

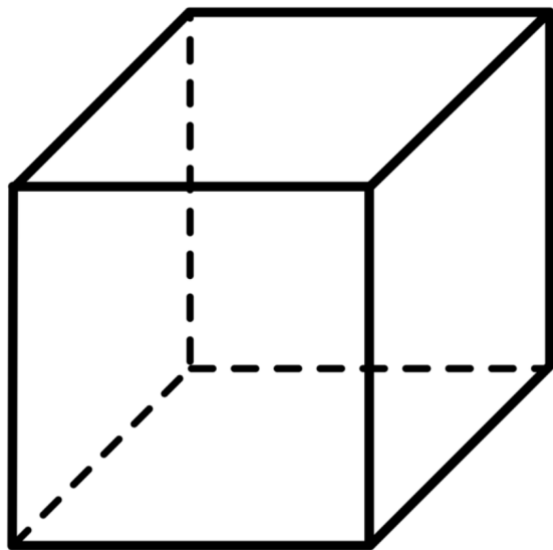
monthly writings in and around mathematics by James Propp

Motley Madness

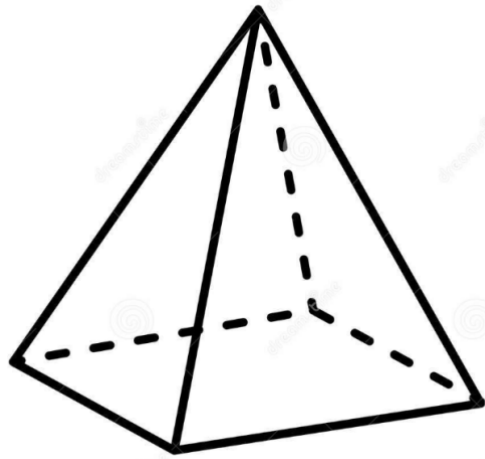
I'd like to tell you about a polyhedron with 24 vertices, 12 edges, and 14 faces, but already those opening words have created a predicament, because some of you are asking "What's a polyhedron?" while others who already know about polyhedra are asking "Doesn't Euler's formula rule out such a thing?" I'd better address the first question first.

A polyhedron is a three-dimensional figure made of polygons (faces) that meet edge-to-edge. A familiar example is the cube; other examples are the square pyramid, tetrahedron, octahedron, cuboctahedron, and rhombic dodecahedron. These polyhedra are all convex: there are no holes or indentations.

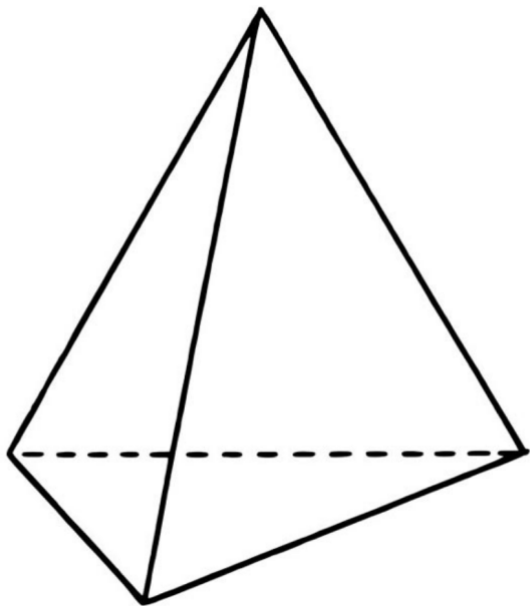
Cube:



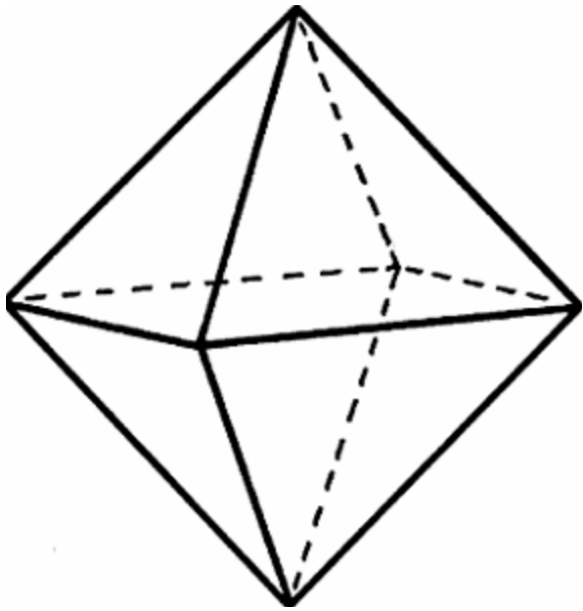
Square pyramid:



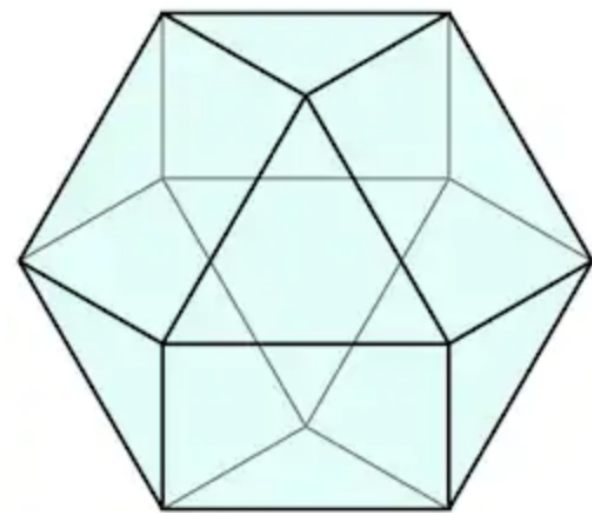
Tetrahedron:



Octahedron:



Cuboctahedron:



Rhombic dodecahedron:



If we take a census of convex polyhedra, counting the vertices, edges, and faces of each, a remarkable pattern emerges.

Polyhedron	# of Vertices	# of Edges	# of Faces
Cube	8	12	6
Square pyramid	5	8	5
Tetrahedron	4	6	4
Octahedron	6	12	8
Cuboctahedron	12	24	14
Rhombic dodecahedron	14	24	12

The number of edges (call it E) is always 2 less than the number of vertices (V) plus the number of faces (F). That is, $E = V + F - 2$, or equivalently, $V - E + F = 2$. This beautiful formula of Leonhard Euler (found earlier by René Descartes but then largely forgotten) is the subject of David Richeson's book "Euler's Gem: The Polyhedron Formula and the Birth of Topology". An older book, "Proofs and Refutations" by Imre Lakatos, shows the subtle interplay between giving a proof of Euler's formula and nailing down precise definitions of the concepts involved.

You may have noticed a different pattern in the table: the first and fourth rows contain the same numbers in a different order (8,12,6 vs. 6,12,8), as do the fifth and sixth (12,24,14 vs. 14,24,12). This shows the notion of duality at play. Every convex polyhedron P has a dual P' : the vertices, edges, and faces of P' correspond respectively to the faces, edges, and vertices of P . The cube and octahedron are dual to one another, as are the cuboctahedron and rhombic dodecahedron, while the square pyramid and tetrahedron are their own duals.

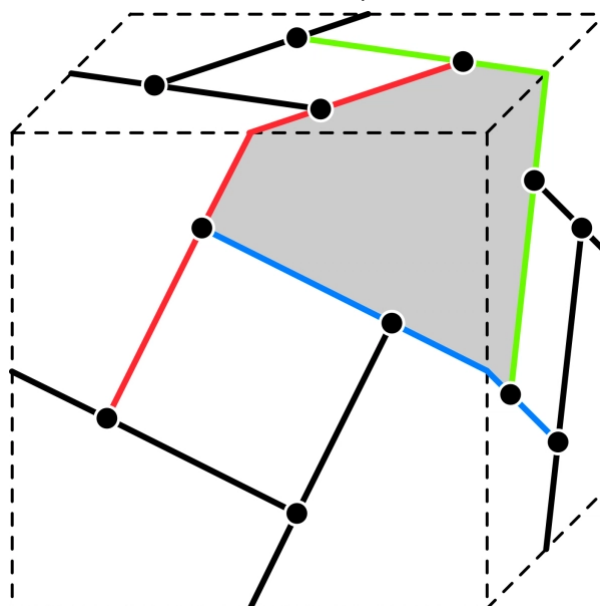
(You may have also noticed that the number of edges of each polyhedron in the table is even, but that's a fluke; for instance, the "wedge" has nine edges.)

Getting back to “I’d like to tell you about a polyhedron with 24 vertices, 12 edges, and 14 faces”: since $24 - 12 + 14$ isn’t 2, the polyhedron can’t be a convex polyhedron of the ordinary kind. It’s what Scott Kim has dubbed a pseudo-polyhedron, and in particular, it’s a pseudo-cuboctahedron. It’s obtained from the cuboctahedron by a curiously twisted duality that turns vertices into edges and vice versa (rather than turning vertices into faces and vice versa the way ordinary duality does).

Kim started down the road leading to pseudo-polyhedra as a sixth grader when he tried to design dissection puzzles for which the the common impulse to join pieces edge-to-edge would lead solvers astray in the worst way possible; he wanted dissections in which *none* of the pieces joined neatly edge-to-edge. Check out Kim’s Gathering 4 Gardner video about what he calls motley dissections and the pseudo-polyhedra that are associated with some of them.

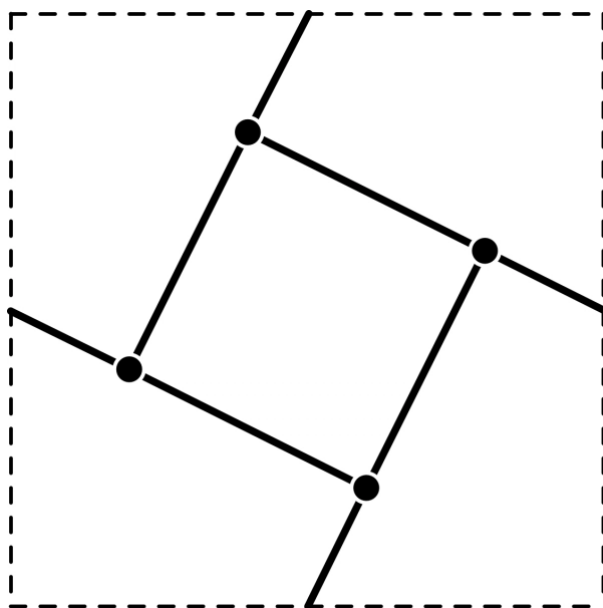


Pseudo-polyhedra satisfy the crazy formula $E^* - V^* + F^* = 2$, where E^* might better be called the number of pseudo-edges. The pseudo-cuboctahedron has six 4-sided pseudo-faces and eight 3-sided pseudo-faces that fit together in an off-kilter way. To see a broken but symmetrical realization of the pseudo-cuboctahedron, examine this motley dissection of the surface of a cube (but ignore the shading for now):

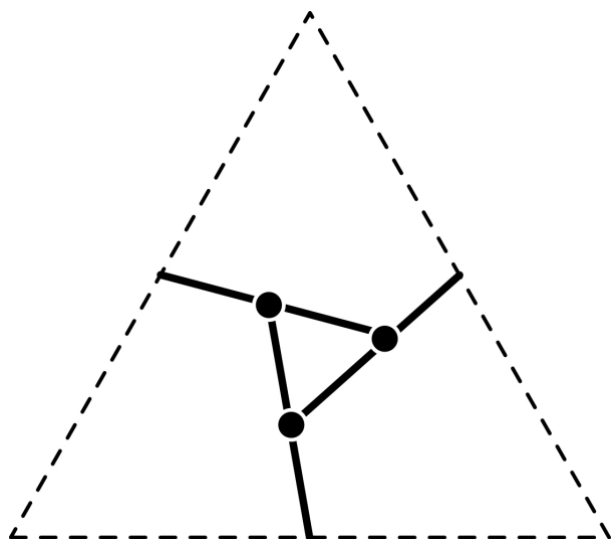


The edges of the cube are shown as dashed lines; they serve as scaffolding, but the pseudo-faces “ignore” them (sometimes trespassing from one face of the cube to another) so you should too. The solid lines show the edges of the pseudo-faces. The cube is opaque, so there are three square faces you can’t see but all six faces are decorated in the same pattern, so the dissection has a square pseudo-face in the middle of each square face. The other eight pseudo-faces are trickier: they’re nonplanar figures, each with six dots on its boundary and covering a corner of the cube. (I’ve shaded one of the eight pseudo-faces.) In this crazy game, those nonplanar pseudo-faces are regarded as triangles (though maybe “trilaterals” would be more accurate): even though each one has six vertices on its boundary, that boundary involves only three pseudo-edges! (For instance, the red, green, and blue pseudo-edges bound the shaded pseudo-face.) Counting things in this quirky way we find 24 vertices, 12 pseudo-edges, and 14 pseudo-faces, as promised. This is no mere fluke; Kim has shown that every convex polyhedron in which each vertex shares an edges with exactly four other vertices (the way the cuboctahedron does) has a pseudo-polyhedron counterpart.

If instead of building a cube out of four squares that look like



we build an octahedron out of eight triangles that look like



we get a motley dissection of the surface of an octahedron with nice-looking triangular pseudo-faces and crumpled square pseudo-faces. This too is a sketch of the pseudo-cuboctahedron: different geometry, same combinatorics!

I haven't told you what the rules of this game are, but I'm not being coy here; Kim is still trying to figure out what the best rules are. He's operating in a "pre-axiomatic" mode of mathematics where the initial focus is on examples, followed by conjectures and proof-sketches, with definitions coming later. In this mode we can sometimes choose the definitions to give the theorems we want; in this case, the theorem I want is the twisted Euler formula $E^* - V^* + F^* = 2$, because even though I love $V - E + F = 2$, I also enjoy an occasional mathematical Feast of Fools.

While I wait for Kim and others to elucidate some sort of general theory of pseudo-polyhedra (and pseudo-polytopes in higher dimensions), I'd like to see a better model of the pseudo-cuboctahedron. Specifically, I'd like to see a curvy pseudo-cuboctahedron based on a motley dissection of the sphere rather than a motley dissection of the surface of a cube or octahedron. Can anyone provide one?

I hope readers will explore the rich landscape that Kim has been hiking through, and maybe build some roads through it! I hope to report new developments in a Mathematical Enchantments essay in late 2019.

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