Counting constrained domino tilings of Aztec diamonds

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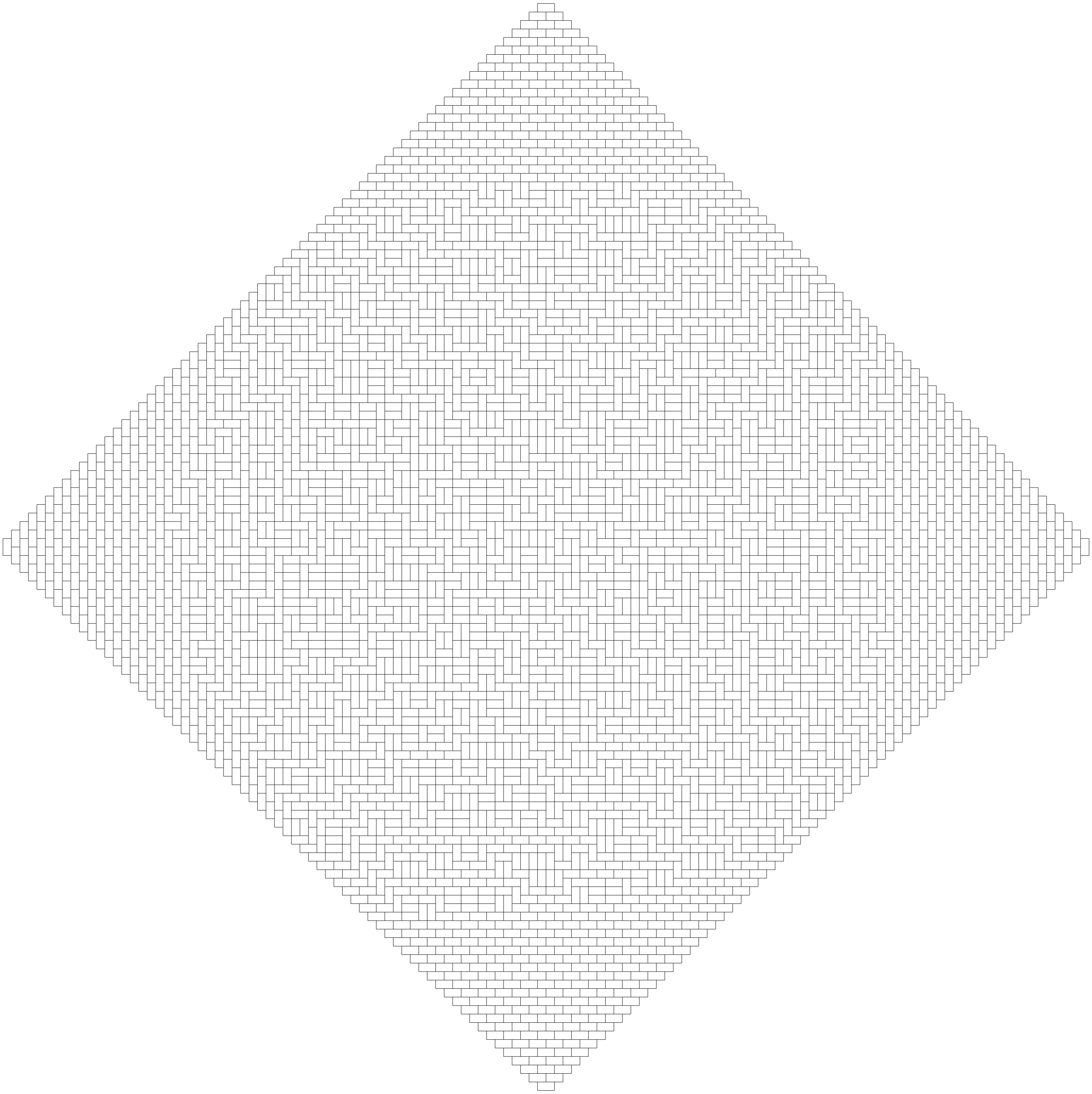
Note: The results described in this presentation will appear in several different articles.

Overview

In this paper, we prove theorems that, as a subsequent paper shows, allow one to obtain precise information about the typical behavior of domino tilings of large Aztec diamonds.

Specifically, we define a three−variable generating function whose coefficients give placement−probabilities associated with random tilings of Aztec diamonds, and we show that this generating function is in fact a rational function.

Our proof makes use of a still−mysterious algorithm called "domino shuffling". New insight into the algebraic significance of this algorithm may permit us to devise extensions of it and to apply these extensions to other problems concerning typical behavior of tilings.

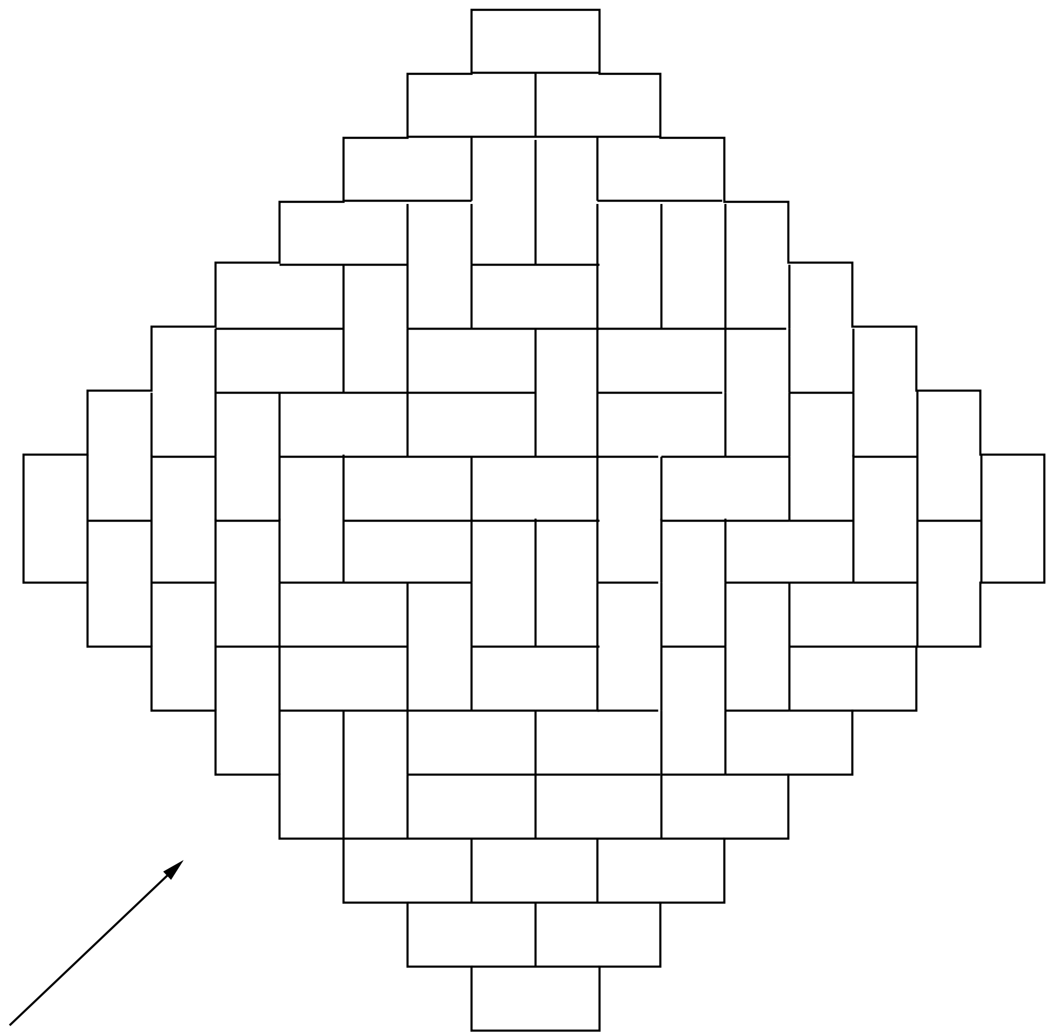


A typical domino tiling of a large Aztec diamond

Theorem (Elkies, Kuperberg, Larsen, and Propp):
The Aztec diamond of order *n* has exactly

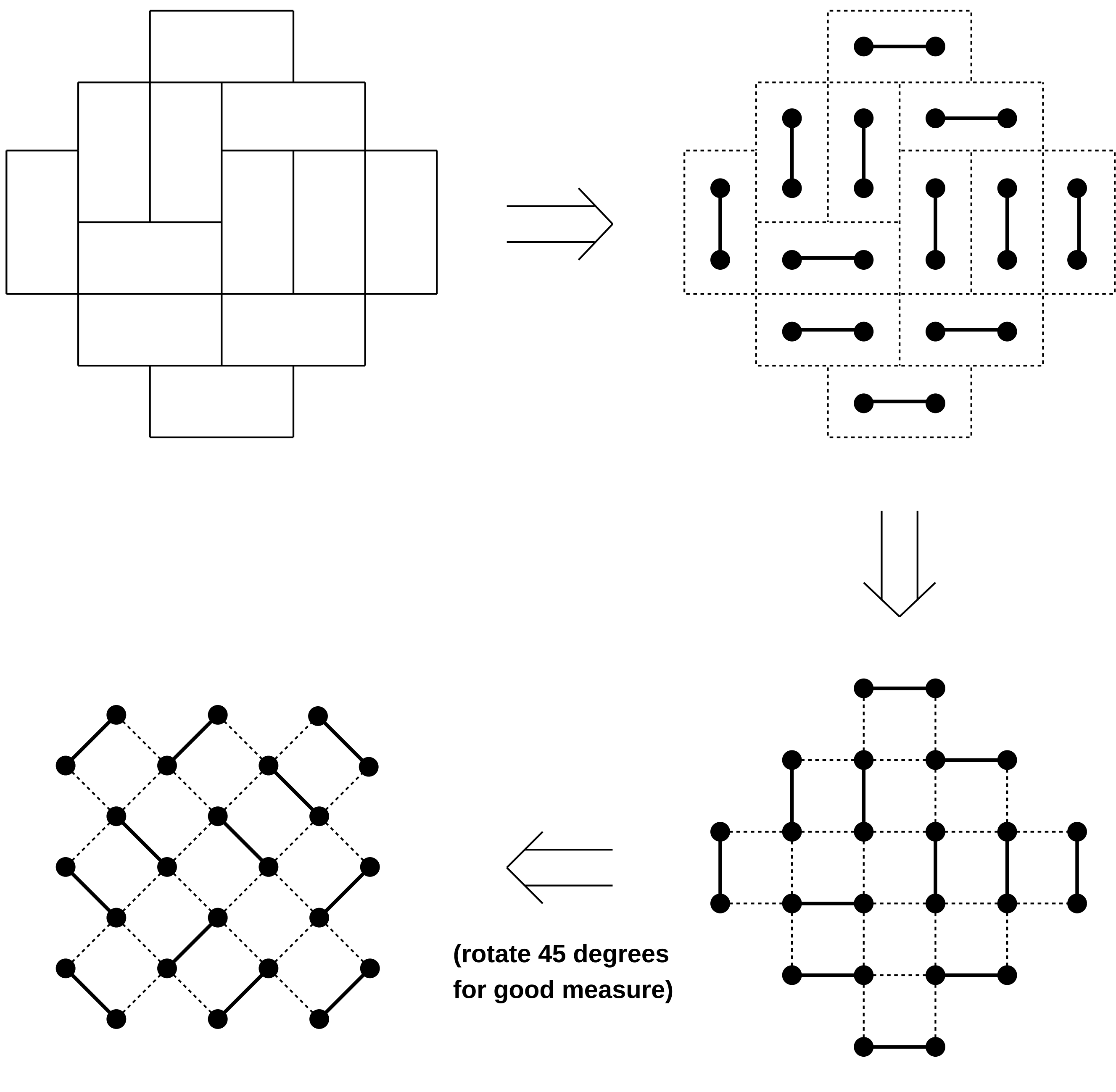
$$2^{n(n+1)/2}$$

tilings by dominoes.



(a domino tiling of the Aztec diamond of order 8)

Duality between domino tilings and perfect matchings



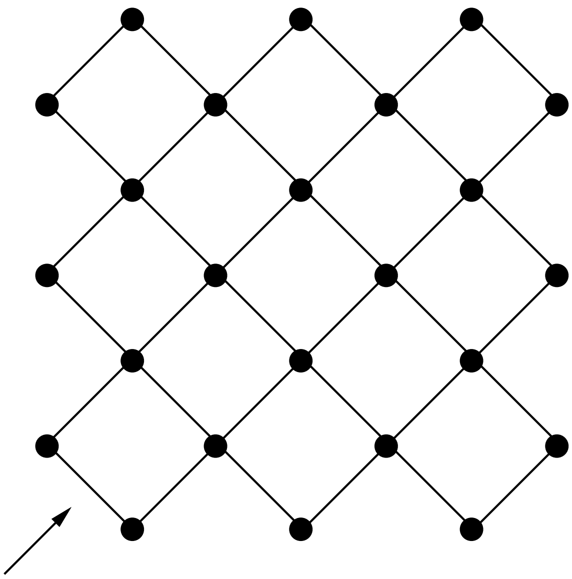
New!

A short proof of the theorem:

Improved!

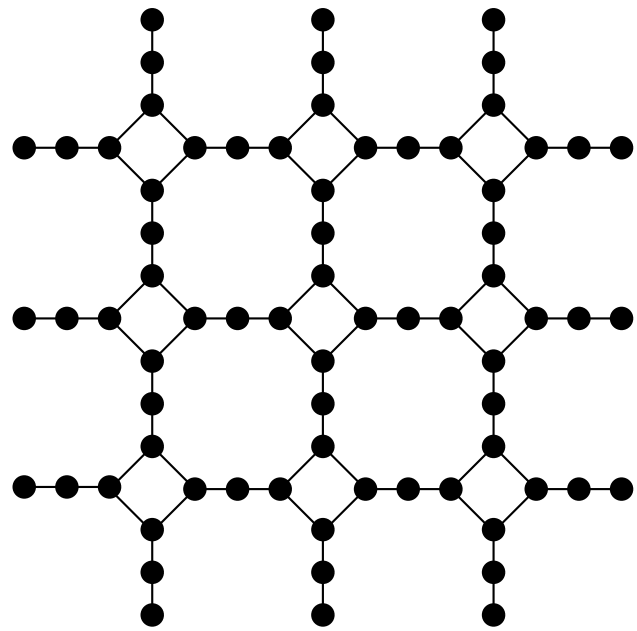
Apply local transformations* to the dual graph of the Aztec diamond of order n, obtaining the dual graph of the diamond of order $n-1$ while picking up an extra weight factor of 2^n.

* see pages 7&8



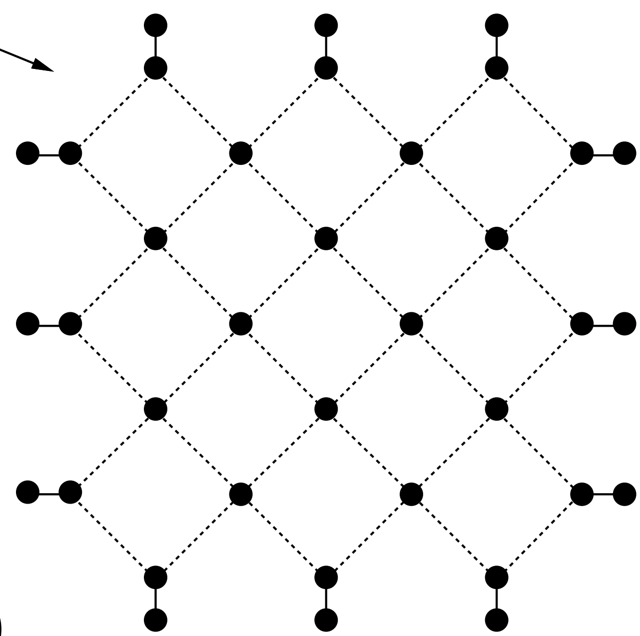
(solid lines denote edges of weight 1)

$\Rightarrow$



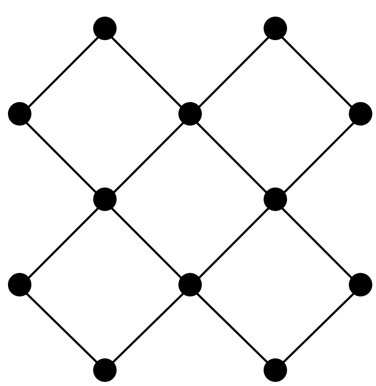
$\Downarrow$ 2^9

(dashed lines denote edges of weight 1/2)



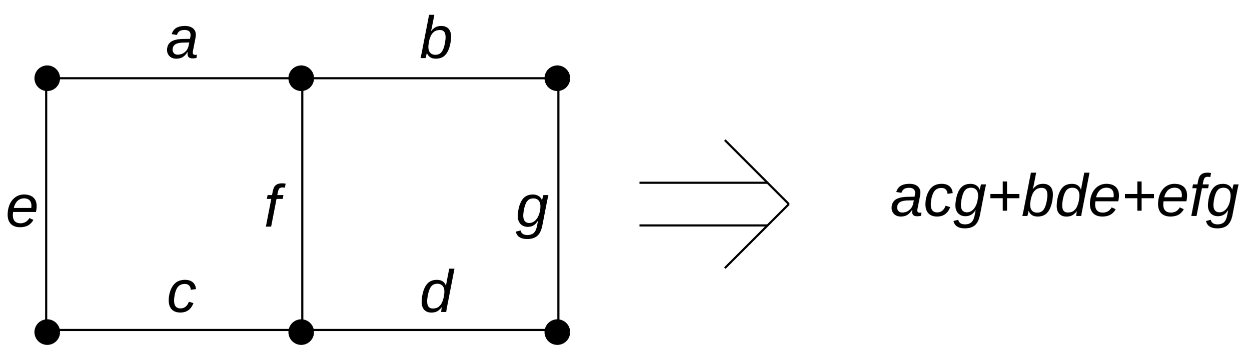
2^{-6} $\Leftarrow$

(two steps in one: pruning leaves and re−scaling weights)



(theorem follows by induction)

Given a graph *G* with non−negative real numbers (or formal variables) called "weights" associated with its edges, we consider the sum of the weights of all the perfect matchings of *G*. For instance:

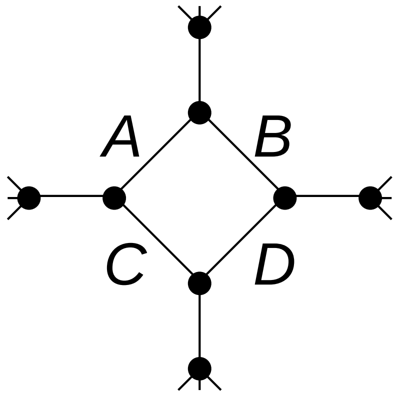


(A *perfect matching* of a graph *G* is a collection of edges of *G* that jointly contain each vertex of *G* exactly once, and the *weight* of a collection of edges is the product of the weights of the edges.)

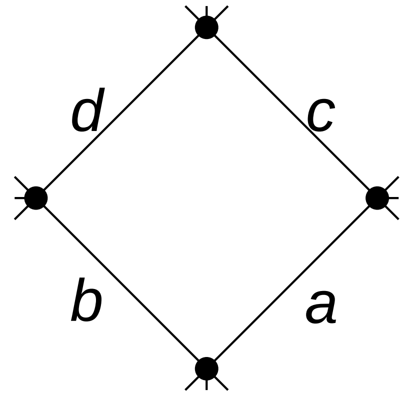
The number of domino tilings of a region like the Aztec diamond is the sum of the weights of all the perfect matchings of the dual graph, where each edge is given weight 1.

The local replacement lemma

The sum of the weights of the matchings of a graph that contains the patch



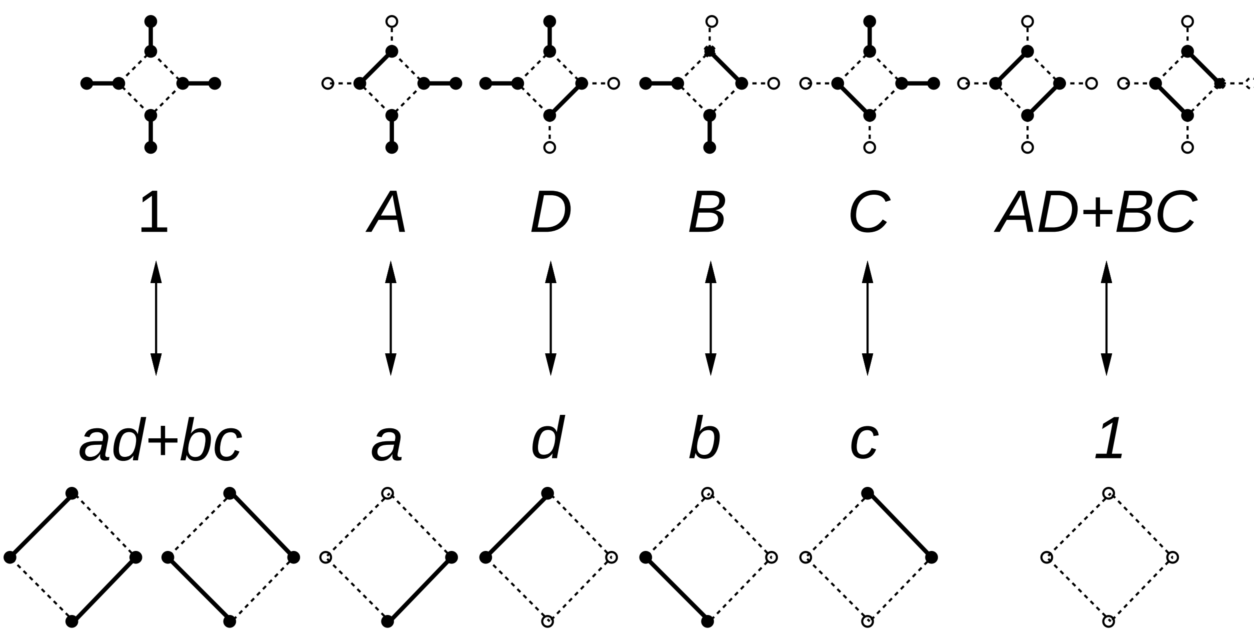
equals R times the sum of the weights of the matchings of the graph obtained by replacing the patch by



where unmarked edges have weight 1,

$R=AD+BC$, $a=A/R$, $b=B/R$, $c=C/R$, $d=D/R$.

Check that the following two lists of numbers differ by a factor of R (note $ad+bc=1/R$):

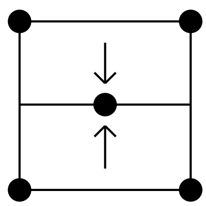


Domino Shuffling

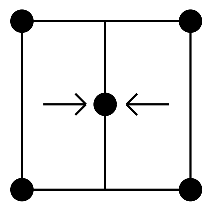
Shuffling is an operation that converts a domino tiling of an Aztec diamond of order n into one of order $n+1$. Assume every other vertex is marked with a dot, as shown. Then, to shuffle:

MARK the dominoes with arrows (arrows pointing towards the dots);

DESTROY all dominoes that belong to 2−by−2 blocks of the form



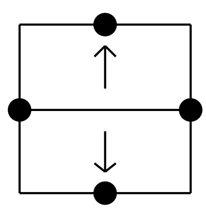
or



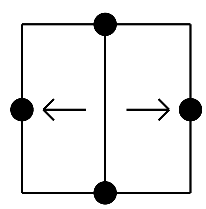
;

SLIDE all remaining dominoes one step in the direction of their arrows; and

CREATE new 2−by−2 blocks of the form

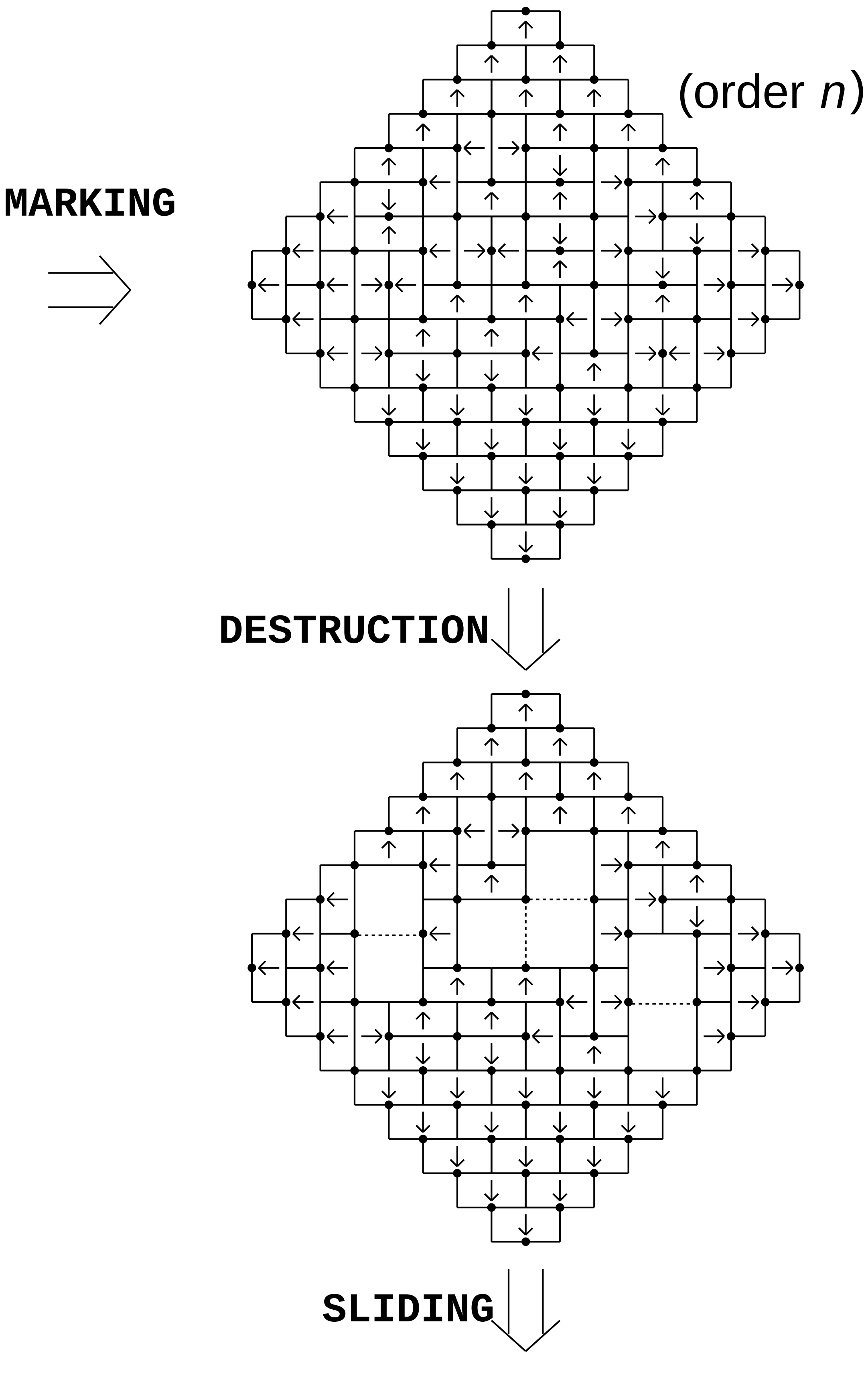


or



in all the new open spaces, using a fair coin to decide which kind goes where.

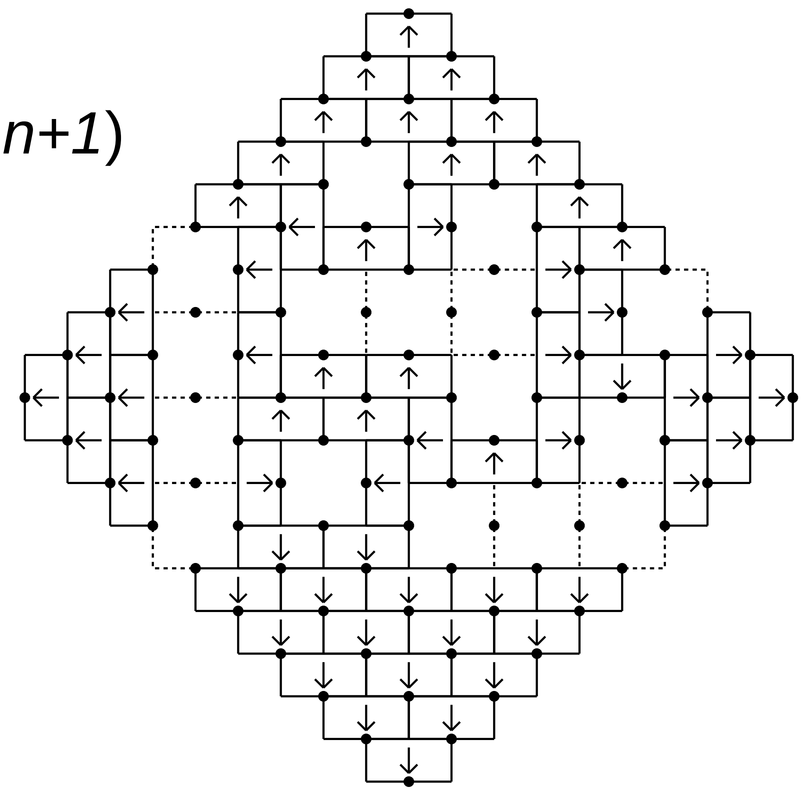
Example of Domino Shuffling



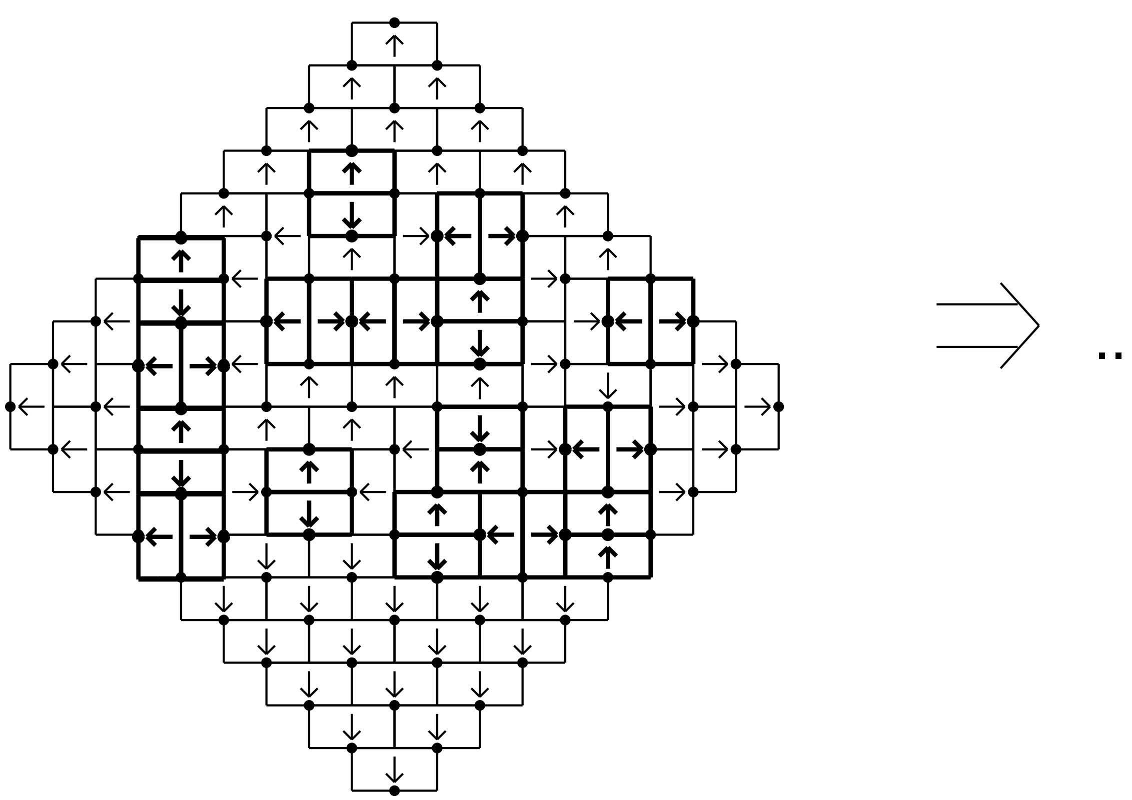
Example of Domino Shuffling (continued)

SLIDING

(order *n+1*)

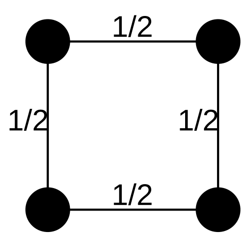


CREATION

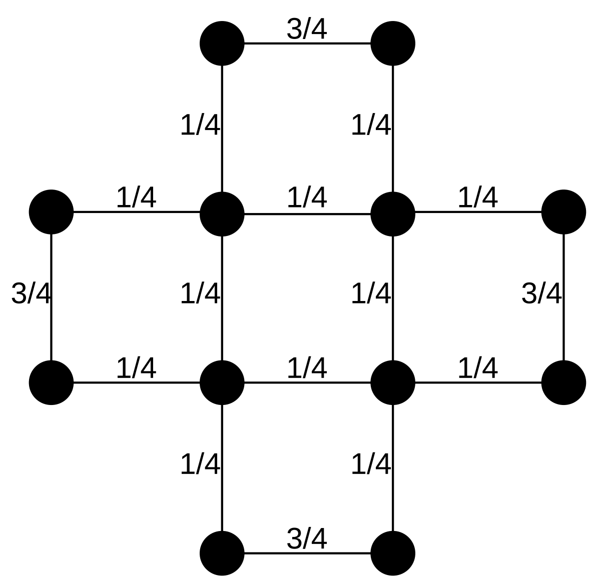


Edge Probabilities

Given an edge *e* of G_n (the dual graph of the *n*th Aztec diamond), define *p(e)* (the "edge probability") as the proportion of matchings of G_n that include the edge *e*.



G_1

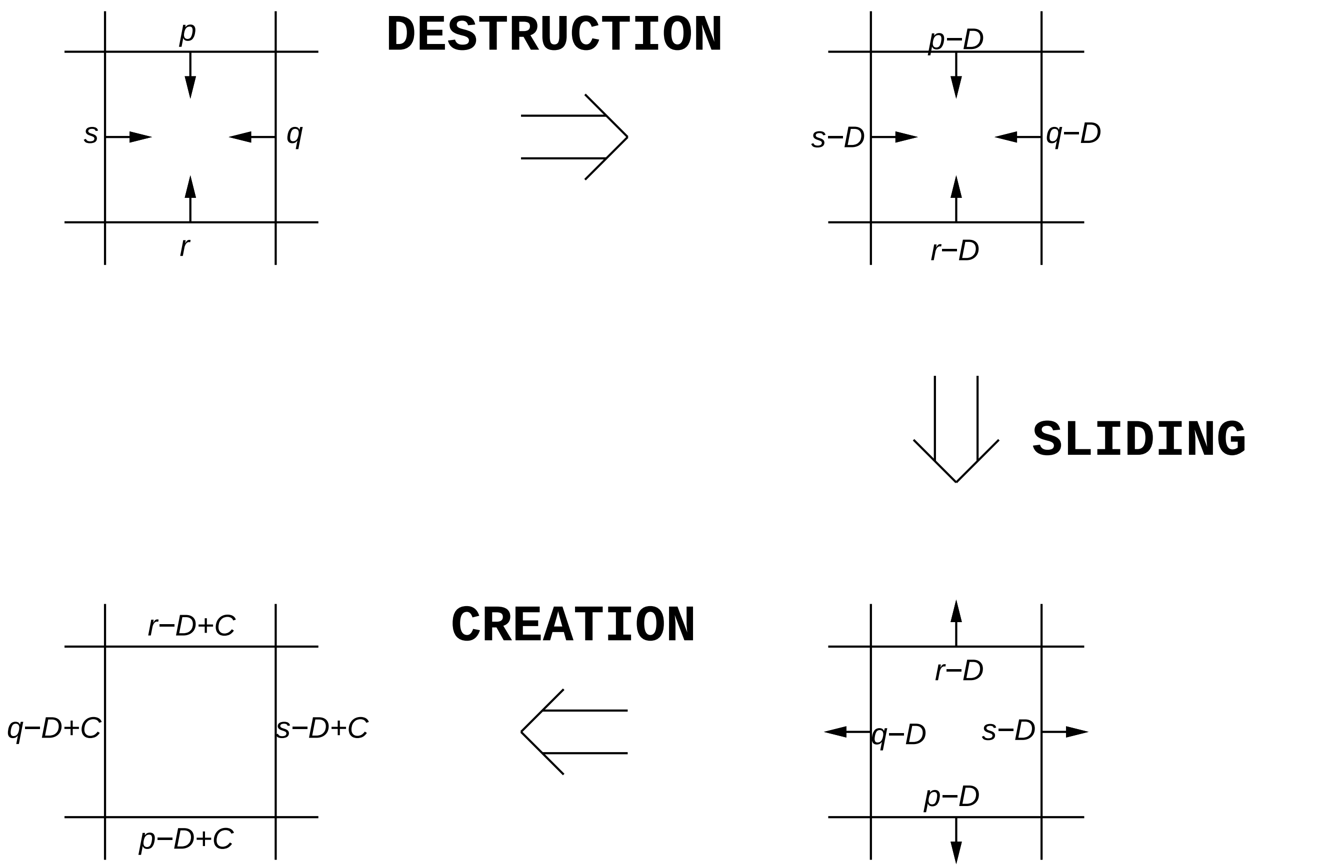


G_2

Note that at each vertex of G_n, the sum of the probabilities of the incident edges is 1.

Shuffling and Edge Probabilities

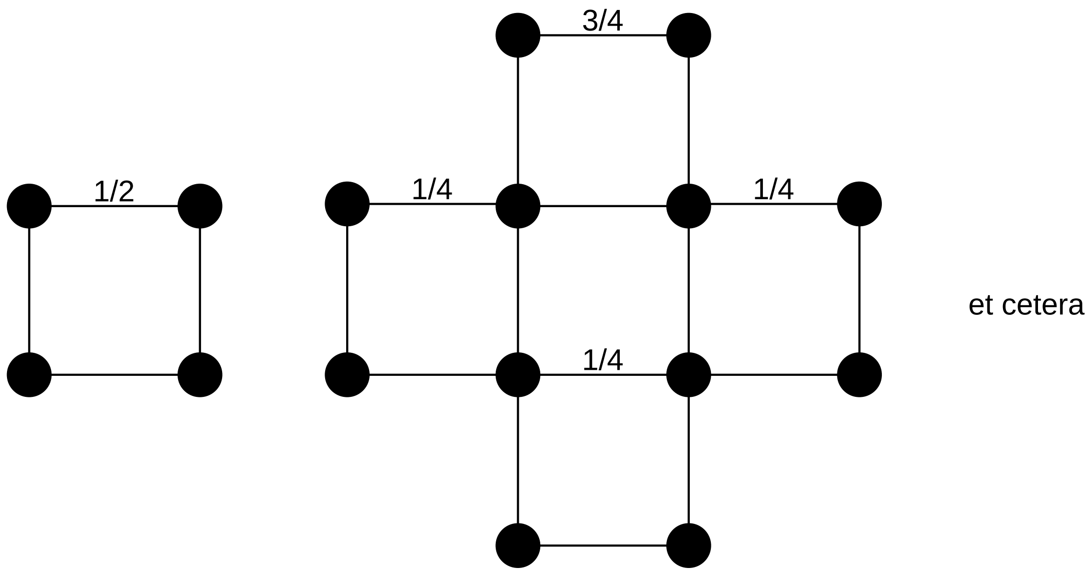
Shuffling applies not only to dominoes, but also to the the probability of a domino existing at a certain place:



This allows us to prove that these probabilities satisfy a linear recurrence relation.

The Generating Function

We can consider, without loss of generality, just the probabilities of the north−going edges of an Aztec diamond (where a north−going edge is defined as one whose arrow points upward).



We encode these via a generating function in three variables:

$$\left(1/2\right)\ z + \left(\begin{array}{ccc} & 3/4\ y & \\ + 1/4\ x^{-1} & & + 1/4\ x \\ & + 1/4\ y^{-1} & \end{array}\right) z^{2} + \ldots$$

(coefficient of $x^{j}y^{k}z^{n}$ equals probability associated with the edge at location *(i,j)*
in the Aztec diamond of order *n*)

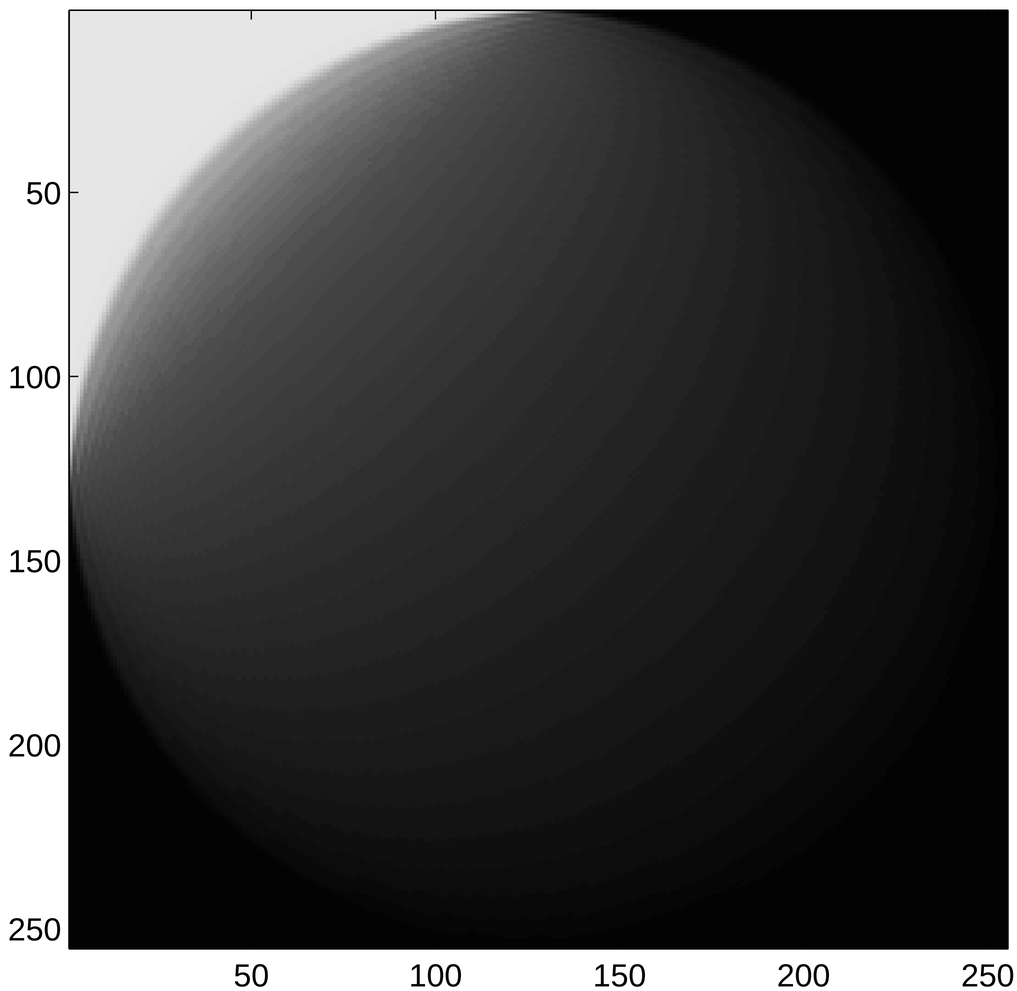
Theorem (Ionescu and Propp): This series in x,y,z **is equal to the rational function**

$$\frac{z/2}{(1-yz)\,(1-(x+x^{-1}+y+y^{-1})z/2+z^{2})}$$

Consequence (Cohn, Elkies, and Propp): The probability that a randomly chosen domino tiling of the Aztec diamond of order n **has a north-going domino at (normalized) location** (s,t) **(with** $|s|+|t|<1$ **) is asymptotic to** $P(s,t)$, **where**

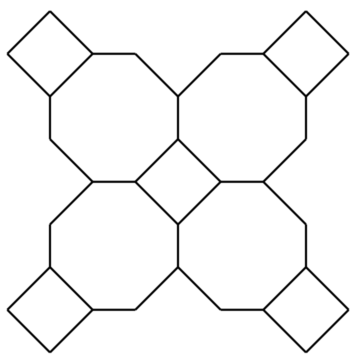
$$P(s,t) = \begin{cases} 0 & \text{if } s^2+t^2 \geq 1/2 \text{ and } t < 1/2 \\ 1 & \text{if } s^2+t^2 \geq 1/2 \text{ and } t > 1/2 \\ \frac{1}{2} + \frac{1}{\pi}\tan^{-1}((2t-1)/\mathrm{sqrt}(1-2s^2-2t^2)) & \text{if } s^2+t^2 < 1/2 \end{cases}$$

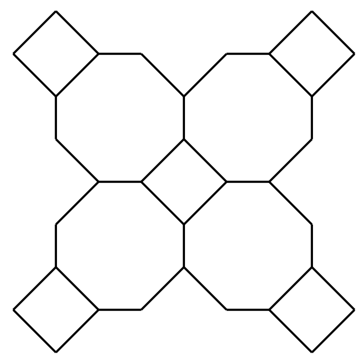
(note sharp cut-offs outside the inscribed circle)

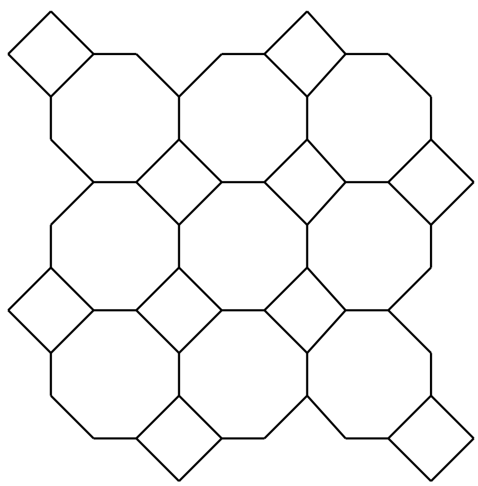

50
100
150
200
250
50
100
150
200
250

Fortresses

Here are some more graphs with the property that the number of perfect matchings they possess is given by a very simple formula (with quadratic exponent):



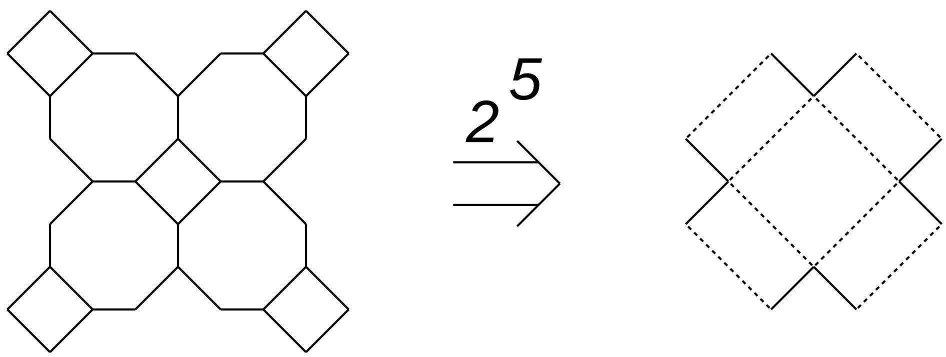




For each such graph, the number of perfect matchings is either a power of 5 or twice a power of 5 (Yang, 1991).

~~Open~~ Problem

The same diagramatic style of proof that worked for Aztec diamonds also works for these graphs. (First step: Reduce to weighted Aztec diamond graphs.)



However, we have not yet figured out an analogue of domino shuffling that applies here.

Such an algorithm would give us an approach to the problem of computing the asymptotic behavior of tilings of these graphs.

Update: Such an algorithm has now been found!

For more details, write to propp@math.mit.edu

or check out http://www−math.mit.edu/~propp.