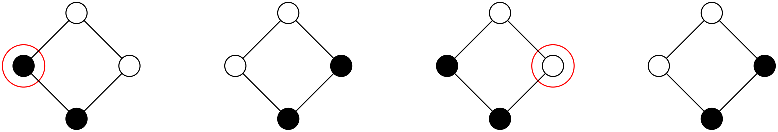

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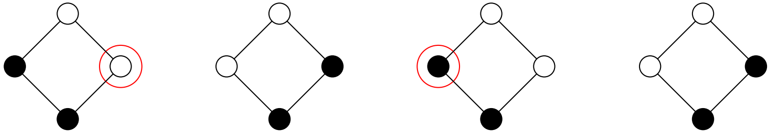

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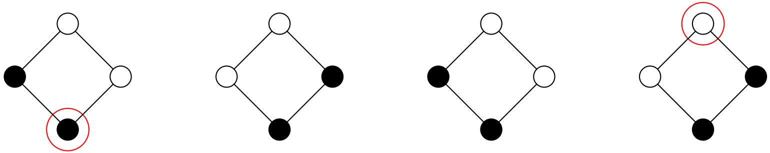

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Question: "Does all this homomesy stuff have anything to do with cyclic sieving?"

Answer: "Not very much, except that they often occur together."

Roby's grad student Mike Joseph is studying a family of combinatorial dynamical systems (X_n, T_n) $(n \geq 1)$ where the orbit sizes have LCM much larger than $|X_n|$. E.g., for $n = 12$, we have $|X_n| = 377$ but the LCM of the orbit sizes of T_n is over 3 million!

All the same, we've found that the examples that manifest the CSP tend to be fertile sources of homomesies.

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