1 Results from Prior NSF Support

During the past five years, the PI has received funding from individual grants from the National Science Foundation and the National Security Agency, as follows: DMS 9500936 (Random Tilings, 1995-1998); DMS 9803249 (Research on Tilings, 1999-2003). As part of these grants, the PI has also received REU Supplements. The PI received additional support from the VIGRE grants awarded by NSF to the University of Wisconsin and Harvard University. He also received funding from the National Security Agency, the most recent of which was MDA904-00-1-0060.

During the funded period (from 1995 to the present), the PI wrote or co-wrote seventeen published articles on the funded topics.

Much of the PI's work (funded through the Statistics and Probability subdivision within the Division of Mathematical Sciences) was probabilistic in nature, originating from the PI's desire to understand the behavior of random tilings. The articles "Exact sampling with coupled Markov chains and applications to statistical mechanics" (with David Wilson, 1996), "Generating random elements of a finite distributive lattice" (1997), "How to get a perfectly random sample from a generic Markov chain and generate a random spanning tree of a directed graph" (with David Wilson, 1998), and "Coupling from the past: a user's guide" (with David Wilson, 1998) developed and promulgated a general approach to the random generation of combinatorial objects such as tilings, which initiated a (still ongoing) flurry of activity by many researchers in the field of Markov Chain Monte Carlo (the new fashion of "exact sampling" or "perfect sampling", spearheaded by David Wilson; see `http://dbwilson.com/exact/` for a comprehensive bibliography). The PI also contributed to combinatorial probability theory by co-organizing (with David Aldous) a workshop on discrete probability and co-editing a volume of workshop proceedings ("Microsurveys in Discrete Probability", 1998).

Having helped develop new methods of random sampling, the PI applied these methods to the empirical study of random tilings, obtaining many conjectures about the ways in which the shape of the boundary of a region can influence the behavior of a random tiling of that region using tiles of a specified kind. Working with other researchers, the PI was able to prove many of these conjectures, in such articles as "Local statistics for random domino tilings of the Aztec diamond" (with Henry Cohn and Noam Elkies, 1996), "The shape of a typical boxed plane partition" (with Henry Cohn and Michael Larsen, 1998), and "A variational principle for domino tilings" (with

Henry Cohn and Richard Kenyon, 2001). The PI also presented this work at a statistical mechanics conference and published an article in a physics journal ("Boundary-dependent local behavior for 2-D dimer models", 1997).

During this period, the PI did other work of a purely combinatorial nature. His article "A pedestrian approach to a method of Conway, or, a tale of two cities" (1997) was on the one hand an attempt to bring some very basic and important ideas of Conway (1990) and Thurston (1990). a broader readership (which he presented successfully to an audience of high-school teachers as part of a Regional Geometry Institute), and on the other hand an attempt to demonstrate the applicability of the method to a new setting (tilings of regions by skew tetrominos). The PI's student Hal Canary (2001) succeeded in proving one of the PI's conjectures, rigorously establishing a bijection between certain sorts of tilings and Baxter permutations. Other articles of a combinatorial nature were "Domino tiling with barriers" (with Richard Stanley, 1999) and "Trees and matchings" (with Richard Kenyon and David Wilson, 2000).

During this period, the PI also contributed to research on alternating-sign matrices. He and his students at MIT and the University of Wisconsin studied alternating-sign matrices via the fully-packed looped (FPL) model on a square grid, as described by Wieland (2000). The PI also contributed to the study of alternating-sign matrices through expository writing. The article "How the alternating-sign matrix conjecture was solved" (with David Bressoud, 1999) brought the story of alternating-sign matrices to a very wide audience of mathematicians, and the article "The many faces of alternating-sign matrices" (2001) brought some of the PI's unproved conjectures to the attention of physicists, who may have better tools than mathematicians for the solution of these problems. In particular, Razumov and Stroganov, upon reading an early version of the latter article, were led to formulate other similar conjectures involving the quantum XXZ model (Razumov and Stroganov, 2000) and the O(1) loop model (Razumov and Stroganov, 2001), and the full story behind these conjectures is likely to be a very deep and interesting one. Another expository article by the PI, "Enumeration of matchings: problems and progress" (1999) not only gives an historical overview of the subject not found elsewhere, but also serves as a source of open problems.

Two other articles published by the PI during this period, "A reciprocity theorem for domino tilings" (2001) and "Generalized domino-shuffling" (2002), play a role in the PI's current proposal, and so will be discussed there, along with unpublished recent work of some of the PI's students.

Between 1995 and 2002, the PI gave invited lectures at twelve conferences in the U.S., Russia, Germany, France, the Netherlands, and Australia. His presentation "Tilings, randomness, and undergraduate research" (a plenary talk at the 1998 annual winter meeting of the Mathematics Association of America) illustrated for a broad audience the rewards of incorporating undergraduates into one's research program. The themes of the conferences to which the PI was invited vary broadly, from probability, combinatorics, and ergodic theory (his main areas of expertise) to cellular automata and quasicrystals. In these settings he has helped carry ideas from one discipline to another.

During the period 1995-2002, the PI worked with dozens of undergraduate research assistants at M.I.T. (the "Tilings Research Group"), the University of Wisconsin (the "Spatial Systems Laboratory"), and Harvard University ("Research Experiences in Algebraic Combinatorics at Harvard"). Using NSF funds, the PI brought four of these undergrads to professional meetings, to present their results and interact with researchers. Throughout this period the PI made a consistent effort to recruit undergraduate women into his research program, as he believes strongly in the importance of bringing gender balance to the profession of mathematics.

The PI also worked with four Ph.D. students who served as Graduate Interns, assisting in the running of the undergraduate research program and receiving explicit guidance in the art of supervising undergraduates; he also helped recruit colleagues to run the Spatial Systems Laboratory during his absence from the University of Wisconsin, and offered guidance to those who accepted. This work was done by the PI in the hopes that his model for the inclusion of undergraduates in mathematical research (during the academic year, operating as a group) will be imitated by others.

During the 2001-2002 academic year, the PI extended his model for supervising undergraduate research by integrating his research group with an academic course. This way of coordinating the activities of student learning and student research, with extensive use of web-support, proved extremely successful at Harvard; the PI plans to repeat the approach during the 2003-2004 academic year at the University of Wisconsin. (See `http://www.math.harvard.edu/~propp/192/` and `http://www.math.harvard.edu/~propp/reach/` to see how things worked with the Harvard group.) In the meantime, the PI expects to continue his work with undergraduates during the 2002-2003 academic year with college students from the Boston area, with Brandeis Ph.D. student Anna Varvak serving as Graduate Intern.

2 New Work

The new research that the PI proposes to conduct commencing in 2003 is an outgrowth of his past work in combinatorics. The bulk of it can be divided into two main strands: combinatorial reciprocity, and integrality/Laurentness/integrability. All of these efforts lie at the interface between algebra and combinatorics, and involve attempts to find new ways to pass back and forth between the two disciplines. At the same time, the work is almost certain to have implications for both exactly solvable statistical mechanics and discrete integrable systems.

2.1 Combinatorial Reciprocity

The PI has already been exploring combinatorial reciprocity phenomena, though so far only the article "A reciprocity formula for domino tilings" (2001) has been published. The term "combinatorial reciprocity theorem" was coined in the context of Ehrhart theory, where it refers to the fact that for certain polynomials $p(x)$, whose values when x is a positive integer have a combinatorial meaning, the values of $p(x)$ when x is a negative integer have a (related) combinatorial meaning as well (see Stanley, 1986). Quite often, in fact, the second combinatorial meaning is identical to the first, up to a shift in indexing and a possible change of sign, yielding a functional equation of the form $p(-n) = (-1)^{f(n)} p(c-n)$ for some fixed integer c and some fixed periodic function $f()$.

The starting point for the PI's recent work on this topic was the observation that similar functional equations often apply when the function $p(x)$ (combinatorially meaningful when x is a positive integer) is not a polynomial but an exponential-polynomial function of x (that is, a function in the linear span of all functions of the form $n \mapsto n^k \alpha^n$ where k is a non-negative integer and α is a complex number). This is true, for instance, if $p(n)$ denotes the number of domino tilings of an m-by-n rectangle, for some fixed m. This particular result concerning domino tilings is not original with the PI; it is a reinterpretation of a result due to Stanley. However, the PI's proof yields a much more general result, pointing the way to a broad understanding of reciprocity phenomena for perfect matchings of graphs.

Since the PI wrote this article, there has been progress on two fronts. First, his undergraduate student David Speyer made some progress towards an even more general picture, applicable even when the graph in question is not the product of a small graph G with a path of length n. Second, physicists Lu and Wu (2001) have used methods like Stanley's to show that the phenomena observed by Stanley for certain special planar graphs also apply to graphs on tori and various simple non-orientable surfaces. One of the goals of the PI is to extend his own methods and Speyer's, obtaining very general reciprocity results for perfect matchings. It would also be desirable to obtain some sort of synthesis between these results and the classic reciprocity theory of Ehrhart and Macdonald, though it is still unclear whether such a synthesis can be made.

One interesting feature of this work concerns the situation in which the function $p(n)$ does not satisfy a reciprocity formula, but nonetheless has the property that $p(n)$ is an integer for all integer values of n (negative as well as positive). In such cases, it is natural to ask whether the values $p(-1), p(-2), \ldots$ have some sort of combinatorial meaning. The PI has developed a methodology for unearthing the underlying combinatorics in such situations (and indeed this methodology was what led him to the signed matchings model used in his 2001 article "A reciprocity theorem for domino tilings"). Making use of the combinatorial interpretation of $p(1), p(2), p(3), \ldots$ (where $p(n)$ counts the number of combinatorial objects of size n), one replaces the sequence of numbers $p(1), p(2), p(3), \ldots$ by a sequence of multivariate polynomials $P_1(x_1, x_2, \ldots), P_2(x_1, x_2, \ldots), P_3(x_1, x_2, \ldots), \ldots$ with the property that each coefficient in $P_n(x_1, x_2, \ldots)$ is 1, and where each monomial in $P_n(x_1, x_2, \ldots)$ corresponds in a natural way to one of the $p(n)$ combinatorial objects of order n. One then finds a recurrence relation governing the multivariate polynomials P_n, so that it becomes possible to run the recurrence in reverse, obtaining multivariate polynomials P_0, P_{-1}, P_{-2}, et cetera. In many cases of interest, these new polynomials have the property that for all $n < 0$, the coefficients of P_n are all equal to 1 (or all equal to -1). Then one seeks a combinatorial characterization of those monomials that contribute to P_n and those that do not; these combinatorial constraints, divorced from their occurrence in P_n and considered in their own right, then become the combinatorial model associated with the numbers $p(0), p(-1), p(-2)$, et cetera.

It should be stressed that the PI does not have at this stage a theoretical explanation for why this method should work, or even a formalized description of the method. But it has been an effective tool for reverse-engineering natural combinatorial interpretations for functions from $\mathbf{Z}$ to $\mathbf{Z}$ that are obtained by algebraic extrapolation of functions from $\mathbf{N}$ to $\mathbf{Z}$ of combinatorial origin. The PI expects to apply this approach to a variety of combinatorial models, starting with the dimer model (aka perfect matchings).

One keys step of the process is pattern-finding: given a sequence of multivariate polynomials, one tries to find a combinatorial recipe for deciding which monomials contribute and what the associated coefficients are. It is at this locus that the PI expects undergraduate research assistants to make the greatest contribution, since patience and general pattern-finding skills (aided by a healthy amount of computer experimentation) will play a bigger role than prior knowledge. Furthermore, the research topics are well suited to segmentation, with each student being given responsibility for a particular sub-topic (e.g., a particular generating function), and with abundant possibilities for fruitful interactions between the students responsible for different sub-topics. Likewise, there is a need for shared software infrastructure, and students could collaborate to create and maintain code.

By studying problems involving combinatorial reciprocity, students will become comfortable with the use of generating functions and the use of computer algebra systems, and more broadly with the whole arsenal of "one-dimensional combinatorics" (the theory of linear recurrence relations and transfer matrices). This background will be essential to those students who tackle some of the deeper problems described in the next sub-section, which involve some inherently two-dimensional combinatorics.

2.2 Integrality, Laurentness, and Integrability

When a sequence of numbers satisfies a linear recurrence equations with integer coefficients, and the initial conditions are integer values, then it is trivial that all the terms of the sequence are integers. Such is not the case for non-linear recurrence equations like $s_n = (s_{n-1}s_{n-3} + s_{n-2}s_{n-2})/s_{n-4}$ (the so-called "Somos-4 recurrence"). That is, it is by no means obvious that all the terms of the sequence of numbers $s_1, s_2, \ldots$ defined by the initial conditions $s_1 = s_2 = \ldots = s_4 = 1$ and the aforementioned recurrence are integers. Yet such is the case, and the third strand of the PI's research proposal involves the quest for a combinatorial understanding of such phenomena.

In the preceding instance, that combinatorial understanding comes from a sequence of graphs $G_1, G_2, \ldots$ with the property that s_n is the number of perfect matchings of G_n. These graphs were constructed by the PI in collaboration with Mireille Bousquet-Mélou and Julian West, in still-unpublished work. (For a very sketchy discussion, but one that at least shows what the graphs G_n look like for some small values of n, see `http://www.math.harvard.edu/~propp/reach/shirt.html`.)

Some of the original motivation for this work arose from the author's exploration of the original context in which alternating-sign matrices arose, in the algebraic work of Robbins and Rumsey (1986) on evaluation of determinants. Robbins and Rumsey explored an algorithm invented by Charles Dodgson, termed condensation of determinants, and found that variants of this iterative scheme have the surprising property of "Laurentness". In brief: a Laurent monomial (in some set of variables) is a product of integer powers of the variables (where each exponent may be positive, negative, or zero), a Laurent polynomial is a linear combination of Laurent monomials, and a recurrence exhibits the Laurent property if repeated application of the recurrence (i.e., composition of the associated birational maps) gives rise to Laurent polynomials ad infinitum. For recurrences in general, the Laurent phenomenon is the exception, not the rule. When do these cancellations occur? This is a purely algebraic question, but it is the PI's belief that whenever the answer is "yes", there is interesting combinatorics lurking nearby.

Another source of inspiration for the PI was the work of Eric Kuo, who while still an undergraduate member of the Tilings Research Group at M.I.T. had found a direct combinatorial proof of the assertion that $t(n)$, the number of domino tilings of the Aztec diamond of order n, satisfies the non-linear recurrence relation $t(n+1)t(n-1) = 2t(n)^2$; see Kuo (2002). The PI saw that this approach could have been used by Robbins and Rumsey to prove their Laurentness result, if they had realized that what they called compatible pairs of ASMs could also be construed as tilings (or equivalently perfect matchings). Kuo also applied his method to rhombus tilings of hexagons ("plane partitions in a box"), and was able to prove a classic result of MacMahon by showing that the number of tilings, viewed as a function of three integer arguments, satisfies a quadratic recurrence relation. (A similar proof, using a different quadratic recurrence relation, was found by Zeilberger (1996).) Kuo's work suggested to the PI that other applications of the method, yielding other quadratic recurrence relations, should be possible (and discoverable by talented undergraduates).

A third source of inspiration was the PI's successful attempt to understand and generalize the domino-shuffling algorithm that he and Greg Kuperberg had developed almost ten years earlier. The PI's article "Generalized domino-shuffling" (published in 2002 but mostly written a half-dozen years earlier) gave a general framework for enumerating perfect matchings of many different sorts of bipartite planar graphs. This complemented Kuo's top-down approach to enumeration of perfect matchings with a bottom-up approach.

A final source of inspiration was the challenge of finding a combinatorial interpretation of sequences like the one mentioned at the beginning of this sub-section. Such sequences were introduced by Michael Somos in the 1980s. For any integer $k \geq 4$, the Somos-k sequence is the unique sequence $s_1, s_2, s_3, \ldots$ satisfying the initial conditions $s_1 = s_2 = \ldots = s_k = 1$ and the recurrence $s_n = (s_{n-1}s_{n-k+1} + s_{n-2}s_{n-k+2} + \ldots + s_{n-\lfloor k/2 \rfloor}s_{n-k+\lfloor k/2 \rfloor})/s_{n-k}$ for all $n > k$. When k is 4, 5, 6, or 7, the Somos-k sequence consists entirely of integers, and it seems plausible that all four sequences should admit combinatorial interpretations, but as of 2001 no one had stumbled upon (or constructed) such interpretations. (For a bibliography on Somos sequences, see `http://mathworld.wolfram.com/SomosSequence.html` or the PI's Somos sequence web-page `http://www.math.wisc.edu/~propp/somos.html`.)

During the Spring 2002 semester, the PI's undergraduate students at Harvard (participants in his "Research Experiences in Algebraic Combinatorics at Harvard", or REACH, program) explored Laurentness phenomena. One student, Gregg Musiker, came up with a good conjecture (Musiker, 2002b) concerning the combinatorial significance of the Markov numbers (integer triples x, y, z satisfying $x^2 + y^2 + z^2 = 3xyz$; see Tom Ace's web-site, listed in the bibliography). Another student, David Speyer, was extremely productive, and (with assistance from other members of the group) came up with a flexible scheme that gave combinatorial interpretations not just for the Somos-4 and Somos-5 sequences but for a host of related sequences (Speyer, 2002).

The basic idea that made this work possible was the idea of taking a one-dimensional recurrence whose terms are numbers and replacing it by (or rather "lifting it to") a three-dimensional recurrence whose terms are Laurent polynomials. The price one pays for this is a profusion of variables, but one gains in return the sort of monicity property mentioned earlier: every coefficient in these Laurent polynomials is equal to 1, and the Laurent monomials themselves can be viewed as algebraic encodings of two-dimensional combinatorial objects. By studying these Laurent monomials, the PI's students were able to deduce the underlying "grammar" and infer what sort of combinatorial model was involved. As it happens, the model required for the Somos-4 recurrence involved nothing more complicated than perfect matchings of suitable graphs.

Although the search for combinatorial interpretations might seem like a purely internal affair for combinatorics, it has broader implications. Specifically, if one can show that a certain quantity counts the elements of some set, one can deduce that the quantity is non-negative. Indeed, the work of the REACH students made possible the first rigorous proof that certain polynomials introduced by Somos have non-negative coefficients. One possible benefit of this research is that it may lead to proofs of positivity of quantities of independent algebraic interest. In this way, the PI expects his work to have fruitful interactions with the continuing work of Fomin and Zelevinsky.

Another burst of progress made by the PI's students concerned the very natural recurrence $c_{i,j,k} = (c_{i-1,j,k}c_{i,j-1,k-1}+c_{i,j-1,k}c_{i-1,j,k-1}+c_{i,j,k-1}c_{i-1,j-1,k})/c_{i-1,j-1,k-1}$ (the "cube-recurrence"). The PI had conjectured that iteration of this recurrence gives rise to Laurent polynomials in which all coefficients are 1. Carroll and Speyer proved this conjecture by introducing a new class of combinatorial objects, which they called "groves". Groves "carry" the Laurentness property for the cube-recurrence just as domino tilings of Aztec diamonds (or, equivalently, compatible pairs of alternating-sign matrices) carry the Laurentness property for the Robbins-Rumsey recurrence. A preliminary write-up is available (Carroll and Speyer, 2002).

These results are listed under future work, rather than past work, inasmuch as the PI and/or his students must write up these results for publication (as of September 2002).

One topic to which these methods can and ought to be applied during the coming years is the burgeoning theory of discrete integrable systems. In this setting, the Laurentness phenomenon is subsumed by the more general phenomenon of "degree-reduction", wherein a composition of rational maps has lower degree than the degrees of the maps being composed would predict; see e.g. Bellon and Viallet (1999), Lafortune et al. (2001), and Ramani et al. (2001). As a test-case, the PI considered the recurrence relation recurrence relation $s_n + t(1 - s_n^2)(s_{n+1} + s_{n-1}) = 0$, (a close relative of the discrete Painlevé II recurrence) with initial conditions $s_{-1} = u$, $s_0 = v$. s_n is clearly expressible as a rational function of t, u, and v, but a priori one would guess that the degree of this rational function would be exponential in n; instead, it is merely quadratic in n. The PI was able to show that this rational function can be written as a ratio of two Laurent polynomials, each of which can be understood in an explicit combinatorial fashion. Specifically, each term in the numerator and denominator can be written as a sum of $2^{n(n+1)/2}$ terms, with each term corresponding in a natural way to one of the $2^{n(n+1)/2}$

domino tilings of the Aztec diamond of order n. This has convinced the PI that other discrete Painlevé recurrence relations, and more broadly, other recurrence relations exhibiting polynomial degree-growth, are in some sense associated with, and governed by, combinatorial models. Moreover, these models will live on integer lattices, and thus will probably turn out to fit in nicely with current work on exactly solvable lattice models in statistical mechanics, much as the work of the PI on domino tilings of Aztec diamonds led to new insights into dimer and ice models on the square grid.

The PI believes that the study of algebraic recurrences of this sort is a natural step beyond the study of linear recurrences that has played such a huge and necessary role in combinatorics to date. Indeed, the PI hopes that certain integer sequences for which no explicit formula is known (e.g., from his own researches, the number of bilaterally symmetric domino tilings of the Aztec diamond of order n) will turn out to satisfy simple algebraic recurrence relations, which is about the best thing one could have, short of a closed-form expression in terms of standard mathematical operations.

Most of the recurrence relations that have turned up so far in the PI's work on higher-degree recurrence relations are special cases of the discrete Hirota equation. (A very preliminary exposé of this link can be found in the slides for my 2002 lecture in Australia, listed in the bibliography.) Thus, the work that is proposed may be conceived as an exploration of the relationship between combinatorics and algebra, but is also likely to be interest to the integrable systems community (and, as remarked above, to the statistical mechanics community). Indeed, much of the recent work on degree-confinement and algebraic entropy has been "numerological" and non-rigorous; combinatorial models could provide the infrastructure needed to provide rigorous demonstrations of patterns that those researchers have observed but not proved.

3 Outreach to the Sciences

The PI and his collaborators, over the past decade and a half, have made a number of discoveries about dimer models on graphs, with an intellectual lineage going back to both mid-20th century statistical mechanics (e.g. Kasteleyn, Fisher, Temperley, Lieb) and early 20th century combinatorics (e.g. MacMahon). However, there has also been an abundant literature on dimer models written by mathematical chemists interested in the quantum chemistry of benzenoid molecules; see Cyvin and Gutman, 1988. Physicists and mathematicians studying dimer models have mostly been ignorant of the work of chemists, and the converse is even more true: Tony Gutman, one of the main experts on Kekulé structures in benzenoid molecules, was until quite recently quite unaware of the work of mathematicians. He has invited the PI to rectify this situation by writing an article on dimer models, to be published in a chemical journal. The PI hopes to write such an article, as it will further the traffic between the two fields and reduce the amount of duplicated effort (which is already substantial — many basic results have been discovered independently in the two research communities).

Meanwhile, the PI intends to continue to interact with mathematical physicists like Razumov and Stroganov. One way to encourage the interaction between workers in different disciplines whose interests overlap in the study of dimer models is to bring them into conversation by an email forum. The PI has done this over the past decade with his "domino forum"; this email discussion group, with a traffic-level of about two messages per week on average, has served as a sort of on-line interdisciplinary "coffee room" and facilitated the creation of a lot of new mathematics. The PI intends to continue to moderate the forum. He also has high hopes for an off-shoot of the domino forum called the "bilinear forum" that has been in existence for the past three years; although so far there have been few postings to this new forum (and most of those have come from the PI), the PI hopes that as the study of bilinear (i.e. quadratic) and higher-degree recurrence equations advances, the on-line conversation will become livelier. (The complete archive of the bilinear forum is available on-line at `http://www.math.wisc.edu/~propp/bilinear/archive`.)

4 Activities Pursuant to the Funded Research

The PI plans to continue using undergrads as research assistants and Ph.D. students as Graduate Interns. Ideally he would like to bring other graduate students who are not serving as interns into the mix, and perhaps postdocs as well. He intends to continue to seek out undergraduate women, and to create an intellectual climate that is conducive to students with varying levels of experience and confidence. (The PI plans to bring in observers to watch group meetings and to solicit their comments, so that he can do a better job of intellectual climate-control.) He hopes to bring some of his students to professional conferences, as a way of giving them a better sense of what a career as a researcher is like. He also plans to increase his efforts in guiding the students to write up their results for publication.

The PI also intends to continue to explore ways to use the Web to enhance the integration of research and teaching. His current model for doing this is to initiate a new group by offering a for-credit class that is focussed on the relevant research tools and videotaping all the lectures. These videotapes, along with other class materials (such as homework problems, solutions, and lesson plans) can be put on the web. In subsequent semesters, these materials can be used to expedite the process whereby new members of the research team are quickly brought up to speed. This worked extremely well at Harvard, and with some minor tinkering the PI expects it to work equally well at the University of Wisconsin.

The Web is also a useful way to illustrate certain sorts of ideas through animation. The PI's web-site illustrating coupling-from-the-past (`http://www.math.wisc.edu/~propp/tiling/www/applets/index.html`) has been a very useful way of quickly explaining how this algorithm works. Given the key role that "shuffling" algorithms play in the PI's work, he would like to create web-animations that demonstrate these algorithms. He expects to be able to hire undergraduates interested in computer science to create the necessary code.

Another possible way to use the Web to transmit knowledge is to make audio lectures available, coordinated with images. Such a web-based "slideshow" would have many of the same virtues as a lecture, and might attract viewers in ways that a traditional article (available as a PostScript or PDF file) might not. The PI wants to explore this idea by making such slideshows out of the lectures he gives in the next few years and making them available over the web.

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