THE FRACTIONAL CHROMATIC NUMBER OF MYCIELSKI’S GRAPHS

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ABSTRACT

The most familiar construction of graphs whose clique number is much smaller than their chromatic number is due to Mycielski, who constructed a sequence $G_n$ of triangle-free graphs with $\chi(G_n) = n$. In this note, we calculate the fractional chromatic number of $G_n$ and show that this sequence of numbers satisfies the unexpected recurrence $a_{n+1} = a_n + \frac{1}{a_n}$.

INTRODUCTION

All our graphs are finite and simple. We write $V(G)$ for the vertex set, $E(G)$ for the edge set, $\chi(G)$ for the chromatic number, and $\omega(G)$ for the clique number of $G$. We denote by $\chi_F(G)$ the fractional chromatic number (or set-chromatic number [2], or ultimate chromatic number [5], or multi-coloring number [6]) of $G$, which is the infimum of all fractions $a/b$ such that, to each vertex of $G$, one can assign a $b$-element subset of $\{1, 2, 3, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets.

The integer programs which compute $\chi$ and $\omega$ are dual to one another. The real relaxation of either of these integer programs is a linear program which computes $\chi_F$. Taking this viewpoint, one sees that the infimum in the definition of $\chi_F(G)$ is always achieved, that $\chi_F(G)$ is always rational, and that $\omega(G) \leq \chi_F(G) \leq \chi(G)$.

A fractional clique is a map $f : V(G) \to [0, 1]$ such that, if $S$ is any independent set of vertices in $V(G)$, $\sum_{v \in S} f(v) \leq 1$. The fractional clique number $\omega_F(G)$ is equal to $\sup \{\sum_{v \in V(G)} f(v)\}$, where the supremum is taken over all fractional cliques $f$. This is just a combinatorial description of the parameter calculated by the real relaxation of the integer program that calculates $\omega(G)$, so $\omega_F(G) = \chi_F(G)$ by the duality theorem of linear programming.

Although there is a duality between $\omega$ and $\chi$, there is a certain lack of symmetry as well. For example, $\chi_F(G)$ is always equal to $\lim_{n \to \infty} \sqrt[n]{\chi(G^n)}$, where the power of $G$ is relative to either the disjunctive or lexicographic product of graphs (see [5], [10], [13]);
on the other hand, it is not true that $\omega_f(G)$ always equals $\lim_{n \to \infty} \sqrt[n]{\omega(G^n)}$. In fact, this limit gives the Shannon capacity of the complement of $G$ [14], which is known to be $\sqrt{5}$ when $G$ is the pentagon $C_5$ [9] (while $\omega_f(C_5) = \frac{5}{2}$).

**THE GRAPH TRANSFORMATION OF MYCIELSKI**

Motivated by [11], given a graph $G$ we define a graph $\mu(G)$ as follows. If $G$ has vertex set $\{v_1, v_2, \ldots, v_m\}$, let $V(\mu(G)) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z\}$ with $x_ix_j \in E(\mu(G))$ if and only if $v_iv_j \in E(G)$, with $x_iy_j \in E(\mu(G))$ if and only if $v_iv_j \in E(G)$, with $y_iz \in E(\mu(G))$ for all $i$ from 1 to $m$, and with $\mu(G)$ having no other edges.

The proofs of parts (a) and (b) of the following theorem are implicit in [11] but are included here for the sake of completeness.

**Theorem.** Suppose that $G$ has at least one edge. Then

(a) $\omega(\mu(G)) = \omega(G)$;

(b) $\chi(\mu(G)) = \chi(G) + 1$;

and (c) $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$.

**Proof:** (a) Since $G$ is an induced subgraph of $\mu(G)$, $\omega(G) \leq \omega(\mu(G))$. To see the opposite inequality, note that the vertex $z$ is in no cliques of size bigger than 2. Also, were $\{x_{i(1)}, x_{i(2)}, \ldots, x_{i(r)}, y_{j(1)}, y_{j(2)}, \ldots, y_{j(s)}\}$ a clique in $\mu(G)$, then the sets $\{i(1), \ldots, i(r)\}$ and $\{j(1), \ldots, j(s)\}$ would be disjoint and $\{v_{i(1)}, v_{i(2)}, \ldots, v_{i(r)}, v_{j(1)}, v_{j(2)}, \ldots, v_{j(s)}\}$ would be a clique. Hence $\omega(G) \geq \omega(\mu(G))$ as well.

(b) If $k : V(G) \to \{1, 2, \ldots, n\}$ is a proper coloring of $G$, then we define a coloring $h : V(\mu(G)) \to \{1, 2, \ldots, n, n + 1\}$ of $\mu(G)$ by setting $h(x_i) = h(y_i) = k(v_i)$ for all $i$, and $h(z) = n + 1$. This is easily seen to be a proper coloring of $\mu(G)$ and uses only one more color than the coloring of $G$. Hence $\chi(\mu(G)) \leq \chi(G) + 1$. On the other hand, if $h$ is any proper coloring of $\mu(G)$, we define a coloring $k$ of $G$ by

$$k(v_i) = \begin{cases} h(x_i) & \text{if } h(x_i) \neq h(z); \\ h(y_i) & \text{if } h(x_i) = h(z). \end{cases}$$

This is a proper coloring of $G$ which does not use the color $h(z)$. So $G$ can be colored with fewer colors than are required to color $\mu(G)$. Hence $\chi(\mu(G)) \geq \chi(G) + 1$ as well.

(c) First, suppose $\chi_f(G) = \frac{a}{b}$ and we have a proper $a/b$-coloring of $G$. We produce an $(a^2 + b^2)/(ab)$-coloring of $\mu(G)$ as follows. Imagine that each of the $a$ colors has $a$ offspring, $b$ male and $a - b$ female. Color $x_i$ with all the offspring of the colors that are associated to $v_i$. Color $y_i$ with all the female offspring of the colors that are associated with $v_i$ and with wholly new colors $\{c_1, c_2, \ldots, c_{b^2}\}$. Color $z$ with all the male offspring of
all the original colors. Note that this set coloring is proper. There are $a^2$ offspring colors and $b^2$ new colors, making $a^2 + b^2$ all told. The resulting coloring of $\mu(G)$ assigns exactly $ab$ colors to each vertex. Hence $\chi_F(\mu(G)) \leq \chi_F(G) + (\chi_F(G))^{-1}$.

To prove the opposite inequality, suppose $f$ is a fractional clique on $G$ that achieves $\omega_F(G)$. We define a map $g : V(\mu(G)) \to [0,1]$ as follows.

$$g(x_i) = \left(1 - \frac{1}{\omega_F(G)}\right) f(v_i),$$

$$g(y_i) = \frac{1}{\omega_F(G)} f(v_i),$$

$$g(z) = \frac{1}{\omega_F(G)}.$$

We now show that $g$ is a fractional clique on $\mu(G)$.

If $M \subseteq V(G)$, let $x(M) = \{x_i : v_i \in M\}$ and let $y(M) = \{y_i : v_i \in M\}$. Let $S$ be an independent set in $\mu(G)$. If $z \in S$, then $S = \{z\} \cup x(M)$ for some independent set $M \subseteq V(G)$, in which case

$$\sum_{v \in S} g(v) = \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) \leq \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) = 1.$$

If $z \notin S$, then $S = x(M) \cup y(N)$ for some (independent) set $M \subseteq V(G)$ and some $N \subseteq V(G)$. Because $S$ is an independent set, $N$ may be partitioned into a subset $A$ of $M$ and a set $B$ of vertices which are neither elements of $M$ nor adjacent to elements of $M$. Then

$$\sum_{v \in S} g(v) = \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in N} f(v) = \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in A} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in B} f(v) \leq \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in B} f(v) = \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in B} f(v).$$

Let $H$ be the subgraph of $G$ induced on $B$. Say that $\omega_F(H) = \chi_F(H) = \frac{a}{b}$ and that we have an $a/b$-coloring of $H$. Then the $a$ color classes $C_1, C_2, \ldots, C_a$ are independent sets in $H$, and the sets of the form $M \cup C_i$ are independent sets in $G$. Because $f$ is a fractional clique,

$$\sum_{v \in M} f(v) + \sum_{v \in C_i} f(v) \leq 1$$
for all \( i \). Adding these \( a \) inequalities gives

\[
a \sum_{v \in M} f(v) + b \sum_{v \in B} f(v) \leq a.
\]

Dividing through by \( a \) yields

\[
\sum_{v \in M} f(v) + \frac{1}{\omega_F(H)} \sum_{v \in B} f(v) \leq 1.
\]

Since \( \omega_F(G) \geq \omega_F(H) \), (*) above is less than or equal to 1, and so \( g \) is a fractional clique.

It follows that

\[
\chi_F(\mu(G)) = \omega_F(\mu(G)) \\
\geq \sum_{v \in V(\mu(G))} g(v) \\
= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in V(G)} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
= \sum_{v \in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
= \omega_F(G) + \frac{1}{\omega_F(G)} \\
= \chi_F(G) + \frac{1}{\chi_F(G)}
\]

and the theorem is proved. \( \square \)

Let \( G_2 \) be \( K_2 \), the complete graph on two vertices, and recursively define \( G_{n+1} = \mu(G_n) \) for \( n \geq 2 \). This definition makes \( G_3 \) the 5-cycle and \( G_4 \) the Grötzsch graph. Our theorem shows that \( G_n \) is triangle-free yet has chromatic number \( n \). (This much was known to Mycielski in 1955. [11]) Our theorem also shows that the fractional chromatic number of \( G_n \) equals \( a_n \), where \( a_2 = 2 \) and \( a_{n+1} = a_n + a_n^{-1} \). It is known that this sequence grows like \( \sqrt{2n} \) in the sense that \( a_n / \sqrt{2n} \to 1 \) as \( n \to \infty \). (See [7], p. 49, [12], problem 60, or [1], problem E3276 for more detailed information about the growth of this sequence.)

This provides a simple example of graphs \( G_n \) with \( \chi(G_n) - \chi_F(G_n) \to \infty \) and \( \chi_F(G_n) - \omega(G_n) \to \infty \). In fact, even the ratios \( \chi(G_n) / \chi_F(G_n) \) and \( \chi_F(G_n) / \omega(G_n) \) approach infinity.

If \( G \) is a graph on \( v \) vertices and \( \chi_F(G) \) is expressed as a fraction in lowest terms, how large can the denominator be? Dave Fisher has pointed out, as a corollary
of our theorem, that $G_n$ breaks the record for the largest denominator, previously held by a sequence of graphs constructed by Chvátal, Garey, and Johnson. [3] The fractional chromatic number of their graphs has a denominator on the order of $e^{\sqrt{n \ln(n)/2}}$, but Fisher calculates that for $G_n$ the denominator is on the order of $c^n$ for a certain constant $c$. [4] Fisher goes on to construct a sequence of graphs with denominators growing like $c^n$ for a larger constant $c$, which gives the best known result of this type.

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References


