q versus λ: plane partitions, alternating sign matrices, and lattice models

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q and Plane partitions

Partitions

$$\lim_{k,\, n-k \to \infty} \binom{n}{k} \;=\; \infty$$

$$\downarrow q$$

$$\begin{aligned}\lim_{k,\, n-k \to \infty} \binom{n}{k}_q &= 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots \\ &= \prod_{n=1}^{\infty} \frac{1}{1-q^n}\end{aligned}$$

where

$$\binom{n}{k}_q = \frac{[n][n-1][n-2]\cdots[n-k+1]}{[k][k-1]\cdots[1]}$$

with

$$[m] = 1 + q + q^2 + \ldots + q^{m-1},$$

so that $\binom{n}{k}_q \longrightarrow \binom{n}{k}$ as $q \longrightarrow 1$.

Gaussian binomial coefficients

$\binom{a+b}{a}$ counts partitions whose Young diagram fits in an a-by-b rectangle, and $\binom{a+b}{a}_q$ counts them by area.

E.g., $\binom{2+2}{2}_q = 1 + q + 2q^2 + q^3 + q^4$:

```
o              o              o
|              |              |
o              o---o          o---o---o
|              |XXX|          |XXX|XXX|
o---o---o      o---o---o      o---o---o

   q^0     +      q^1     +      q^2     +

o---o          o---o          o---o---o
|XXX|          |XXX|          |XXX|XXX|
o---o          o---o---o      o---o---o
|XXX|          |XXX|XXX|      |XXX|XXX|
o---o---o      o---o---o      o---o---o

   q^2     +      q^3     +      q^4
```

Alternative product formula

$$\binom{a+b}{a}_q = \prod_{i=0}^{a-1} \prod_{j=0}^{b-1} \frac{[i+j+2]}{[i+j+1]}.$$

Note: Putting $b = 1$ gives

$$\prod_{i=0}^{a-1} \frac{[i+2]}{[i+1]} = [a] = 1 + q + \ldots + q^{a-1}$$

which counts "1-dimensional Young diagrams". E.g., for $a = 3$:

```
o   o   o         1


o---o   o       + q


o---o---o       + q^2
```

Plane partitions and 3-dimensional Young diagrams

MacMahon (1912): For all $a, b, c \geq 1$,

$$M(a,b,c;q) = \prod_{i=0}^{a-1} \prod_{j=0}^{b-1} \prod_{k=0}^{c-1} \frac{[i+j+k+2]}{[i+j+k+1]}$$

is a polynomial in q with positive integer coefficients, with

$$\lim_{a,\, b,\, c \to \infty} M(a,b,c;q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}.$$

(Note: $M(a,b,1;q) = \binom{a+b}{a}_q$.)

The product $\prod_{n=1}^{\infty} 1/(1-q^n)^n$ counts plane partitions by sum of parts, or equivalently, 3-dimensional Young diagrams by volume.

$M(a,b,c;q)$ counts only the 3-dimensional Young diagrams that fit inside an a-by-b-by-c box.

Example: $M(1,2,1;q)$.

```
  o---o---o
 / \   \   \
o   o---o---o         1
 \ /   /   /
  o---o---o


  o---o---o
 /   / \   \
o---o   o---o       + q
 \   \ /   /
  o---o---o


  o---o---o
 /   /   / \
o---o---o   o       + q^2
 \   \   \ /
  o---o---o
```

Solid partitions and 4-dimensional Young diagrams?

Note:

$$\prod_{n=1}^{\infty} 1/(1-q^n)^{n(n-1)/2}$$

is not the enumerator for solid partitions (even though the first few terms of the q-series agree).

Also note:

$$\prod_{i=0}^{a-1}\prod_{j=0}^{b-1}\prod_{k=0}^{c-1}\prod_{l=0}^{d-1}\frac{[i+j+k+l+2]}{[i+j+k+l+1]}$$

is usually not a polynomial; in fact,

$$\prod_{i=0}^{a-1}\prod_{j=0}^{b-1}\prod_{k=0}^{c-1}\prod_{l=0}^{d-1}\frac{i+j+k+l+2}{i+j+k+l+1}$$

is usually not an integer. E.g., $a=b=c=d=2$ gives $500/3$.

Young diagrams and tilings

3-dimensional Young diagrams that fit in an a-by-b-by-c box are equivalent to tilings of an equi-angular hexagon with sides of length a, b, c, a, b, c, using unit rhombuses as tiles:

```
      o---o---o---o
     /   / \   \   \
    o---o   o---o---o
   /   / \ /   /   / \
  o---o   o---o---o   o
 / \   \ /   / \   \ / \
o   o---o---o   o---o   o
 \ / \   \   \ /   / \ /
  o   o---o---o---o   o
   \ /   /   / \   \ /
    o---o---o   o---o
     \   \   \ /   /
      o---o---o---o
```

We can read off the plane-partition from the stacks of cubes:

```
      o---o---o---o
     / 3 / \   \   \
    o---o   o---o---o
   / 3 / \ / 2 / 2 / \
  o---o   o---o---o   o            3   2   2
 / \   \ / 2 / \   \ / \
o   o---o---o   o---o   o   --->   3   2   1
 \ / \   \   \ / 1 / \ /
  o   o---o---o---o   o            1   1   0
   \ / 1 / 1 / \   \ /
    o---o---o   o---o
     \   \   \ / 0 /
      o---o---o---o
```

So this tiling has q-weight

$$q^{3+2+2+3+2+1+1+1+0} = q^{15}.$$

Local moves

Note that it is possible to get from any 3-dimensional Young-diagram in a box to any other by means of removing or adding a cube.

Therefore it is possible to get from any (unit-) rhombus-tiling of a hexagon to any other by means of the move

```
  o---o            o---o
 / \   \          /   / \
o   o---o  <-->  o---o   o
 \ /   /          \   \ /
  o---o            o---o
```

Tilings and matchings

A rhombus-tiling of a hexagon is also equivalent to a perfect matching of the graph dual to the dissection of the hexagon into unit triangles:

E.g., start with a tiling of the $a = 2$, $b = 2$, $c = 2$ hexagon:

```
        o-------o-------o
       /       / \       \
      /       /   \       \
     /       /     \       \
    o-------o       o-------o
   / \       \     /       / \
  /   \       \   /       /   \
 /     \       \ /       /     \
o       o-------o-------o       o
 \     /       / \       \     /
  \   /       /   \       \   /
   \ /       /     \       \ /
    o-------o       o-------o
     \       \     /       /
      \       \   /       /
       \       \ /       /
        o-------o-------o
```

Put a * in the center of each unit triangle, and join two *'s by an edge if they lie in the same tile:

```
        o-------o-------o
       /    *  / \  *    \
      /   /   /   \   \   \
     /  *    /  * \     *  \
    o-------o   |   o-------o
   / \  *    \  *  /    *  / \
  /   \   \   \   /   /   /   \
 /  *  \    *  \ /  *    /  *  \
o   |   o-------o-------o   |   o
 \  *  /    *  / \  *    \  *  /
  \   /   /   /   \   \   \   /
   \ /  *    /  * \     *  \ /
    o-------o   |   o-------o
     \  *    \  *  /    *  /
      \   \   \   /   /   /
       \    *  \ /  *    /
        o-------o-------o
```

Erase the tiling, and what remains is a perfect matching of a honeycomb graph (the dual of the dissection of the original hexagon into unit triangles):

```
    *   *
   /     \
  *   *   *
      |
  *   *   *
   \     /
*   *   *   *
|           |
*   *   *   *
   /     \
  *   *   *
      |
  *   *   *
   \     /
    *   *
```

Physicists call this a dimer covering of a graph.

Rotate by 30 degrees clockwise, for clarity; the faces that were the tops of cube-stacks are now the horizontal edges in the (perfect) matching.

```
        *---*

  *   *       *   *
 /     \     /     \
*       *   *       *

  *---*       *---*

*       *   *       *
 \     /     \     /
  *   *       *   *

        *---*
```

Local moves

Any such matching can be obtained from any other by means of local moves:

```
  *   *               *---*
 /     \
*       *   <--->   *       *
                     \     /
  *---*               *   *
```

In particular, every matching can be obtained from the "reference-matching"

```
        *   *
       /     \
  *   *       *   *
 /                 \
*       *   *       *
       /     \
  *   *       *   *
 /                 \
*       *---*       *

  *---*       *---*

        *---*
```

Matchings and q-weights

To bring q back into the story, label all the horizontal edges of the honeycomb graph as shown:

```
       *-3-*
      /    \
  *-2-*     *-2-*
 /    \    /    \
*      *-2-*     *
 \    /    \    /
  *-1-*     *-1-*
 /    \    /    \
*      *-1-*     *
 \    /    \     /
  *-0-*     *-0-*
      \    /
       *-0-*
```

Then for any matching, the sums of the labels of the matched horizontal edges equals the volume of the associated 3-dimensional Young diagram, plus a correction constant (in this case, 1).

Proof:

(1) It's true for the reference matching.

(2) The move that adds a cube to the Young diagram (increasing the volume by 1) translates into a local move on matchings that increases the sums of the labels of the matched edges by 1.

(3) These moves suffice to turn any Young diagram (or matching) into any other, so the claim follows.

Other ways of weighting the edges will work as well; e.g.,

```
        *-3-*
       /     \
  *-2-*       *-3-*
 /     \     /     \
*       *-2-*       *
 \     /     \     /
  *-1-*       *-2-*
 /     \     /     \
*       *-1-*       *
 \     /     \      /
  *-0-*       *-1-*
       \     /
        *-0-*
```

All that matters is that the sum of the labels should increase by 1 when one does the local move

```
  *   *                 *---*
 /     \
*       *   <--->   *       *
                     \     /
  *---*               *   *
```

Symmetrical tilings

Note that the sample tiling has a lot of symmetry:

```
        o-------o-------o
       /       / \       \
      /       /   \       \
     /       /     \       \
    o-------o       o-------o
   / \       \     /       / \
  /   \       \   /       /   \
 /     \       \ /       /     \
o       o-------o-------o       o
 \     /       / \       \     /
  \   /       /   \       \   /
   \ /       /     \       \ /
    o-------o       o-------o
     \       \     /       /
      \       \   /       /
       \       \ /       /
        o-------o-------o
```

There are two such tilings of the $a = b = c = 2$ hexagon.

How many such tilings are there for the $a = b = c = 2n$ hexagon? These are equivalent to "TSSCPPs" (Totally Symmetric Self-Complementary Plane Partitions) of order n.

Andrews (1994): The number of TSS-CPPs of order n is

$$A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

The sequence $(A_n)_{n \geq 1}$ goes

$$\mathbf{1}, \mathbf{2}, \mathbf{7}, \mathbf{42}, \mathbf{429}, \mathbf{7436}, \ldots$$

(remember this for later).

λ and Aztec diamonds

Connected minors

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n-1} & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n-1} & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \dots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & \dots & m_{n,n-1} & m_{n,n} \end{pmatrix}$$

$$M_C = \begin{pmatrix} m_{2,2} & \dots & m_{2,n-1} \\ \vdots & \ddots & \vdots \\ m_{n-1,2} & \dots & m_{n-1,n-1} \end{pmatrix} \qquad \text{(“center”)}$$

$$M_{TL} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n-1} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \dots & m_{n-1,n-1} \end{pmatrix} \qquad \text{(“top left”)}$$

with the $n-1$ by $n-1$ minors M_{TR} (“top right”), M_{BL} (“bottom left”), and M_{BR} (“bottom right”) defined similarly.

Desnanot-Jacobi identity

$$\det(M)\ \det(M_C)$$
$$= \det(M_{TL})\ \det(M_{BR})$$
$$- \det(M_{TR})\ \det(M_{BL})$$

Dodgson condensation

To compute the determinant of an n-by-n matrix, iteratively use this identity to compute the determinants of the connected minors of orders 1, 2, ..., n. (A "connected" k-by-k minor is formed by taking k consecutive rows and k consecutive columns.)
Note: a minor of order 0 has determinant 1.
Example:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 1 & 6 & 36 \\ 1 & 12 & 144 \end{pmatrix} \to$$

$$\begin{pmatrix} 2 & 12 \\ 2 & 48 \end{pmatrix} \to \begin{pmatrix} 12 \end{pmatrix}$$

Dodgson pyramids

Stacking the matrices in layers, we get a relation of discrete Hirota type, relating the values associated with the vertices of an octahedron:

```
                                   a
                                b------c
Rule: For the 3D pattern       /      /   ,
                              d------e
                                   f

      we have af = be - cd.
```

For $1 \leq m \leq n-1$, the determinant of the n-by-n matrix M can be expressed as a rational function of the determinants of the $(n-m)$-by-$(n-m)$ and $(n-m-1)$-by-$(n-m-1)$ connected minors of M.

(We can arrange these determinants in the form of an $(m+1)$-by-$(m+1)$ and an $(m+2)$-by-$(m+2)$ matrix; we can superimpose these two matrices, obtaining a "bimatrix".)

Robbins and Rumsey (1986): This rational function (the "determinant" of the bimatrix) is formally a Laurent polynomial in the entries of the bimatrix (the determinants of the connected minors of order $n-m$ and $n-m-1$).

$m = 1$:

$$\ldots \rightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} j & k \\ l & m \end{pmatrix} \rightarrow \begin{pmatrix} X \end{pmatrix} \;,$$

$$\begin{aligned} X &= (jm - kl)/e \\ &= j^1 m^1 e^{-1} - k^1 l^1 e^{-1} \\ &= e^{-1} j^1 k^0 l^0 m^1 - e^{-1} j^0 k^1 l^1 m^0 \\ &= \left(\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}\right) \;. \end{aligned}$$

$m = 2$:

$$\ldots \longrightarrow \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \longrightarrow \begin{pmatrix} q & r & s \\ t & u & v \\ w & x & y \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} X \end{pmatrix} \;,$$

$m = 2$ (continued):

X = an alternating sum of eight Laurent monomials

$$
\begin{aligned}
= \quad & \left(\begin{pmatrix} 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \end{pmatrix}\right) \\
& - \left(\begin{pmatrix} 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \end{pmatrix}\right) \\
& + \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 0 & & 0 & & 1 \\ & -1 & & 0 & \\ 1 & & 0 & & 0 \end{pmatrix}\right) + \left(\begin{pmatrix} 0 & & 0 & & 1 \\ & 0 & & -1 & \\ 1 & & 0 & & 0 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \end{pmatrix}\right) \\
& + \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & -1 & & 0 & \\ 1 & & -1 & & 1 \\ & 0 & & -1 & \\ 0 & & 1 & & 0 \end{pmatrix}\right) - \left(\begin{pmatrix} 0 & & 1 & & 0 \\ & 0 & & -1 & \\ 1 & & -1 & & 1 \\ & -1 & & 0 & \\ 0 & & 1 & & 0 \end{pmatrix}\right).
\end{aligned}
$$

Determinants and tilings

Robbins and Rumsey (1986), continued: For general m, this rational function (the "determinant" of the bimatrix) is an alternating sum of $2^{m(m+1)/2}$ Laurent monomials.

Elkies, Kuperberg, Larsen, and Propp (1992): The Laurent monomials correspond to to domino-tilings of Aztec diamonds.

Aztec diamonds

Here is a typical domino-tiling of the Aztec diamond of order 2:

```
    o---o---o
    |       |
o---o---o---o---o
|   |   |       |
o   o   o---o---o
|   |   |       |
o---o---o---o---o
    |       |
    o---o---o
```

Add “spurs”:

```
            o
            |
        o---o---o
        |       |
    o---o---o---o---o
    |   |   |       |
o---o   o   o---o---o---o
    |   |   |       |
    o---o---o---o---o
        |       |
        o---o---o
            |
            o
```

Rotate 45 degrees clockwise:

```
o   o   o   o
 \ / \ / \ /
  o   o   o
 /   / \   \
o   o   o   o
 \ /   / \ /
  o   o   o
 / \ / \   \
o   o   o   o
 \   \   \ /
  o   o   o
 / \ / \ / \
o   o   o   o
```

At each internal vertex, write down the number of incident edges minus 3:

```
o   o   o   o
 \ / \ / \ /
  0   1   0            0   1   0
 /   / \   \
o  -1   0   o           -1   0
 \ /   / \ /
  1   0   0       -->  1   0   0
 / \ / \   \
o   0  -1   o            0  -1
 \   \   \ /
  0   0   1            0   0   1
 / \ / \ / \
o   o   o   o
```

This gives the bimatrix of exponents associated with one of the terms in the Dodgson expansion of an n-by-n matrix in terms of its $n-2$-by-$n-2$ and $n-3$-by-$n-3$ connected minors.

Tilings and matchings

Domino tilings of Aztec diamonds can also be viewed as perfect matchings of a dual graph:

```
    o---o---o
    |       |                     *---*
o---o---o---o---o
|   |   |       |             *   *   *---*
o   o   o---o---o   <--->     |   |
|   |   |       |             *   *   *---*
o---o---o---o---o
    |       |                     *---*
    o---o---o
```

(This is the dimer model on a square grid.)

λ-determinants

Robbins and Rumsey showed that "Laurentness" continues to hold if the Dodgson recurrence $af = be - cd$ is replaced by the recurrence $af = be + cd$ or the more general

$$af = be + \lambda cd.$$

This is called the λ-determinant of the bimatrix.

Example: If we superimpose an m-by-m Vandermonde matrix with an $m+1$-by-$m+1$ matrix of 1s, the λ determinant of the bimatrix is

$$\prod_{j<k} (x_k + \lambda x_j).$$

λ-determinants and matchings

The λ-determinant of a bimatrix is a sum of

$$2^{m(m+1)/2}$$

monomials, in which each coefficient is a power of λ, and the exponents of all the entries equal $+1$, 0, or -1.

Each term is associated with a matching/tiling, and we've already seen how to read off the exponents from the tiling. The exponent of λ can be read off from the matching if we assign labels to the horizontal edges. Here's one scheme that works:

```
            -5
          *-----*

      -3    +4    -5
    *-----*-----*-----*

  -1    +2    -3    +4    -5
*-----*-----*-----*-----*-----*

   0    -1    +2    -3    +4
*-----*-----*-----*-----*-----*

       0    -1    +2
    *-----*-----*-----*

             0
          *-----*
```

We add the labels of the edges in a matching and add $n(n+1)(2n+1)/6$ to find the associated λ-exponent. E.g., for

```
             -3
          *------*


    -1       +2       -3
 *        *        *------*
 |        |
 |   0    |  -1      +2
 *        *        *------*


             0
          *------*
```

the matched horizontal edges sum to -4; adding $(2)(3)(5)/6 = 5$ gives 1. So in the expansion of the λ-determinant of a "3-by-3-and-2-by-2" bimatrix, the term associated with the above perfect matching has coefficient λ^1.

The λ-determinant of the m-by-m-and-$m+1$-by-$m+1$ bimatrix containing only 1s is a weight-enumerator of the domino-tilings of Aztec diamonds, and is given by the exact formula

$$D(m) = (1+\lambda)^m (1+\lambda^3)^{m-1} \dots (1+\lambda^{2m-1})^1.$$

λ and q

Suppose we take the labelled grid and delete all the horizontal edges whose weight is odd.

```
                  *     *
                  |     |
                  | +6  |
            *     *-----*     *
            |     |     |     |
            | +4  |     | +6  |
      *     *-----*     *-----*     *
      |     |     |     |     |     |
      | +2  |     | +4  |     | +6  |     |
*     *-----*     *-----*     *-----*     *
|     |     |     |     |     |     |     |
|  0  |     | +2  |     | +4  |     | +6  |
*-----*     *-----*     *-----*     *-----*
      |     |     |     |     |     |
      |  0  |     | +2  |     | +4  |
      *-----*     *-----*-    *-----*
            |     |     |     |
            |  0  |     | +2  |
            *-----*     *-----*
                  |     |
                  |  0  |
                  *-----*
```

This graph has only 1 perfect matching.

But now let us remove some edges:

```
        +6
      *-----*
      |     |
  +4  |     | +6
*-----*     *-----*
|     |     |     |
|     | +4  |     |
*     *-----*     *
|     |     |     |
| +2  |     | +4  |
*-----*     *-----*
|     |     |     |
|     | +2  |     |
*     *-----*-    *
|     |     |     |
|  0  |     | +2  |
*-----*     *-----*
      |     |
      |  0  |
      *-----*
```

Drawing the graph differently, we get

```
        *-6-*
       /     \
  *-4-*       *-6-*
 /     \     /     \
*       *-4-*       *
 \     /     \     /
  *-2-*       *-4-*
 /     \     /     \
*       *-2-*       *
 \     /     \      /
  *-0-*       *-2-*
       \     /
        *-0-*
```

which is exactly twice the labelling we saw in the section on plane partitions.

(The point here is not that λ is $\sqrt{q}$, but that they both correspond to a spatially-varying external field on a two-dimensional dimer model.)

Octahedron relations

Another link between the two dimer models comes from the octahedron relation.

We already saw one octahedron relation, namely $af = be + \lambda cd$, in the context

```
        a
     b-------c
    /       /
   d-------e
        f
```

$M(a,b,c;q)$ satisfies the octahedron relations

$$M(a,b,c)M(a,b-1,c-1)$$
$$= M(a+1,b-1,c-1)M(a-1,b,c)$$
$$+q^a M(a,b-1,c)M(a,b,c-1)$$

and

$$M(a,b,c)M(a,b,c-2)$$
$$= M(a,b,c-1)M(a,b,c-1)$$
$$-q^{c-1}M(a-1,b+1,c-1)M(a+1,b-1,c-1).$$

The enumerator

$$D(m;\lambda) = (1+\lambda)^m(1+\lambda^3)^{m-1}\dots(1+\lambda^{2m-1})^1$$

described above satisfies the octahedron recurrence

$$D(m)D(m-2)$$
$$= D(m-1)D(m-1)$$
$$+\lambda^{2m-1}D(m-1)D(m-1).$$

Eric Kuo (2003) has found direct combinatorial proofs of these quadratic relations. The exact formulas for $D(m;\lambda)$ and $M(a,b,c;\lambda)$ follow from these relations by trivial induction. Kuo's method also applies to the enumeration of Transpose-Complement Plane Partitions, giving a new proof of a theorem of Proctor (1988).

Alternating-sign matrices

Robbins and Rumsey had a different understanding of the bimatrices that arise in the study of Dodgson condensation (and its λ-generalization). Specifically, each term in the Laurent polynomial corresponds to a compatible pairs of ASMs (Alternating-Sign Matrices) of order m and $m + 1$. The entries in an ASM of order $m+1$ and the negatives of the entries in a compatible ASM of order m are superimposed to form a bimatrix.

A square matrix is an Alternating-Sign Matrix if:
(a) it consists of $+1$s, 0s and -1s;
(b) the entries in each row or column sum to 1; and
(c) the non-zero entries in each row and column alternate in sign, beginning and ending with a $+1$.

E.g., an ASM of order 4 is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The ASMs of order 3, and the ASMs of order 2 they are respectively compatible with:

```
(1 0 0) (0 1 0) (1 0 0) (0  1 0) (0 0 1) (0 1 0) (0 0 1)
(0 1 0) (1 0 0) (0 0 1) (1 -1 1) (1 0 0) (0 0 1) (0 1 0)
(0 0 1) (0 0 1) (0 1 0) (0  1 0) (0 1 0) (1 0 0) (1 0 0)
      \       \     |       / \       |     /       /
         \     \    |     /     \     |    /     /
            \   \   |   /         \   |   /   /
               \ \ | /              \ | / /
                (1 0)                (0 1)
                (0 1)                (1 0)
```

I won't define compatibility here, except to say that compatible pairs of ASMs are precisely those that arise from domino tilings of Aztec diamonds, and that there are $2^{m(m+1)/2}$ such pairs.

Enumerating ASMs

Zeilberger (1996): The number of ASMs of order n is

$$A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

(the same as the number of TSSCPPs of order n: $\mathbf{1}, \mathbf{2}, \mathbf{7}, \mathbf{42}, \mathbf{429}, \mathbf{7436}, \ldots$).

This was originally conjectured by Mills, Robbins and Rumsey.

Zeilberger's proof uses Andrews' theorem about TSSCPPs; in particular, Zeilberger shows that the number of ASMs of order n equals the number of TSSCPPs of order n.

Question: Is there a natural bijection between ASMs and TSSCPPs?

If we define $n!_q = [1][2]\ldots[n]$ (with $[n] = 1 + q + q^2 + \ldots + q^{n-1}$), then

$$A_n(q) = \prod_{k=0}^{n-1} \frac{(3k+1)!_q}{(n+k)!_q}$$

is a polynomial in q.

Question: Is there a natural combinatorial statistic such that the number of ASMs of order n for which the statistic takes the value m is the coefficient of q^m in $A_n(q)$?

Square ice

Greg Kuperberg (1996) found a different (and simpler) proof of Zeilberger's result, making use of the relationship between ASMs and states of the "six-vertex" or "square-ice" model.

Here is one of the seven states of the square-ice model in the case $n = 3$ (with "domain-wall boundary conditions"):

```
             o             o             o
             |             |             |
             |             |             |
            /|\           /|\           /|\
             |             |             |
      \      |     \       |      /      |      /
o------>-----o------>-----o-----<------o-----<------o
      /      |     /       |      \      |      \
             |             |             |
            /|\           \|/           /|\
             |             |             |
      \      |     \       |     \       |      /
o------>-----o------>-----o------>-----o-----<------o
      /      |     /       |     /       |      \
             |             |             |
            /|\           \|/           \|/
             |             |             |
      \      |      /      |      /      |      /
o------>-----o-----<------o-----<------o-----<------o
      /      |      \      |      \      |      \
             |             |             |
            \|/           \|/           \|/
             |             |             |
             |             |             |
             o             o             o
```

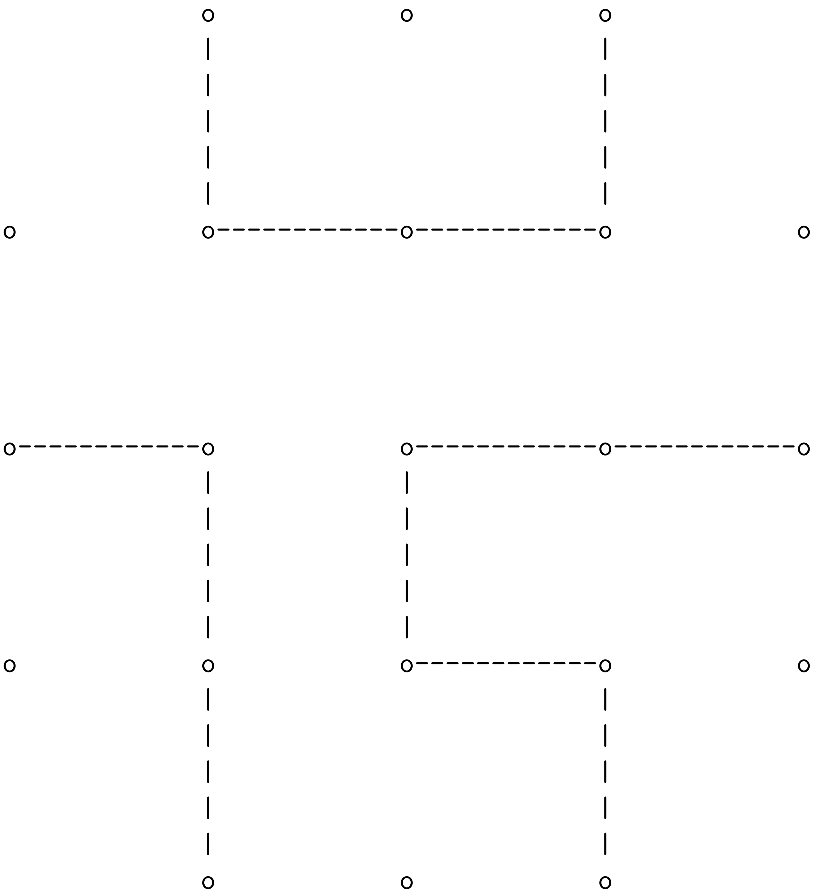
At each internal vertex, there are equal numbers of incoming and outgoing edges (the “ice condition”).

The edges along the top and bottom are oriented outward, and the edges along the sides are oriented inward.

Fully packed loops

A different statistical mechanics model related to Alternating-Sign Matrices is the Fully Packed Loop (FPL) model.

Here is one of the seven states of the FPL model in the case $n = 3$ (with domain-wall boundary conditions):



At each internal vertex, there are exactly two incident edges. At external vertices, the number of incident edges alternates between 0 and 1 (starting with the leftmost of the top row of external vertices, which is incident with 1 edge).

Number the $2n$ "terminals" (external vertices of degree 1) 1 through $2n$. Then each FPL consists of n non-crossing paths that pair up the terminals in some fashion, plus some number closed loops, so that each internal vertex belongs to exactly one path or closed loop.

In particular, each FPL determines a pairing π of the numbers 1 through $2n$, corresponding to the way it joins up the terminals. Let $N(n;\pi)$ denote the number of FPLs of order n associated with the pairing π. E.g., for $n = 3$ and $\pi : 1 \leftrightarrow 2, 3 \leftrightarrow 4, 5 \leftrightarrow 6$, we have $N(3;\pi) = 2$.

Wieland (2000): $N(n;\pi)$ is invariant under the action of the dihedral group of symmetries of the $2n$-gon, acting on the set $\{1,2,\ldots,2n\}$ in the natural way. (This was originally conjectured by Bosley and Fidkowski.)

E.g.,
$N(4; 1 \leftrightarrow 2,\ 3 \leftrightarrow 4,\ 5 \leftrightarrow 6,\ 7 \leftrightarrow 8) = N(4; 2 \leftrightarrow 3,\ 4 \leftrightarrow 5,\ 6 \leftrightarrow 7,\ 8 \leftrightarrow 1) = 7.$

```
    1       2                          1       2
    |       |                          |       |
    o---o---o   o---3                  o---o   o---o---3
                |                          |
8---o   o---o   o                      8---o---o   o---o
    |   |   |   |      versus                      |   |
    o   o   o   o---4                      o---o   o   o---4
    |   |   |                              |   |   |
7---o   o   o---o                      7---o   o   o---o
        |       |                              |       |
        6       5                              6       5
```

$$N(2; 1 \leftrightarrow 2,\ 3 \leftrightarrow 4) = \mathbf{1}$$
$$N(3; 1 \leftrightarrow 2,\ \ldots, 5 \leftrightarrow 6) = \mathbf{2}$$
$$N(4; 1 \leftrightarrow 2,\ \ldots, 7 \leftrightarrow 8) = \mathbf{7}$$
$$N(5; 1 \leftrightarrow 2,\ \ldots, 9 \leftrightarrow 10) = \mathbf{42}$$
$$N(6; 1 \leftrightarrow 2,\ \ldots, 11 \leftrightarrow 12) = \mathbf{429}$$
$$N(7; 1 \leftrightarrow 2,\ \ldots, 13 \leftrightarrow 14) = \mathbf{7436}$$

Conjecture:

$$N(n; 1 \leftrightarrow 2,\ 2n-1 \leftrightarrow 2n) = A_{n-1}$$

Conjecture (David Wilson):

$$N(n; 1 \leftrightarrow 2) = \frac{3}{2}\ \frac{n^2+1}{4n^2-1}\ A_n$$

Neither Zeilberger's techniques nor Kuperberg's are immediately applicable, since the information contained in π is so nonlocal.

This is cause for optimism.

I believe there is some completely different way to count ASMs that will prove these conjectures and will incidentally give an even simpler proof of the original ASM conjecture than Kuperberg's,

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See also links accessible from the following web-pages:
`www.math.wisc.edu/~propp/somos.html`
`www.math.wisc.edu/~propp/bilinear.html`
`www.math.harvard.edu/~propp/reach`