

**Rational maps,
Laurent polynomials,
and combinatorics**

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These notes available on-line at
www.math.wisc.edu/~propp/ratmaps.pdf

I. Prologue: the Lyness recurrence

The recurrence relation $a_n = (a_{n-1} + 1)/a_{n-2}$ has period 5:

$$a_1 = x$$

$$a_2 = y$$

$$a_3 = (y + 1)/x$$

$$a_4 = (x + y + 1)/xy$$

$$a_5 = (x + 1)/y$$

$$a_6 = x$$

$$a_7 = y$$

...

That is, the map $F : (x, y) \mapsto (y, (y + 1)/x)$ from $\mathbf{C} \times \mathbf{C}$ to itself has period 5 (except where the composite map is undefined).

Note that for all n , a_n is a **Laurent polynomial** in x and y , i.e., a polynomial in x , $1/x$, y , and $1/y$. For $n = 5, 6$, and 7 , this property requires a “fortuitous” cancellation: e.g., we would expect to see a factor of $y + 1$ in the denominator of a_5 .

Every Laurent polynomial in x and y can be written as a sum of **Laurent monomials** of the form cx^ay^b , where a and b are (not necessarily positive) integers. Alternatively, it can be written as an ordinary polynomial divided by a monomial.

E.g.,

$$(y + 1)/x = x^{-1}y^1 + x^{-1}y^0.$$

The Lyness map F has the property that

$$F^{(n)}(x, y) = (a_{n+1}(x, y), a_{n+2}(x, y))$$

is a pair of Laurent polynomials in x and y , for all n .

The Laurent phenomenon: **More often than we can explain, iterating a Laurent polynomial mapping leaves us in the domain of Laurent polynomial mappings.**

See Fomin and Zelevinsky, “The Laurent phenomenon”, [math.CO/0104241](https://arxiv.org/abs/math/0001042).

What's the natural domain for the Lyness map that avoids the singularities at $x = 0$ and $y = 0$?

It's not enough to look at

$$\{(x, y) : x \neq 0, y \neq 0\},$$

since this is not invariant under the map.

We can fix some of the singularities by moving to the projective plane:

$$(t : x : y) \mapsto (tx : xy : ty + t^2)$$

Here $(t : x : y) \in \mathbf{P}^2 = \mathbf{CP}^2 = (\mathbf{C}^3 \setminus (0, 0, 0)) / \sim$, where $u \sim v$ iff $v = cu$ for some $c \neq 0$.

We think of the original affine 2-space as $\{(t : x : y) \in \mathbf{P}^2 : t \neq 0\}$. Each such element of \mathbf{P}^2 can be written uniquely as $(1 : x : y)$.

We can recover the affine map from the projective map by setting $t = 1$: $(1 : x : y) \rightarrow (x : xy : y + 1) = (1 : y : (y + 1)/x)$.

We can construct the projective map from the affine map by allowing the latter to act on $(x/t, y/t)$.

$$(x/t, y/t) \mapsto (y/t, (y/t + 1)/(x/t))$$

$$(1 : x/t : y/t) \mapsto (1 : y/t : (y/t + 1)/(x/t))$$

$$(t : x : y) \mapsto (tx : xy : t(y + t))$$

$F : (t : x : y) \mapsto (tx : xy : ty + t^2)$ is a rational map from \mathbf{P}^2 to itself. It is not defined on all of \mathbf{P}^2 (e.g., it is not defined at $(0 : 0 : 1)$), but it is defined on a Zariski dense subset.

(If we want to get rid of all the singularities of the map, we must blow up the projective plane at four points in general linear position, obtaining the “Del Pezzo surface of degree 5”, but let’s settle for projectivizing today.)

The mapping F is birational: it has a rational inverse mapping, at least on a Zariski dense subset.

The map $F : (t : x : y) \mapsto (tx : xy : ty + t^2)$ has (algebraic) degree 2.

More generally, every rational map $F : \mathbf{CP}^n \rightarrow \mathbf{CP}^n$ can be written in the form $F(x_1 : x_2 : \dots : x_{n+1}) = (p_1 : p_2 : \dots : p_{n+1})$ where p_1, \dots, p_{n+1} are homogeneous polynomials of some common degree d in the variables x_1, \dots, x_{n+1} having no common polynomial factor; this representation is unique up to scaling, and d is called the algebraic degree of the map.

Geometrically, the degree of a rational map $F : \mathbf{CP}^n \rightarrow \mathbf{CP}^n$ is the number of points in $F(A) \cap B$, where A is a generic \mathbf{CP}^1 in \mathbf{CP}^n and B is a generic \mathbf{CP}^{n-1} in \mathbf{CP}^n . This relates to the work of Arnold on intersection-based dynamical complexity.

II. The map $(x, y) \mapsto (y, (y^2 + 1)/x)$

Generalizing the Lyness sequence, Fomin and Zelevinsky have studied the recurrence

$$f(n) = \begin{cases} (f(n-1)^b + 1)/f(n-2) & \text{if } n \text{ is even,} \\ (f(n-1)^c + 1)/f(n-2) & \text{if } n \text{ is odd.} \end{cases}$$

where b, c are positive integers. They showed that recurrences of this form exhibit the Laurent phenomenon. See Fomin and Zelevinsky, “Cluster algebras I: Foundations”, [math.RT/0104151](#).

The case $b = c = 1$ is the Lyness recurrence.

It is conjectured that for all $b, c \geq 1$, the distinct Laurent polynomials arising from the (b, c) recurrence are associated with elements of the dual canonical basis for the semisimple Kac-Moody Lie algebra of rank 2 with generalized Cartan matrix

$$\begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}.$$

When $bc < 4$, the (b, c) recurrence is related to classical root systems, and the period of the recurrence is equal to the Coxeter number of the associated reflection group.

For $bc \geq 4$, there's still a (partly conjectural) relationship with infinite reflection groups. When $bc = 4$, the reflection group is Euclidean; when $bc > 4$, the reflection group is hyperbolic.

Let's look at the case $b = c = 2$, the simplest case in which the recurrence is not periodic.

The associated projective map $(w : x : y) \mapsto (wx : xy : w^2 + y^2)$ is of degree 2. The n th iterate of this map has degree $d_n = 2n$, so the **algebraic entropy** of the map (defined by Bellon and Viallet as $\lim_{n \rightarrow \infty} \log d_n$) is equal to zero.

Put $a_1 = x$, $a_2 = y$, and $a_n = (a_{n-1}^2 + 1)/a_{n-2}$ for $n \geq 3$:

$$a_1 = x$$

$$a_2 = y$$

$$a_3 = (y^2 + 1)/x$$

$$a_4 = (y^4 + x^2 + 2y^2 + 1)/x^2y$$

$$a_5 = (y^6 + x^4 + 2x^2y^2 + 3y^4 + 2x^2 + 3y^2 + 1)/x^3y^2$$

...

Note that all coefficients are positive.

Positivity of the coefficients is not a consequence of Laurentness (consider $(x^3 + y^3)/(x + y) = x^2 - xy + y^2$).

We “lift” the 1-dimensional recurrence

$$\begin{array}{l} A \\ B \\ C \end{array} \qquad C = (B^2 + 1) / A$$

to the 2-dimensional recurrence

$$\begin{array}{l} A \\ B \quad B' \\ C \end{array} \qquad C = (B B' + 1) / A$$

(a two-dimensional algebraic cellular automaton).

$$\begin{array}{ccccccccc}
A_0 & & A_2 & & A_4 & & A_6 & & A_8 \\
& B_1 & & B_3 & & B_5 & & B_7 & \\
& & C_2 & & C_4 & & C_6 & & \\
& & & D_3 & & D_5 & & & \\
& & & & E_4 & & & & \\
& & & & & \dots & & &
\end{array}$$

$$A_0 = x_0, A_2 = x_2, \dots$$

$$B_1 = y_1, B_3 = y_3, \dots$$

$$C_2 = (B_1 B_3 + 1)/x_2 = x_2^{-1} y_1 y_3 + x_2^{-1}, \dots$$

$D_3 = (C_2 C_4 + 1)/y_3 =$ a sum of 5 Laurent monomials in $x_2, x_4, y_1, y_3, y_5, \dots$

$E_4 = (D_3 D_5 + 1)/C_4 =$ a sum of 13 Laurent monomials in $x_2, x_4, x_6, y_1, y_3, y_5, y_7, \dots$

Note that if the sequences A_0, A_2, A_4, \dots and B_1, B_3, B_5, \dots are constant (i.e., $A_n = x$ and $B_n = y$ for all n) then each succeeding row is also constant, with $C_n = (y^2 + 1)/x$, $D_n = (((y^2 + 1)/x)^2 + 1)/y$, etc.: our original 1-dimensional recurrence.

What's bad about

$$D_3 = (y_1 y_3^2 y_5 + x_2 x_4 + y_1 y_3 + y_3 y_5 + 1) / x_2 x_4 y_3$$

in comparison with the x, y version

$$(y^4 + x^2 + 2y^2 + 1) / x^2 y$$

is that D_3 has more terms.

What's good about D_3 is that it has smaller coefficients. In fact, all coefficients are equal to 1.

This remains true in all subsequent rows: each Laurent monomial that occurs has coefficient 1.

Fact: Every entry in the n th row of this table (counting the top two rows as the -1 st and 0 th) is a Laurent polynomial in the variables x_k (with $|k| < n$ and $k \not\equiv n \pmod{2}$) and the variables y_k (with $|k| \leq n$ and $k \equiv n \pmod{2}$) in which all exponents are between -1 and $+1$ and all coefficients are equal to 1 . Moreover, there is a simple bijection between the Laurent monomials that occur and the domino tilings of the rectangle $[-n, +n] \times [-1, +1]$ in \mathbf{R}^2 .

Example ($n = 2$):

$\begin{array}{ccccccc} o & - & o & - & o & - & o \\ & & & & & & \\ - & 4 & - & 2 & - & 4 & - & 2 & - & 4 \\ & & & & & & \\ o & - & o & - & o & - & o \end{array}$	$\begin{array}{ccccc} 1 & -1 & 1 & -1 & 1 \\ y & x & y & x & y \\ -2 & -1 & 0 & 1 & 2 \end{array}$
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$\begin{array}{ccccccc} o & - & o & - & o & - & o \\ & & & & & & \\ - & 4 & - & 2 & - & 3 & 2 & 3 \\ & & & & & & \\ o & - & o & - & o & - & o \end{array}$	$\begin{array}{ccccc} 1 & -1 & 0 & -1 & 0 \\ y & x & y & x & y \\ -2 & -1 & 0 & 1 & 2 \end{array}$
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$\begin{array}{ccccccc} o & - & o & - & o & - & o \\ & & & & & & \\ - & 3 & 2 & 3 & - & 2 & - & 4 \\ & & & & & & \\ o & - & o & - & o & - & o \end{array}$	$\begin{array}{ccccc} 0 & -1 & 0 & -1 & 1 \\ y & x & y & x & y \\ -2 & -1 & 0 & 1 & 2 \end{array}$
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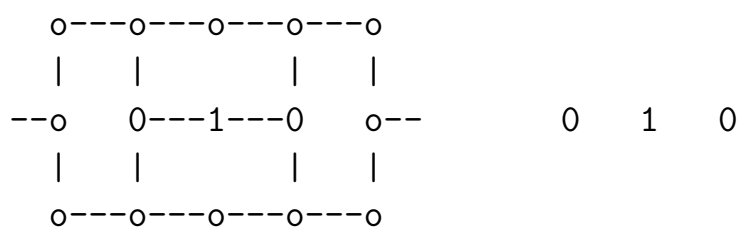
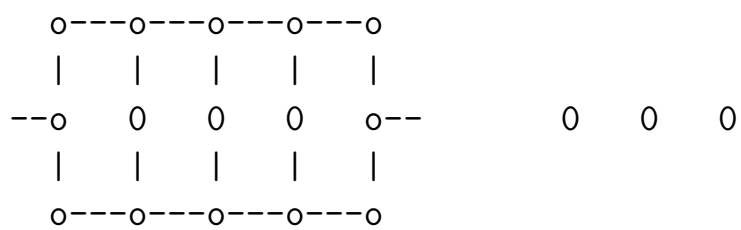
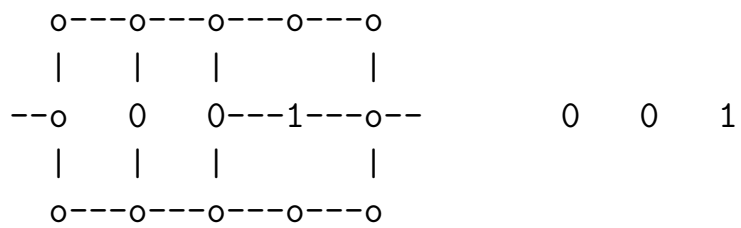
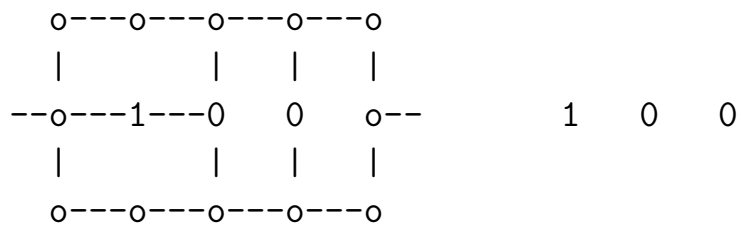
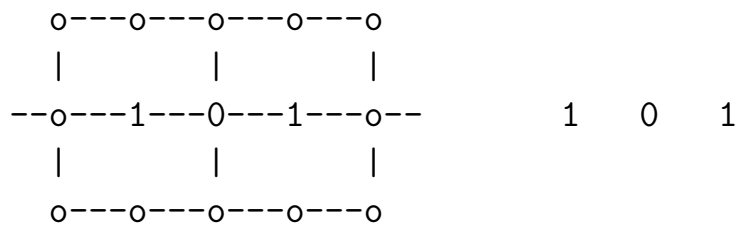
$\begin{array}{ccccccc} o & - & o & - & o & - & o \\ & & & & & & \\ - & 3 & 2 & 2 & 2 & 3 \\ & & & & & & \\ o & - & o & - & o & - & o \end{array}$	$\begin{array}{ccccc} 0 & -1 & -1 & -1 & 0 \\ y & x & y & x & y \\ -2 & -1 & 0 & 1 & 2 \end{array}$
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$\begin{array}{ccccccc} o & - & o & - & o & - & o \\ & & & & & & \\ - & 3 & 3 & - & 2 & - & 3 & 3 \\ & & & & & & \\ o & - & o & - & o & - & o \end{array}$	$\begin{array}{ccccc} 0 & 0 & -1 & 0 & 0 \\ y & x & y & x & y \\ -2 & -1 & 0 & 1 & 2 \end{array}$
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Each numerical marking in the tiling tells how many tile-edges meet at the marked vertex (where we add two notional tile-edges at the left and right for convenience).

Each exponent in the Laurent polynomial is 3 less than the corresponding vertex-marking.

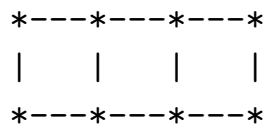
Furthermore, each domino-tiling of the 2 -by- $2n$ rectangle can be coded by a sequence of $2n - 1$ bits, subject to the rule that no two 1 's can appear in a row: we put 0 's along vertical break-lines in the tiling, 1 's everywhere else.

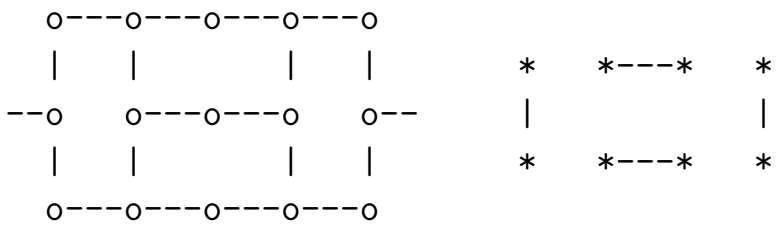
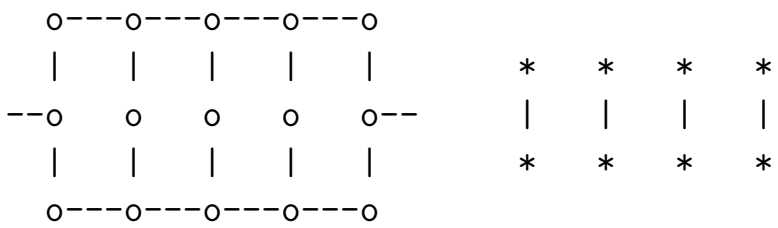
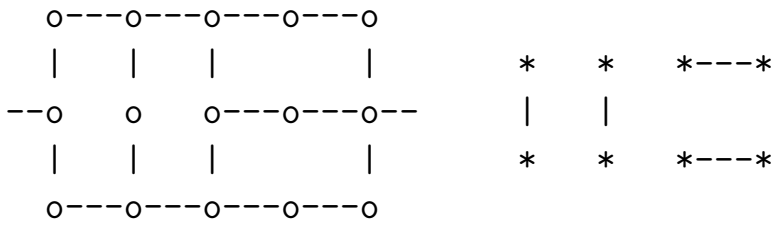
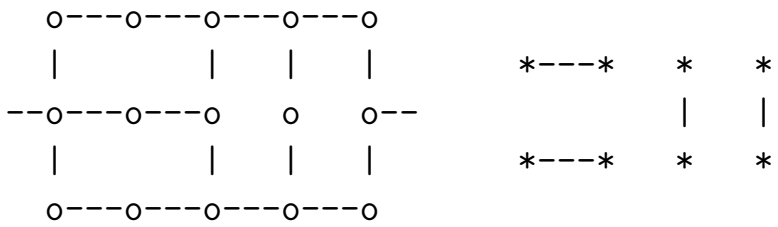
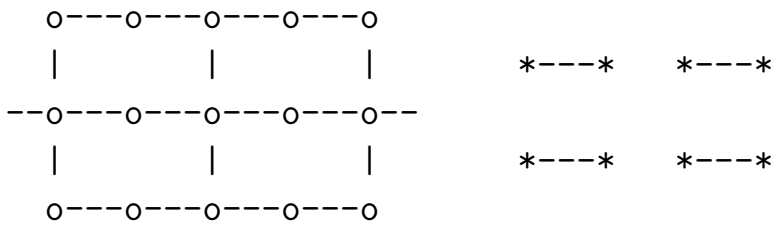


So the Laurent polynomial is a sum of monomials, which individually represent the different words of length $2n - 1$ in the golden mean shift.

It turns out to be useful to think of the domino tilings of the 2-by- $2n$ rectangle as perfect matchings on the 2-by- $2n$ grid-graph, known in the statistical physics literature as dimer configurations. These are configurations consisting of some subset of the edges of a graph such that each vertex occurs in exactly one edge in the subset.

E.g., domino-tilings of the 2-by-4 rectangle correspond to dimer configurations on the 2-by-4 grid-graph





The different perfect matchings are states, the associated monomials are Boltzmann weights, and the sum of these monomials is the partition function for this model.

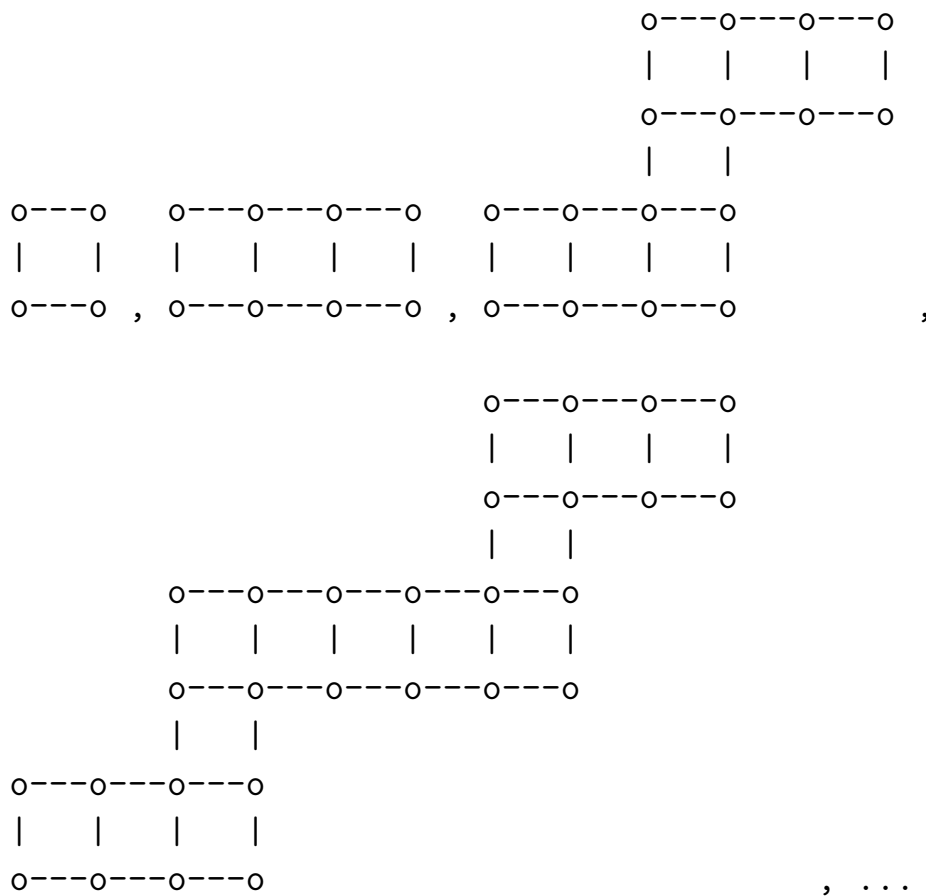
III. $(x, y, z) \mapsto (y, z, (y^2 + z^2)/x)$

As in the previous cases, the map can be constructed as the composition of an involution that affects the first term of the tuple (in this case $(x, y, z) \mapsto ((y^2 + z^2)/x, y, z)$) with a cyclic shift of the arguments that puts the modified term at the end of the tuple.

The associated projective map $(w : x : y : z) \mapsto (wx : xy : xz : y^2 + z^2)$ is of degree 2. Its iterates have degrees 2, 4, 8, 14, 24, 40, 66, 108, ...; each term of the degree sequence is 2 less than twice a Fibonacci number, and the algebraic entropy is the log of the golden ratio.

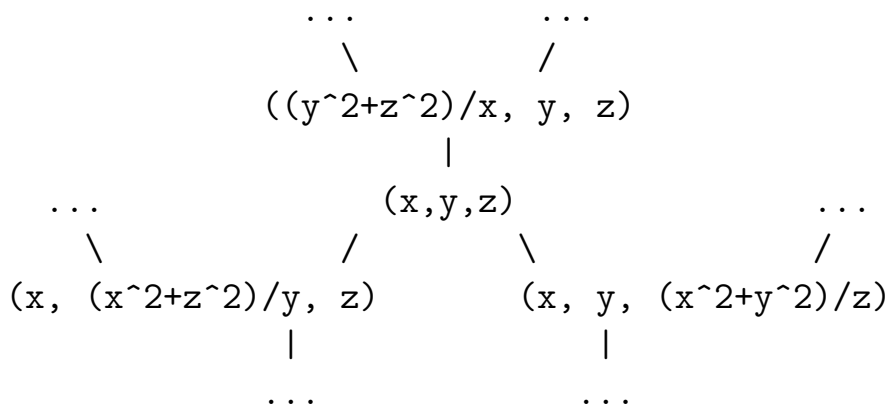
Open question (Bellon-Viallet): Is the algebraic entropy of a rational map always the log of an algebraic integer?

Iteration of the original (affine) map yields Laurent polynomials that are partition functions for bent versions of the 2-by- $2n$ grid:



The n th graph in the sequence is gotten by taking two copies of the $n - 1$ st graph and overlapping the end of one with the start of the other, gluing the two graphs together along a copy of the $n - 3$ rd graph.

The right way to lift the map $(x, y, z) \mapsto (y, z, (y^2 + z^2)/x)$ is to put it on an infinite 3-valent tree. At the root of the tree, we put (x, y, z) . We 3-color the edges of the tree so that each vertex has one incident edge of each color, and the three colors correspond to the three involutions $(x, y, z) \mapsto ((y^2 + z^2)/x, y, z)$, $(x, y, z) \mapsto (x, (x^2 + z^2)/y, z)$, $(x, y, z) \mapsto (x, y, (x^2 + y^2)/z)$.



Putting $x = y = z = 1$, we get a triple of numbers (a, b, c) satisfying the relation $a^2 + b^2 + c^2 = 3abc$ (a Markoff triple).

This is also related to the Farey tree. See links to preprints at

<http://www.math.wisc.edu/~propp/reach/>.

IV. $(w, x, y, z) \mapsto (x, y, z, (xz + y^2)/w)$

This example arose from the study of the Somos-4 sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, \dots$$

satisfying the relation

$$a_n = (a_{n-1} a_{n-3} + a_{n-2} a_{n-2}) / a_{n-4}$$

(see article by Gale).

The projective version is $(v : w : x : y : z) \mapsto (vw : wx : wy : wz : xz + y^2)$, whose degree sequence grows quadratically rather than linearly or exponentially.

This recurrence exhibits the Laurent property, and positivity of coefficients. The only known proof of positivity makes use of a combinatorial interpretation of the Laurent polynomial in terms of dimer configurations on suitable graphs.

Speyer and Bousquet-Mélou–Propp–West (2002):
 The rational functions $a_{n,i,j}(\dots)$ ($n, i, j \in \mathbf{Z}$)
 defined by the initial conditions

$$a_{n,i,j} = x_{n,i,j} \quad (1 \leq n \leq 4)$$

and the recurrence relation

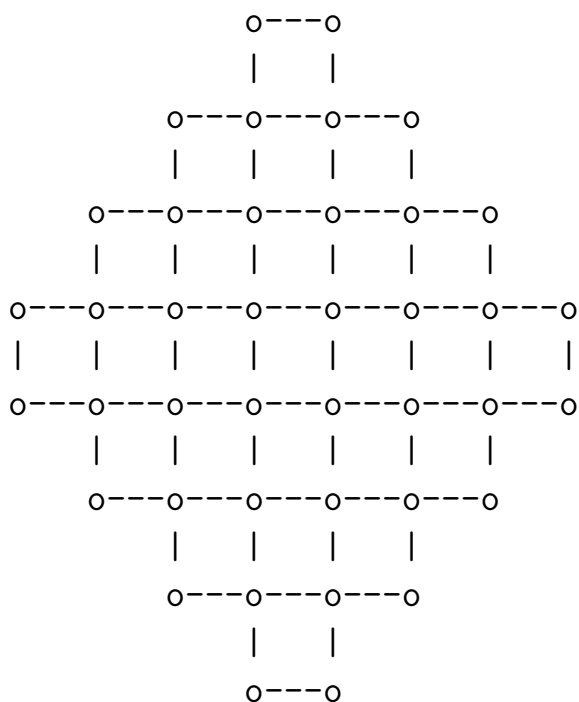
$$a_{n,i,j} = \frac{(a_{n-1,i-1,j} a_{n-3,i+1,j} + a_{n-2,i,j-1} a_{n-2,i,j+1})}{a_{n-4,i,j}}$$

(for $n > 4$) are Laurent polynomials with positive coefficients, and in fact, with all coefficients equal to 1.

The first result of this kind was proved back in the mid-1980s by Robbins and Rumsey. They proved an analogous claim for the recurrence

$$a_{n,i,j} = (a_{n-1,i-1,j}a_{n-1,i+1,j} + a_{n-1,i,j-1}a_{n-1,i,j+1}) / a_{n-2,i,j}$$

For this recurrence, the relevant dimer-graphs turned out to be the **Aztec diamond graphs** (Elkies, Kuperberg, Larsen, and Propp, 1992):



Aztec diamond graphs have been an important tool in the study of the dimer model on a square grid.

For both of these three-dimensional recurrences, note that the subscript-triples, viewed as points in 3-space, form the vertices of a centrally symmetric octahedron. Equations of this shape are instances of the discrete Hirota equation from the theory of integrable systems (see Zabrodin).

IV. Beyond dimer models

The two-dimensional algebraic cellular automaton

$$a_{i,j,k} = \left(a_{i-1,j,k} a_{i,j-1,k-1} + a_{i,j-1,k} a_{i-1,j,k-1} + a_{i-1,j,k} a_{i,j-1,k-1} \right) / a_{i-1,j-1,k-1}$$

(the “cube recurrence”), like the octahedron recurrence, exhibits quadratic degree-growth.

Here the relevant combinatorial models are not dimer models but stat mech models involving constrained spanning-trees called groves. (See Carroll and Speyer.)

V. Stability

A kind of p -adic numerical stability seems to go hand-in-hand with the Laurent property.

Recall that the Somos-4 recurrence is associated with the rational map

$$(w, x, y, z) \rightarrow (x, y, z, (xz + y^2)/w)$$

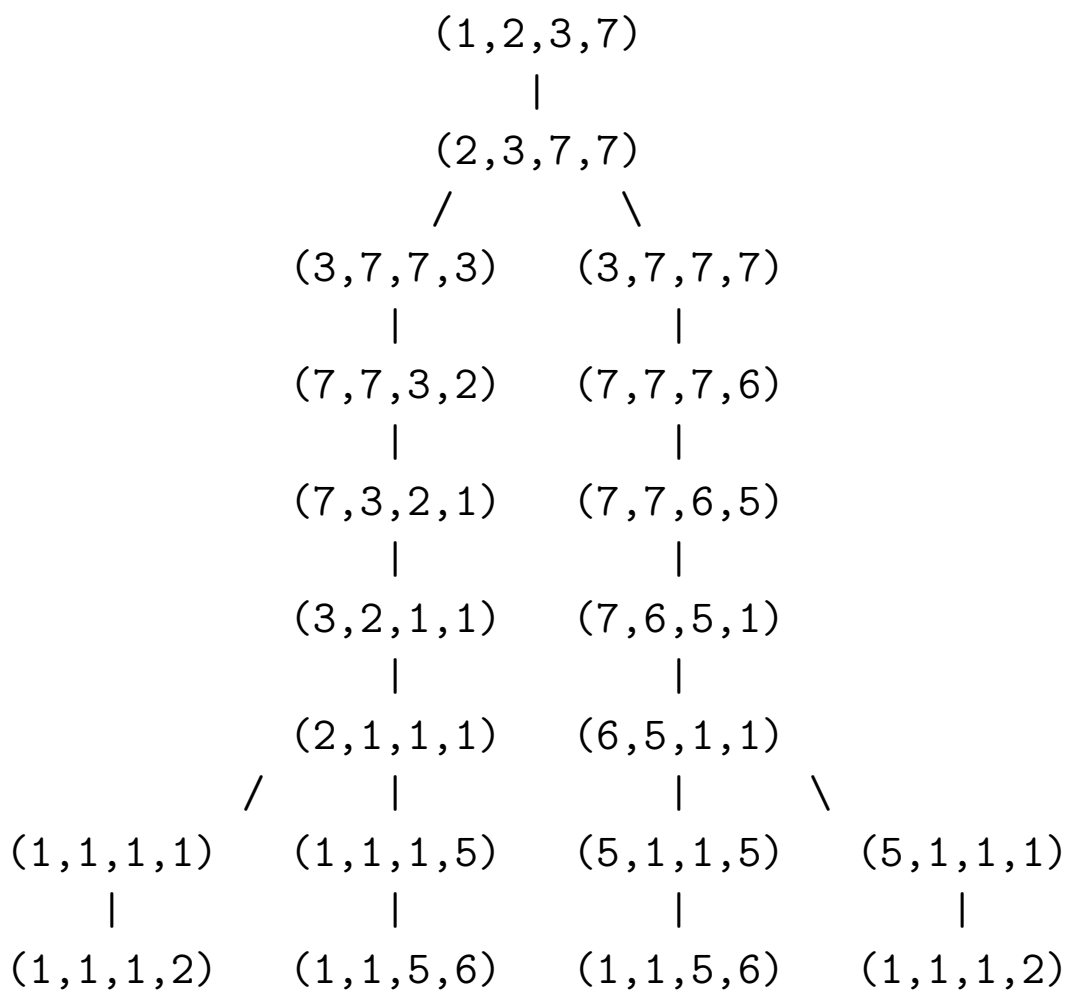
from \mathbf{C}^4 to itself (with singularities).

How might we compute the Somos-4 sequence modulo 8 (say)?

Can we replace \mathbf{C} by $\mathbf{Z}/8\mathbf{Z}$?

View modular division as a multi-valued function; e.g., $4 / 2$ is 1 or 5 mod 8.

If we only keep track of the terms of the Somos-4 sequence mod 8, divergence occurs when we divide an even number by an even number, which is ambiguous mod 8.



But note that re-convergence occurs too. This happens more often than we can explain.

See Kedlaya and Propp, “In search of Robbins stability”, [math.NT/0409535](https://arxiv.org/abs/math/0409535).

VI. Summary

When iterates of a rational map from affine n -space to itself have the Laurent property (i.e., for all n the n th iterate of the map is given by an n -tuples of Laurent polynomials), there is often some way to view each Laurent polynomial as the partition function for an exactly solvable statistical mechanics model. The size of the model is given roughly by the degree of the associated iterated mapping from projective n -space to itself.

Sometimes the natural setting for a one-dimensional recurrence is a higher-dimensional recurrence.

The link between dynamics and combinatorics can yield proofs of the Laurent property as well as proofs of positivity of the coefficients. (Laurentness can also be proved by cluster algebra methods; positivity cannot.)

All the stat mech models obtained from recurrences in this fashion appear to be exactly solvable (e.g., bulk entropy can be expressed exactly via integrals and infinite series).

Wild guess: If one can find a recurrence that has the Laurent property, positive coefficients, and third-power growth in its degree-sequence, it should correspond to an exactly solvable three-dimensional stat mech model (of which very few are known).

Very little systematic theory has been developed for dynamical systems theory in the category of algebraic geometry and rational/birational maps, and many foundational questions remain open. E.g., is the algebraic entropy of a rational map always the logarithm of an algebraic integer?

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See also links accessible from the following web-pages:

www.math.wisc.edu/~propp/somos.html
www.math.wisc.edu/~propp/bilinear.html
www.math.wisc.edu/~propp/reach/