A counterexample to integration by parts

Alexander Kheifets
Department of Mathematics
University of Massachusetts Lowell
Alexander_Kheifets@uml.edu

James Propp
Department of Mathematics
University of Massachusetts Lowell
James_Propp@uml.edu

The integration-by-parts formula

$$\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g'(x)\ dx$$

carries with it an implicit quantification over functions f, g to which the formula applies. So, what conditions must f and g satisfy in order for us to be able to apply the formula?

A natural guess — which some teachers might even offer to a student who raised the question — would be that this formula applies whenever f and g are differentiable. Clearly this condition is necessary, since otherwise the integrands $f'(x)g(x)$ and $f(x)g'(x)$ are not defined. But is this condition sufficient? We will show that it is not. That is, we will give an example of two differentiable functions f, g on $[0, 1]$ for which the definite integrals $\int_0^1 f'(x)g(x)\ dx$ and $\int_0^1 f(x)g'(x)\ dx$ do not exist (the former is $-\infty$ and the latter is $+\infty$); it follows that the functions $f'(x)g(x)$ and $f(x)g'(x)$ do not have antiderivatives on the interval $[0, 1]$, so that the indefinite integrals $\int f'(x)g(x)\,dx$ and $\int f(x)g'(x)\,dx$ do not exist.

A cautious teacher might instead reply that the theorem holds whenever f and g are differentiable and $f'g$ and fg' are integrable. While this version of the theorem is true, it cannot be applied in cases where one does not know ahead of time that the integral one is trying to compute actually exists.

One wants an integration-by-parts theorem that includes the integrability of $f'(x)g(x)$ as part of its conclusion, not as part of its hypothesis.

Before we give our counterexample to the naive interpretation of the integration by parts formula, we point out that the formula holds if either f' or g' is continuous. For instance, if f' is continuous, then (since g is continuous) the product $f'g$ is continuous; but then the function $f'g$ must have an antiderivative h, and consequently the function fg' must have an antiderivative too, namely $fg - h$. So any counterexample to the naive interpretation of integration by parts must feature differentiable functions f, g whose derivatives are not continuous, such as the famous function $x^2 \sin 1/x$ (extended to a function on all of $\mathbf{R}$ by continuity) and its relatives. Moreover, it will not do to let f and g be the same function of this sort, since the function ff' always has an antiderivative, namely $\frac{1}{2}f^2$.

Our counterexample is the pair of functions

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^4}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^4}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

on the interval $[0, 1]$. Both functions are continuous on $[0, 1]$ and differentiable on $[0, 1]$. Indeed, if we consider f and g as defined above to be defined on all of $\mathbf{R}$, both functions are differentiable everywhere; for, away from 0 we can use the chain rule, while at 0 we have $|(f(h) - f(0))/(h - 0)| = |f(h)/h| \leq |h^2/h| = |h|$ so that $f'(0) = \lim_{h\to 0}(f(h) - f(0))/(h - 0) = 0$, and likewise $g'(0) = 0$. Obviously, the integral

$$\int_0^1 [f(x)g(x)]' dx$$

exists. However, we will show that both integrals

$$\int_0^1 f'(x)g(x)dx \quad \text{and} \quad \int_0^1 f(x)g'(x)dx$$

are divergent. It suffices to show that the first integral is divergent. For $x \neq 0$,

$$f'(x) = 2x \sin\left(\frac{1}{x^4}\right) - 4x^2 \cos\left(\frac{1}{x^4}\right)\frac{1}{x^5}.$$

The first term in this representation of $f'(x)$ is continuous, and $g(x)$ is continuous, so their product is continuous and therefore integrable. So, we focus on the second term times $g(x)$, namely

$$\begin{aligned} -4\int_0^1 x^2 \cos\left(\frac{1}{x^4}\right)\frac{1}{x^5}\, g(x)dx &= -4\int_0^1 x^4 \cos^2\left(\frac{1}{x^4}\right)\frac{1}{x^5}\, dx \\ &= \int_0^1 x^4 \cos^2\left(\frac{1}{x^4}\right) d\left(\frac{1}{x^4}\right). \end{aligned}$$

After the substitution

$$u = \frac{1}{x^4}$$

the integral turns into

$$-\int_1^\infty \frac{1}{u}\cos^2(u)du$$

(with the minus sign coming from the interchange of upper and lower limits of integration). To show that this integral diverges, let k be a positive integer. Then for every u in the interval $[2\pi k - \frac{\pi}{4}, 2\pi k]$ we have

$$\cos^2(u) \geq \frac{1}{2} \quad \text{and} \quad \frac{1}{u} \geq \frac{1}{(2\pi k)}.$$

Therefore,

$$\int_1^\infty \frac{1}{u}\cos^2(u)du \geq \sum_{k=1}^\infty \int_{2\pi k - \frac{\pi}{4}}^{2\pi k} \frac{1}{u}\cos^2(u)du \geq \sum_{k=1}^\infty \frac{1}{(2\pi k)}\frac{1}{2}\frac{\pi}{4} = \frac{1}{16}\sum_{k=1}^\infty \frac{1}{k} = \infty$$

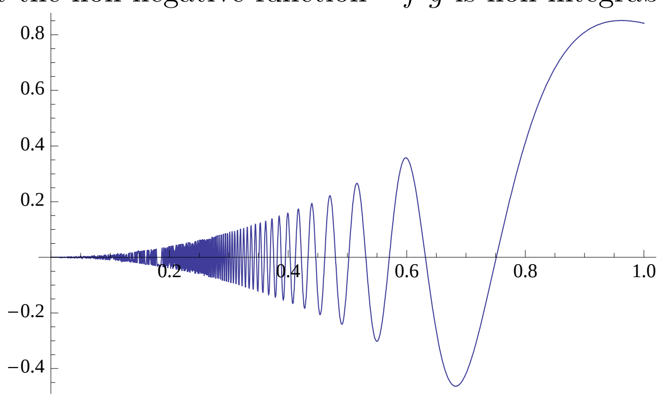
This completes the proof.

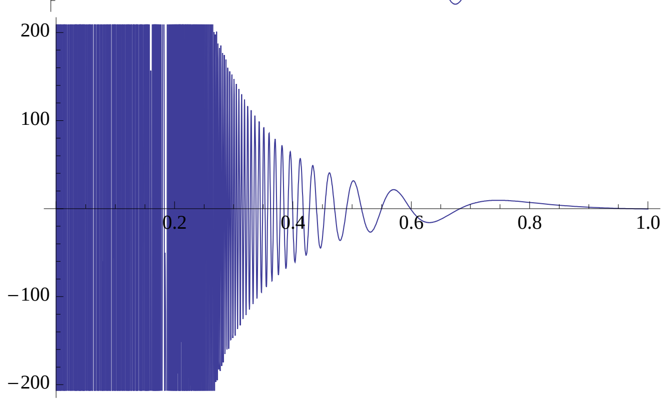
Our analysis shows that the (improper) definite integrals $\int_0^1 f'(x)g(x)\, dx$ and $\int_0^1 f(x)g'(x)\, dx$ do not exist. This in turns shows that the functions

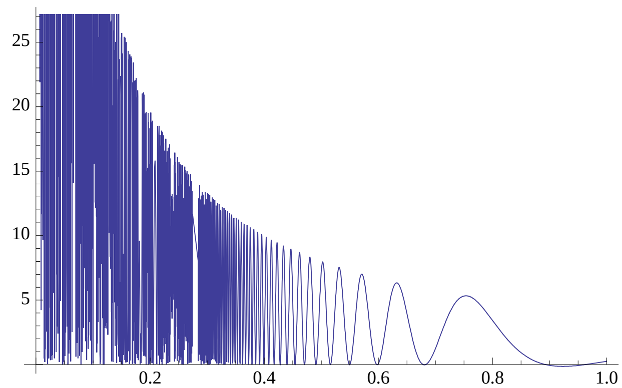
$f'(x)g(x)$ and $f(x)g'(x)$ do not have antiderivatives on $[0,1]$. For, if these functions had antiderivatives, the fundamental theorem of calculus would yield finite values for the definite integrals.

We have shown that the functions $f'g$ and fg' are not integrable over $[0,1]$. It is worth noting that $|f'|$ and $|g'|$ are not integrable over $[0,1]$ either, as can be shown by a similar method. On the other hand, the function f' is integrable over $[0,1]$ in the sense that the improper Riemann integral $\int_0^1 f'(x)\,dx$ exists: for all $\epsilon > 0$ the Fundamental Theorem of calculus implies $\int_\epsilon^1 f'(x)\,dx = f(1) - f(\epsilon)$, which converges to $f(1) - f(0)$ as $\epsilon \to 0$, implying that $\int_0^1 f'(x)\,dx$ exists and equals $f(1) - f(0)$. Likewise g' is integrable over $[0,1]$.

The following three pictures (created with the help of Mathematica) illustrate what is going on: they depict the (truncated) graphs of f, f', and $-f'g$ (we show $-f'g$ rather than $f'g$ so that the function will be non-negative rather than non-positive). The continuous function f is integrable, and the discontinuous function f' is integrable because its oscillations balance out, but the non-negative function $-f'g$ is non-integrable.







Some might be inclined to say that our example is actually a vindication of an extended integration by parts theorem that asserts, as important special cases, that if $\int_a^b f'(x)g(x)$ is ∞ then $\int_a^b f(x)g'(x)$ is $-\infty$ and vice versa (and likewise with the signs reversed), and that if either of these integrals "diverges by oscillation" (as in the case for the functions f, g on $[-1, 1]$ given by $x^2 \sin(1/x^4)$, $x^2 \cos(1/x^4)$ on $[0, 1]$ and $-x^2 \sin(1/x^4)$, $-x^2 \cos(1/x^4)$ on $[-1, 0]$, respectively) then so does the other. However, to the extent that one might be inclined to treat the integration by parts formula as implicitly asserting that the integrals are well-defined, our example provides a corrective.

Is this corrective needed? We have not found any calculus texts that present a mistaken statement of the integration by parts theorem, but we have found some widely-used web sites that do so (e.g.: "Let u and v be differentiable functions, then $\int uv'dx = uv - \int u'vdx$"). More common are books and web-sites that present the integration by parts formula and give examples without specifying the conditions under which the formula applies. A provocative treatment of other pedagogical aspects of the integration by parts theorem is [**2**].

For a more advanced course (an honors calculus class or an introductory real analysis class), the example can be used to motivate the notion of bounded variation, since the lack of bounded variation of the derivatives of the functions near the origin is the source of the problem. We also mention that, in lieu of adopting the hypothesis that f and g are continuously differentiable, one might require that f be Riemann-Stieltjes integrable with respect to dg. Then it can be shown that the integration by parts formula (where the integrals now are Riemann-Stieltjes integrals) is valid, and it is part of the conclusion that g will be Riemann-Stieltjes integrable with respect to df (see [**1**]).

Finally, we mention that if the functions f' and g' are assumed to be integrable in the sense that $\int_0^1 f'(x)\,dx$ and $\int_0^1 g'(x)\,dx$ exist as strict Riemann integrals (and not just as improper Riemann integrals), then the conclusion of the integration by parts theorem applies. Indeed, we only need to know that at least one of f' and g' is Riemann integrable. For, Lebesgue's Theorem states that a (measurable) function is Riemann integrable if and only if it is bounded and its set of discontinuity has Lebesgue measure zero. If g is continuous and f' is Riemann integrable (i.e. it is bounded and its set of discontinuity has Lebesgue measure zero), then so is $f'g$, and the integration by parts theorem applies. Hence it is an essential feature of our counterexample that the functions f' nor g' are not just discontinuous but also non-integrable in the Riemann sense.

This work was stimulated by conversations with the honors freshman calculus class at UMass Lowell, and also benefited from conversations with Lee Jones of UMass Lowell (who found a different counterexample), Zbigniew Nitecki of Tufts University, and two anonymous referees of an earlier version of this article.

REFERENCES

[1] Thomas Apostol, *Mathematical Analysis.*

[2] Jonathan Lewin, "Integration by Parts: Another Example of Voodoo Mathematics," http://science.kennesaw.edu/∼jlewin/fb/integration-by-parts.pdf.