Engel Machines

James Propp, UMass Lowell

& Mathematical Enchantments (blog)
& Barefoot Math (YouTube)

MOVES
August 8, 2017

Slides at http://jamespropp.org/moves17.pdf

A number puzzle

Can we write the number 8 as a sum of powers of $3/2$ (1, $3/2$, $9/4$, $27/8$, etc.)?

A number puzzle

Can we write the number 8 as a sum of powers of $3/2$ (1, $3/2$, $9/4$, $27/8$, etc.)?

What if we can't use any particular power of $3/2$ more than twice?

A number puzzle

Can we write the number 8 as a sum of powers of $3/2$ $(1, 3/2, 9/4, 27/8, \text{etc.})$?

What if we can't use any particular power of $3/2$ more than twice?

Yes: $8 = (3/2)^2 + (3/2)^2 + (3/2)^1 + (3/2)^0 + (3/2)^0$.

8 is nothing special!

Theorem: Every positive integer has a representation as a sum of powers of 3/2, with no power appearing more than twice. E.g.,

$$\begin{aligned} 2017 &= \mathbf{2} \times (3/2)^{15} + \mathbf{1} \times (3/2)^{14} + \ldots + \mathbf{0} \times (3/2)^{1} + \mathbf{1} \times (3/2)^{0} \\ &= \mathbf{2120222122002201}_{3/2} \end{aligned}$$

The inductive proof is embodied by the $\textsc{Sesquiac}$ computer: Feed it *n* balls and it'll output the sesquinary representation of the number *n* (similar to, but different from, the β-expansion of *n* with $\beta = 3/2$).

How it works

How would we take the base-3/2 representation of 8 and turn it into the base-3/2 representation of 9?

Key fact: $3 \times (3/2)^k = 2 \times (3/2)^{k+1}$.
Hence when $b \geq 3$, we can trade

$$\ldots \ \mathbf{a}\ \mathbf{b} \ \ldots$$

for

$$\ldots \ \mathbf{a{+}2}\ \mathbf{b{-}3} \ \ldots$$

Example: $8 = \mathbf{212}_{3/2}$, so

$$\begin{aligned} 9 &= \mathbf{213}_{3/2} \\ &= \mathbf{230}_{3/2} \\ &= \mathbf{400}_{3/2} \\ &= \mathbf{2100}_{3/2} . \end{aligned}$$

To seventeen and beyond

The SESQUIAC can add $8 = \mathbf{212}_{3/2}$ to $9 = \mathbf{2100}_{3/2}$, obtaining
$17 = \mathbf{21012}_{3/2} =$
$\mathbf{2} \times (3/2)^4 + \mathbf{1} \times (3/2)^3 + \mathbf{0} \times (3/2)^2 + \mathbf{1} \times (3/2)^1 + \mathbf{2} \times (3/2)^0.$

Unsolved problem: Are there infinitely many palindromes among the sequinary representations of the positive integers?

Engel machines

The SESQUIAC is an example of an *Engel machine*, which its inventor, Arthur Engel, called *the probabilistic abacus*.

Engel's abacus is a board on which a human operator moves identical pieces under certain constraints.

The pieces are called *chips*, and the locations at which chips reside are called *states*. The chips slide from one state to another according to certain rules.

States

THE PROBABILISTIC ABACUS 17

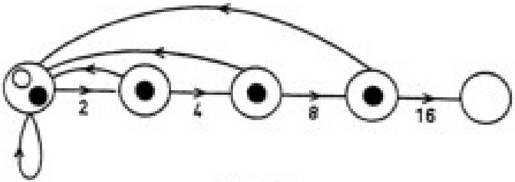


Fig. 25.

The states come in two types: *absorbing* states with no outgoing arrows, and *nonabsorbing* states with outgoing arrows pointing toward one or more other states.

How chips move

When a chip arrives at an absorbing state, it stays there.

When the number of chips at a nonabsorbing state equals or exceeds the number of outgoing arrows, the operator gets to send one chip along each of the outgoing arrows.

This is called *firing the chips* at that state.

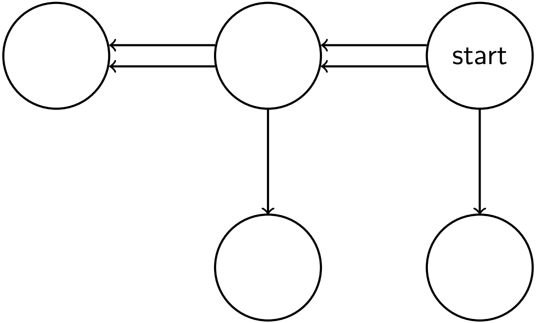
Where chips come from

There's a special nonabsorbing state called the *starting state*, or *source*. The operator gets to feed chips into the machine by adding chips to the source while the computation is taking place.

There are also some chips placed at non-source states before the computation starts. Engel tells us to preload the machine so that each nonabsorbing state is one chip shy of being able to fire. This is called the *critical loading*.

A small example

Here's a small Engel machine that will remind you of the SESQUIAC once we get it going:



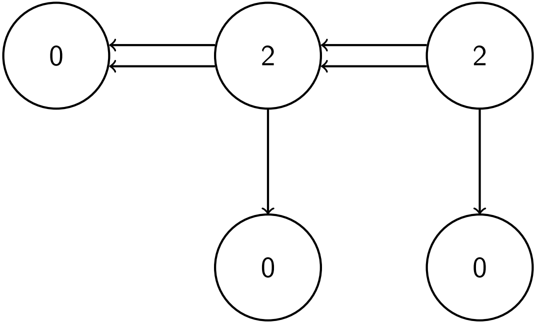
We feed in chips at the upper right. Each time we add a chip, we fire all the chips we can, until no more chips can be fired.

Recurrence

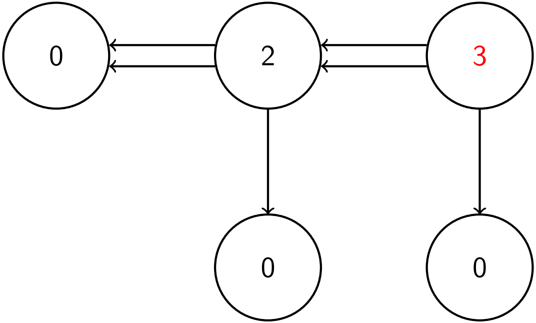
Theorem (Scheller): In an Engel machine started from the critical loading, the nonabsorbing states will eventually return to the critical loading.

Nowadays this result from the 1970s is understood as part of the theory of chip-firing and sandpiles (developed in the 1980s).

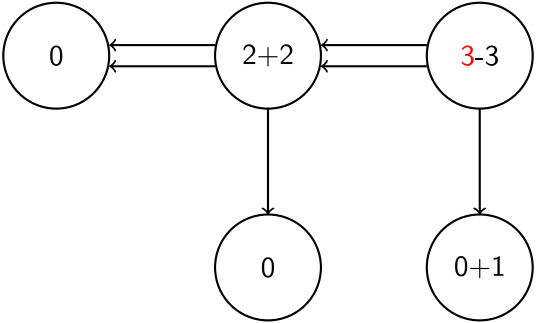
An example



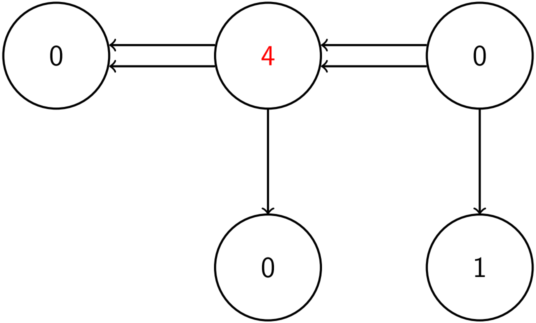
An example



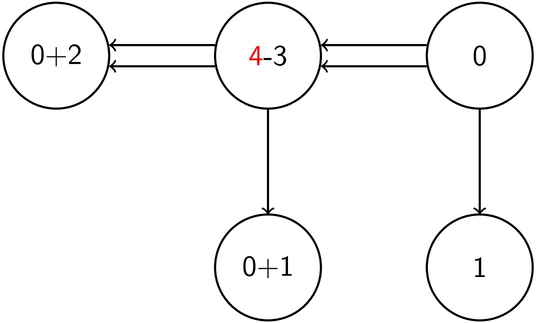
An example



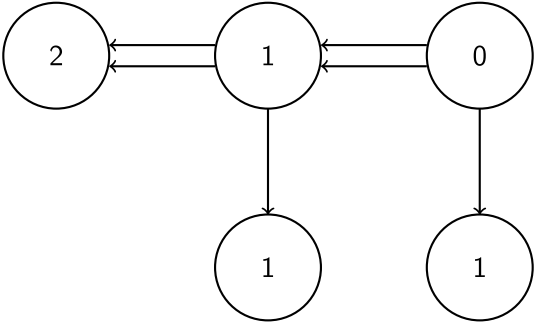
An example



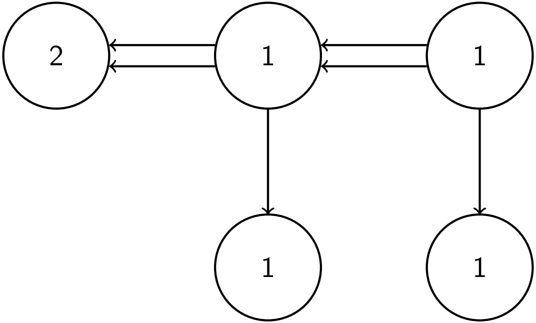
An example



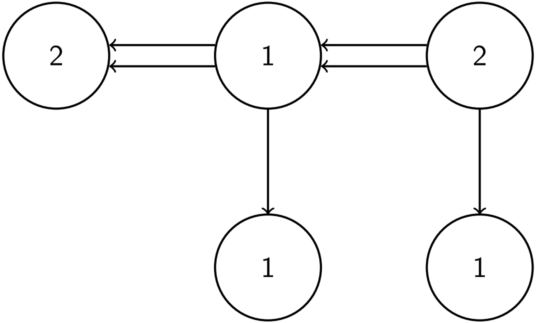
An example



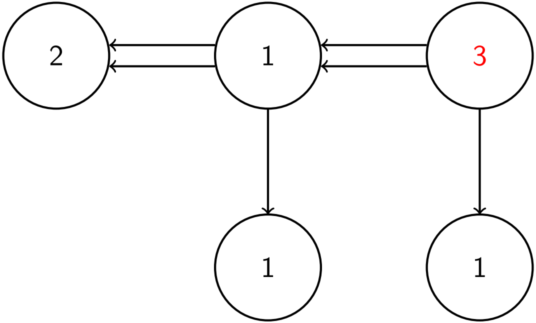
An example



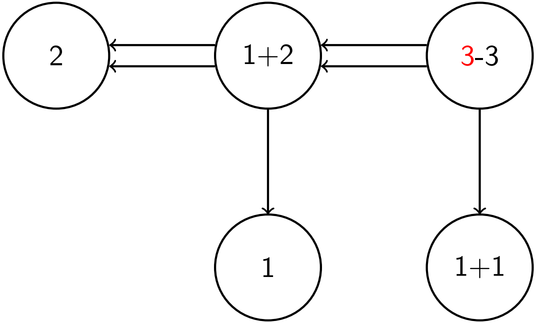
An example



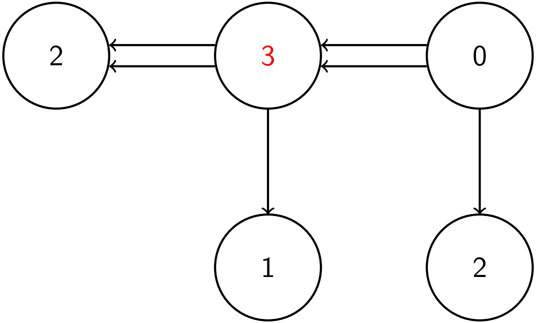
An example



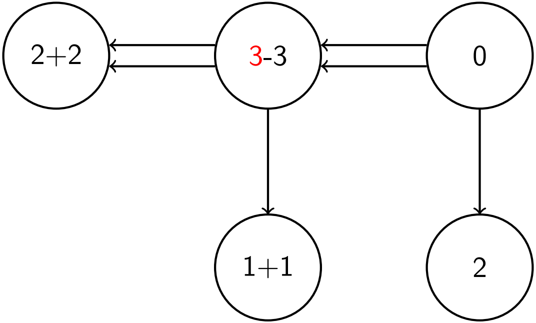
An example



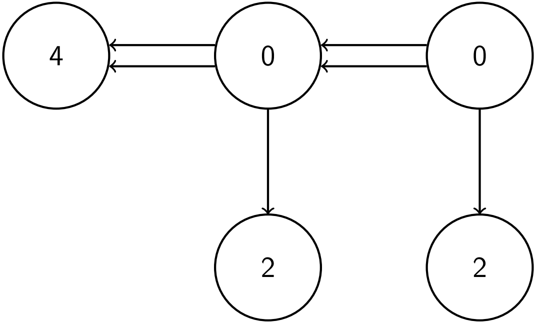
An example



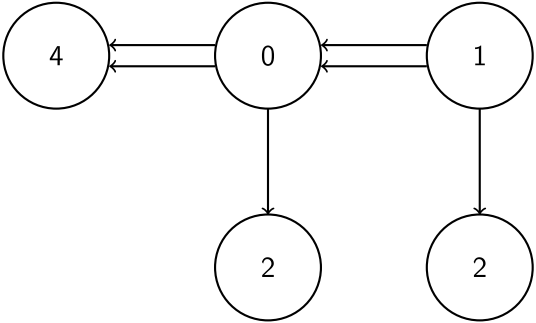
An example



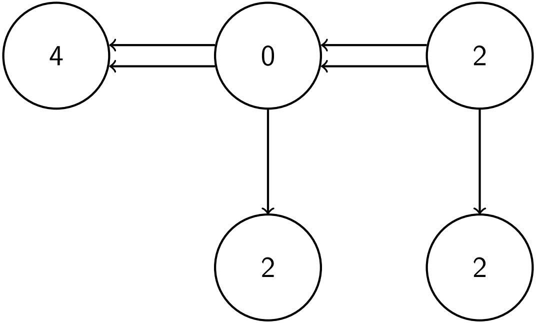
An example



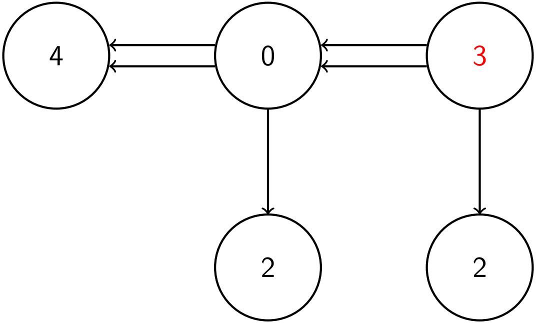
An example



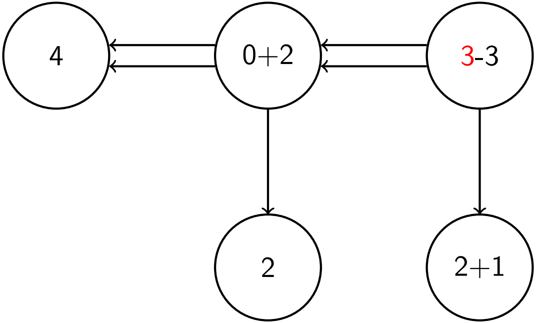
An example



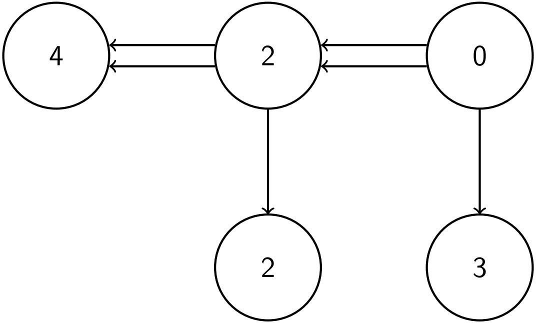
An example



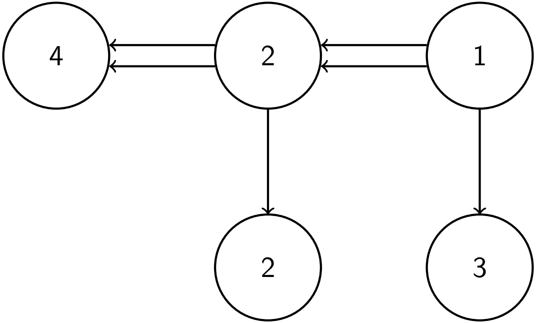
An example



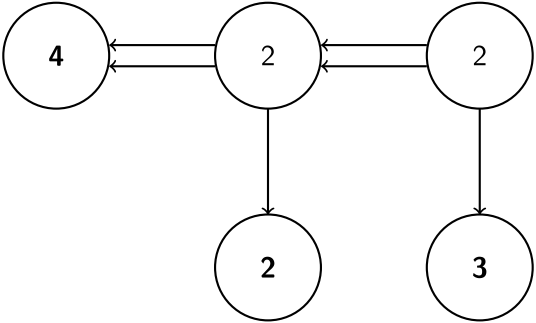
An example



An example



An example



The answer

When the critical loading recurs, the computation is over, and we read out the answer by counting the chips in the absorbing states: in this case, the answer is "**4**:**2**:**3**".

But what question is "**4**:**2**:**3**" the answer to?

The question that goes with the answer

We reinterpret the directed graph as the setting for a random walk.

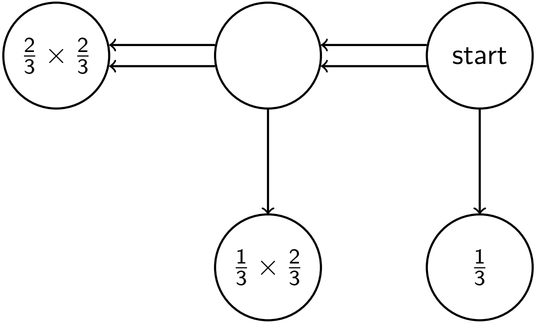
Imagine a solitaire game in which we start at the source, choose a random arrow from that state, follow that arrow to a new state, then choose a random arrow from the new state, and so on, until we end up at a sink.

What is the chance that we end up at a particular sink?

Theorem (Engel): Let p be the probability that a random walker started at S will eventually be absorbed at S'. Then p is also the proportion of the chips that end up at a particular sink S' during the duty cycle of an Engel machine being fed chips through source state S.

The meaning of 4:2:3

We can check that Engel's theorem is true for our particular directed graph, which gave us $4:2:3$ i.e. $\frac{4}{9}:\frac{2}{9}:\frac{3}{9}$.



A step beyond

What's cool is that Engel machines can compute answers to problems in discrete probability even when the Markov chain is complicated and the state-diagram contains cycles.

For instance, we can use a small Engel machine to solve a problem due to Bruce Torrence that was posted as a Riddler puzzle on the FiveThirtyEight blog just under a year ago. Go to the Barefoot Math YouTube channel to see how this goes, or follow the links from the Barefoot Math homepage at http://barefootmath.org.

Engel machines can also answer questions like "What's the expected time it takes for the Markov chain to enter an absorbing state?", and with some cleverness one can also get Engel machines to answer questions like "What is the expected square of the time it takes for the Markov chain to enter an absorbing state?"

Tanton, Diaconis, Bhargava

In the last few minutes, I'll connect Engel machines to work of James Tanton, Persi Diaconis, and Manjul Bhargava.

And then we'll see what happens to the SESQUIAC when we try to compute 23 plus 1.

Tanton's Exploding Dots, a la Engel

Exploding Dots: Exploding soon all across a planet near you!

When there are ten dots in a box, replace them by one dot in the next box over. Thus for instance

$$\boxed{1} \quad \boxed{7} \quad \boxed{17}$$

becomes

$$\boxed{1} \quad \boxed{8} \quad \boxed{7}$$

From dots to dice

Add a sink and paraphrase:

When there are ten or more chips sharing a state, send nine chips to a sink and one chip to the next state.

Directed graph: Draw ten arrows from a nonabsorbing state: nine go to a sink, and one goes to another nonabsorbing state.

Probabilistic interpretation: Roll a fair ten-sided die. With probability $9/10$, the game ends; with probability $1/10$, the game continues.

Think globally

Check out the Global Math Project!

The Diaconis-Fulton Game

In the 1980s, Diaconis and Fulton came up with a game played with identical chips on a directed graph. When there are two or more chips at the same vertex, **one** of them follows a **random** arrow from that vertex. (Compare with Engel's game.)

If there are several vertices with two or more chips, you get to choose which vertex to deal with first. You might think that the order in which you deal with the overloaded vertices matters, but:

Theorem (Diaconis and Fulton): The probability of a given outcome is independent of the order of choices.

Confluence

Engel machines have the same confluence property, as does Exploding Dots. Consider for example

$$\boxed{0} \quad \boxed{17} \quad \boxed{17}$$

You can explode from left to right, or from right to left; you get to the same final state

$$\boxed{1} \quad \boxed{8} \quad \boxed{7}$$

either way.

Why does the Engel abacus work?

Consider two ways of routing chips on the board:

A. Run the Engel machine as described above, many many times, gradually putting millions of chips into the source.

B. Load millions of chips into the source at the start. Fire "nearly all" of them (that is, fire as many of them as possible). Then fire nearly all the chips that have moved once. Then fire nearly all the chips that have moved twice. Etc.

Scenario A distributes chips among the sinks in the same proportions as Engel's process.

Scenario B distributes chips among the sinks in the same proportions as the random walk process.

But confluence tells us that the two scenarios distribute chips among the sinks in the same proportions!

Making it musical

Here's a musical representation of the SESQUIAC counting to 23.

The repeated C note in the piano signals another chip being fed into the system (and the final low C signals that the piece is over).

The pitches corresponding to the five digit-positions (from right to left) ascend by pitch-ratios of 3:2 (a perfect fifth), from C to G to D to A to E. So *pitch* corresponds to *place value*.

The volume of a pitch corresponds to the digit in that position.

0

0 0 0 0 0

0 0 0 0 1

0 0 0 0 1

0 0 0 0 2

2

0 0 0 0 2

3

0 0 0 0 3

3

0 0 0 2 0

0 0 0 2 0

4

0 0 0 2 1

0 0 0 2 1

5

0 0 0 2 2

0 0 0 2 2

6

0 0 0 2 3

6

0 0 0 4 0

6

0 0 2 1 0

6

0 0 2 1 0

7

0 0 2 1 1

0 0 2 1 1

0 0 2 1 2

8

$$0\ 0\ 2\ 1\ 2$$

0 0 2 1 3

0 0 2 3 0

0 0 4 0 0

0 2 1 0 0

9

0 2 1 0 0

10

0 2 1 0 1

10

0 2 1 0 1

11

0 2 1 0 2

11

0 2 1 0 2

12

0 2 1 0 3

0 2 1 2 0

12

0 2 1 2 0

13

0 2 1 2 1

13

0 2 1 2 1

14

0 2 1 2 2

14

0 2 1 2 2

15

0 2 1 2 3

15

0 2 1 4 0

15

0 2 3 1 0

0 4 0 1 0

15

2 1 0 1 0

15

2 1 0 1 0

2 1 0 1 1

2 1 0 1 1

17

2 1 0 1 2

17

2 1 0 1 2

18

2 1 0 1 3

18

2 1 0 3 0

18

2 1 2 0 0

18

2 1 2 0 0

19

2 1 2 0 1

2 1 2 0 1

20

2 1 2 0 2

20

2 1 2 0 2

21

2 1 2 0 3

21

2 1 2 2 0

21

2 1 2 2 0

22

$$2\ 1\ 2\ 2\ 1$$

22

2 1 2 2 1

23

2 1 2 2 2

23

2 1 2 2 2

Thanks!

Slides for this talk are at http://jamespropp.org/moves17.pdf
Videos about Engel machines coming soon to
http://barefootmath.org.