monthly writings in and around mathematics by James Propp

The Square Root of Pi

> *Bella: You gotta give me some answers.*
>
> *Edward: "Yes"; "No"; "To get to the other side"; "1.77245…"*
>
> *Bella (interrupting): I don't want to know what the square root of pi is.*
>
> *Edward (surprised): You knew that?*
>
> — *Twilight*

March 14 (month 3, day 14) is the day math-nerds celebrate the number π (3.14…), and you might be one of them. But if you're getting tired of your π served plain, why not spice things up by combining the world's favorite nerdy number with the world's favorite nerdy operation?

The square root of π has attracted attention almost as long as π itself. When you're an ancient Greek mathematician studying circles and squares and playing with straightedges and compasses, it's natural to try to find a circle and a square that have the same area. If you start with the circle and try to find the square, that's called squaring the circle. If your circle has radius r=1, then its area is $\pi r^2 = \pi$, and a square with side-length s has the same area as the circle of radius 1 if $s^2 = \pi$, that is, if s = sqrt(π). It's well-known that squaring the circle is impossible in the sense that, if you use classic Greek tools in the classic Greek manner, you can't construct a square whose side-length is sqrt(π) (even though you can approximate it as closely as you like); see David Richeson's new book (listed in the References) for lots more details about this. But what's less well-known is that there are (at least!) two other places in mathematics where the square root of π crops up: an infinite product that on its surface makes no sense, and a calculus problem that you can use a surface to solve.

A NONSENSICAL FACTORIAL

Factorials are a handy shortcut in many counting problems. If someone asks you how many different ways there are to order the 52 cards of a standard deck, the answer is "52 factorial" (meaning 1 × 2 × 3 × … × 50 × 51 × 52, often written as "52!"); this answer takes a lot less time to say than "eighty thousand six hundred and fiftyeight vigintillion, … five hundred thousand quadrillion". But how should we define *n*! if *n* is not a counting number? What if, say, *n* is negative one-half?

A practical-minded person might jibe that someone who wants to know how many ways there are to order a stack consisting of negative one-half cards isn't playing with a full deck. But this kind of craziness is often surprisingly fruitful; the symbolisms we mathematicians come up with sometimes take on a life of their own, and living things want to grow. Or, as the great mathematician Leonhard Euler wrote [ENDNOTE #1: Can anyone provide a source, and/or the original non-English version of the aphorism?], "My pen is a better mathematician than I am," meaning that notations sometimes precede understanding. By trying to extend the factorial function to numbers like −1/2 that aren't counting numbers, Euler was led to invent the gamma function, which is important not just in pure mathematics but in applications.

Specifically, Euler showed that when *n* is a positive integer, *n*! is equal to the integral

$$\int_0^\infty x^z e^{-x}\, dx$$

(which I'm going to treat as a black box, so don't worry if you don't know what an integral is or what that expression means). The cool thing about that expression is that it makes sense when *n* is −1/2, and gives the value sqrt(π). What's more, if you find other integrals that coincide with *n*! when *n* is a positive integer, they'll give you sqrt(π) when you plug in $n = -1/2$. Metaphorically you could say that even though "1 times 2 times … times −1/2" isn't equal to any number, the number it's *trying* to equal is sqrt(π).

Whatever can I mean by "trying to equal"? Here's a simpler example. If *r* is a positive number that's less than 1, then the alternating sum $1-r+r^2-r^3+\ldots$ (an alternating geometric series) converges to $1/(1+r)$ in the sense that the partial sums 1, $1-r$, $1-r+r^2$, $1-r+r^2-r^3$,… get ever-closer to $1/(1+r)$ as the number of terms gets ever-bigger. (ENDNOTE: Provide proof.) Now, if *r* is 1, then $1/(1+r)$ makes sense and equals 1/2, but the alternating sum 1−1+1−1+… does not get ever-closer to 1/2 or to any other number; the partial sums 1, 1−1, 1−1+1, 1−1+1−1, … just vacillate between 1 and 0 instead of converging. But what if we change the definition of convergence? Mathematician Ernesto Cesàro proposed a more permissive definition of convergence in which we take a sequence of numbers that annoyingly refuses to converge and replace it by a new, more tractable sequence whose *n*th term is the *average* of the first *n* terms of the old sequence. If we do that with the sequence 1,0,1,0,… we get 1/1, (1+0)/2, (1+0+1)/3, (1+0+1+0)/4, … which sure enough converges to 1/2. And remember, 1/2 is exactly what we got from $1/(1+r)$ when we replaced *r* by 1. [CARTOON: "She loves me, she loves me not, …" "This could take a while. Can we just say she half-loves you?"]

So I've shown you two different shady methods of assigning the non-converging sum 1−1+1−1+… a value. One method considers the more general sum $1-r+r^2-r^3+\ldots$, finds an expression for it that's valid for all positive $r<1$, and then substitutes $r=1$. The other method changes the meaning of convergence. But what's interesting is that even though both methods are shady (and are shady in seemingly different ways), they give the same value.

This phenomenon − different shady methods arriving at the same answer − crops up a lot at the frontiers of math, and it often points to mathematical concepts that have not yet come into view. And when we've climbed the hill that hid those concepts from view, we find that there's nothing illegitimate about them; they're just more mathematically technical than, and not as attention-grabbing as, a formula like "1−1+1−1+...=1/2".

An infamous example of this is "1+2+3+... = −1/12" (and its less-celebrated sibling "1+1+1+... = −1/2"). When equations like these are hyped outside of their proper context, math-fans sometimes respond with anger: "Those bastards are changing the rules again!" What often isn't explained (because it's fairly technical) is what the new rules are. What can and should be explained is that this rules-change is analogous to what you'd see if you were watching American football on TV and then changed the channel to watch Australian football. Well-educated sports fans don't fume "They're playing it wrong! Why, even my gym teacher knows more about football than these players do!"; they recognize that the Australians are playing a different sport. Same here, except that the name of the sport is "zeta-function regularization", it's played with numbers, and the players are analytic number theorists, only some of whom are Australian.

SOLVING A PROBLEM BY MAKING IT MORE COMPLICATED

The physicist William Thomson, better known as Lord Kelvin, was a big fan of mathematics, calling it "the etherealization of common sense" and "the only good metaphysics". According to an anecdote recounted by his biographer S. P. Thompson, Kelvin was a bit of an awestruck fanboy when it came to mathematicians themselves:

Once when lecturing to a class he [Lord Kelvin] used the word "mathematician," and then interrupting himself asked his class: "Do you know what a mathematician is?" Stepping to the blackboard he wrote upon it:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Then putting his finger on what he had written, he turned to his class and said: "A mathematician is one to whom that is as obvious as that twice two makes four is to you."

I don't think I know a single mathematician who regards this formula as obvious, so I don't think Kelvin's definition is a good one. Here is an alternative definition for your consideration: A mathematician is one who recognizes the difference between what is obvious and what is merely familiar. Or: A mathematician is one who recognizes the difference between what is obvious and what one has come to understand in stages, by means of a nontrivial chain of trivial steps.

This seems like an ungrateful way to treat a scientist who, when all is said and done, was just trying to praise my kind. But a mathematician is one to whom flattery is annoying if it is inaccurate. (See, I can overgeneralize too!)

The expression

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx$$

represents the area under the curve $y = e^{-x^2}$, where e = 2.718… is the other famous nerdy number, dominating the world of exponentials and logarithms the way π rules the world of sines and cosines. The $\int$ notation is due to Leibniz, who chose a stylized version of the letter "*s*" to commemorate the fact that the way we can compute such areas is by first approximating them as sums, and by then atoning for the error of the approximation by using ever-better approximations and seeing what the approximations converge to.

Let's talk about sums for a minute. Specifically, consider the problem of adding together all the numbers in the first four rows and first four columns of the multiplication table:

1 2 3 4

2 4 6 8

3 6 9 12

4 8 12 16

A shortcut for computing the sum is to consider that the product (1 + 2 + 3 + 4) × (1 + 2 + 3 + 4), if expanded out by the distributive law, would have these sixteen numbers as its constituent terms, so that the sum of the sixteen numbers must be 10 × 10, or 100. ("A mathematician is someone who works hard at being lazy", said Murray Klamkin, riffing on George Pólya.)

You might want to apply the same idea to sum the numbers in the infinite array

1 1/2 1/4 1/8 …

1/2 1/4 1/8 1/16 …

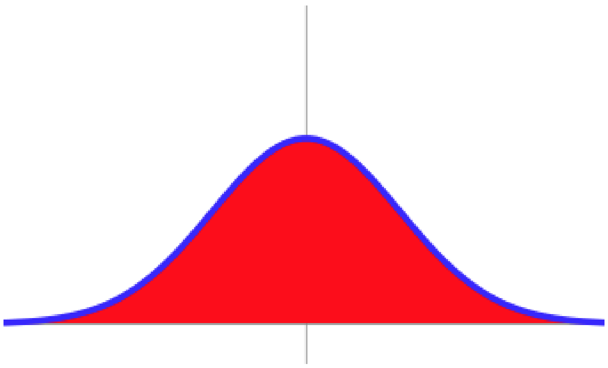
1/4 1/8 1/16 1/32 …

1/8 1/16 1/32 1/64 …

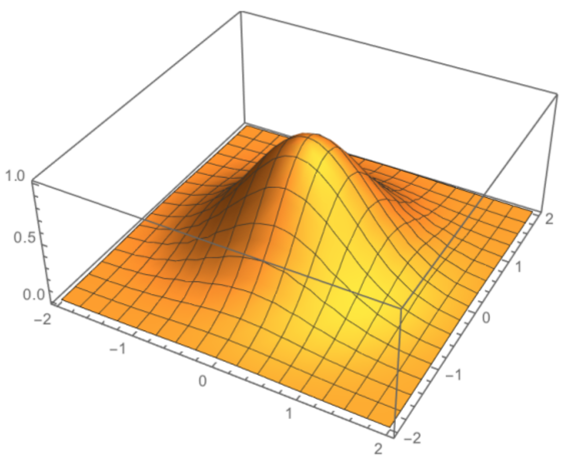
…

(see Endnote #X).

This way of summing the numbers in an array is a cute trick, and it's the sort of trick you see quite often in mathematics, where you reduce a two-dimensional problem to a one-dimensional problem, or more broadly solve a problem by reducing it to something simpler. But how often do you solve a problem by reducing it to something that's more complicated? That counterintuitive tactic provides the nicest way I know to prove the famous formula Lord Kelvin quoted.



Let A be the area between the x-axis and the curve $y = e^{-x^2}$. Then multivariate calculus can be used to show that A^2 is the volume between the x,y-plane and the surface $z = e^{-x^2} e^{-y^2}$.



Using the laws of exponents, we can rewrite that right-hand side as $e^{-(x^2+y^2)}$. This crucial step reveals that the surface has a surprising symmetry: it's rotationally symmetric around the z-axis. That fact enables calculus students to treat the space between the x,y-plane and the surface $z = e^{-(x^2+y^2)}$ as a solid of revolution, and to compute its volume using the method of cylindrical shells. One gets $A^2 = \pi$, from which it follows that A = sqrt(π). (For more details, see John Cook's writeup.)

I would modify Kelvin's adage to say that a mathematician is someone who, having learned the preceding story, may not be able to remember the details, but remembers that circles play a role, and that the value of the integral involves π.

The formula Lord Kelvin wrote on his blackboard isn't just a mathematical curiosity: the expression e^{-x^2} is the famous bell-shaped curve of statistics, and the fact that the area beneath it is sqrt(π) explains where the π in basic statistical formulas comes from.

To go back to the classic problem with which this essay began: You can't square the circle, but you can do something much more important, namely, you can prove

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

by interpreting the square of the left hand side as a volume and then computing that volume using circles.

To give Lord Kelvin his due, let me credit him for praising mathematicians for conceptual understanding, even if his praise strikes me as excessive. His adage is a lot better than something along the lines of "A mathematician is someone who knows the first half-dozen digits of the square root of π."

ENDNOTES

The sum arises as the expansion of 1+1/2+1/4+1/8+… times 1+1/2+1/4+1/8+…, and since 1+1/2+1/4+1/8+… equals 2 (in the sense that the partial sums converge to 2), the sum of all those numbers equals 2 times 2, and as Lord Kelvin said, "twice two makes four". On the other hand, we can take that table of numbers and group it by diagonals to get 1×1 + 2×1/2 + 3×1/4 + 4×1/8 + … So we've shown that the infinite sum 1/1 + 2/2 + 3/4 + 4/8 + 5/16 + … (where the numerators increase by adding 1 and the denominators increase by doubling) converges to 4. A mathematician is someone who thinks this is cute. Not "obvious" — just cute.

REFERENCES

David Richeson, Tales of Impossibility: The 2000-Year Quest to Solve the Mathematical Problems of Antiquity, Princeton University Press.

Videos about "−(1/2)!"

Presh Talkwalkar: "What is the Factorial of 1/2?"

blackandredpen: "The Gamma function & the Pi function"

blackandredpen: "But can you do negative factorial?"

Euler's Academy: "Gamma Function: (−1/2)!"

Videos about 1+2+3+… = −1/12 (see also [LINK TO BLOG POSTS])

[LINK TO NUMERPHILE VIDEO]

[LINK TO MATHOLOGER VIDEO]

This entry was posted in Uncategorized on March 1, 2020 [https://mathenchant.wordpress.com/?p=3300] .