Combinatorial Interpretations for the Markov Numbers

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Abstract

We need to put in an abstract, but we can do this last.

1 Markov Polynomials

We define Markov polynomials to be triples of polynomials (f(x,y,z), g(x,y,z), h(x,y,z)) that satisfy the relation

$$f^2 + g^2 + h^2 = \frac{x^2 + y^2 + z^2}{xyz} fgh$$

Equivalently, we may rewrite this as

$$\frac{f^2 + g^2 + h^2}{fgh} = K(x, y, z), \quad K(x, y, z) := \frac{x^2 + y^2 + z^2}{xyz}$$

Clearly the triple (x,y,z) satisfies this relation. To generate another Markov triple given some other triple of Markov polynomials (f,g,h), we take (f,g,h) to (f, g, h') where $h' = \frac{f^2+g^2}{h}$ from which it can be verified algebraically that (f,g,h') satisfies the relation. Note that by symmetry, we may also create new triples (f', g, h) and (f, g', h) in a similar manner and that (f,g, h") = (f,g,h) where h" is the polynomial created by taking (f,g,h) to (f,g, h') to (f,g,h"). From these basic facts, we may derive two relations between h and h' that will be used later.

$$hh' = f^2 + g^2 \tag{1}$$

$$h + h' = h + \frac{f^2 + g^2}{h} = K(x, y, z)fg \tag{2}$$

From (2), $h' = K(x,y,z)fg - h$, so h' is the product and sum of Laurent polynomials by induction, so we have that h' will be a Laurent polynomial as well.

Define three operators u,v, and w that act on triples of Markov polynomials (f,g,h):

$$u((f,g,h)) = (\frac{g^2+h^2}{f}, g, h) = (f', g, h)$$

$$v((f,g,h)) = (f, \frac{f^2+h^2}{g}, h) = (f, g', h)$$

$$w((f,g,h)) = (f, g, \frac{f^2+g^2}{h}) = (f, g, h')$$

Thus from taking the initial Markov triple (x,y,z) as our starting point, we encode a Markov triple generated in the manner above as a string of letters consisting of u's, v's, and w's. Furthermore since uu(f,g,h) = (f,g,h") = (f,g,h), we also may define a reduced word as a sequence of moves starting from (x,y,z) that take us to a given Markov polynomial (f,g,h) without consecutive u's, v's, or w's. For convenience, let the word $x_n x_{n-1} \cdots x_1$ mean $x_1(x_2(\cdots x_n((x,y,z))\cdots))$. Furthermore, we have a notion of the reduced word $x_{n'} x_{n'-1} \cdots x_1$ of $x_n x_{n-1} \cdots x_1$ since we may cancel out repeated terms and our word still represents the same triple of polynomials.

2 Snake Graphs

These are the graphs that will be our combinatorial model for Markov polynomials. As with the Markov polynomials, we define triples of snake graphs (F,G,H) inductively. For our initial snake graphs, we let (F,G,H) be the three empty graphs (, ,). As in the case with Markov polynomials, we introduce the operators U,V, and W that act on (F,G,H) a similar manner as they do on Markov polynomials:

$$U((F,G,H)) = (F', G, H)$$

$$V((F,G,H)) = (F, G', H)$$

$$W((F,G,H)) = (F, G, H')$$

These operators take a triple of graphs (F,G,H) and create a new triple of graphs (F',G,H), (F,G',H), or (F,G,H'). As before, we take $X_n X_{n-1} \cdots X_1 :=$

$X_n(X_{n-1}(\cdots(X_1(,,)\cdots))$. However, we disallow that $X_k = X_{k+1}$, that is that we use the same operator twice. $X_n \cdots X_1$ then represents a series of moves or steps in creating a graph.

Now define W, VW , and UW according to figure 1.

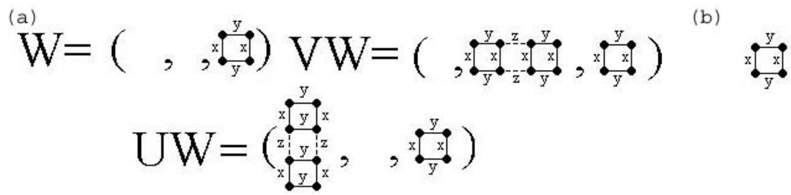

Figure 1: (a)The base cases W, VW, and UW (b) a unit for these cases.

Note that for reasons that will become apparent, there are labeled x-edges, y-edges, and z-edges. X-edges are always vertical; y-edges are always horizontal. Z-edges can be either vertical or horizontal. Also, there is a natural way of embedding these graphs in a square grid. We call the graph consisting of four vertices connected by edges to form a square a unit (see figure 1(b)). Thus the triple W is two empty graphs for the first two graphs, and a unit in the third graph.

We now define how to construct all other triples of graphs that are of the form $X_n \cdots VW$ or $X_n \cdots UW$. Let $X_{n-1} \cdots VW or X_{n-1} \cdots UW = (F, G, H)$ as schematically represented below.

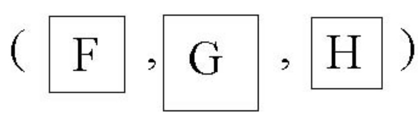

Figure 2: (F,G,H)

Suppose without loss of generality that we can take (F,G,H) to (F,G,H'). That is that $X_{n-1} \neq W$. Then to generate H', adjoin F, G, and a unit as follows.

Place F and G on a square grid so that the upper right corner of F and the lower left corner of G are on opposite corners of a square. We either place the unit just above the end of F and connect it to G and F using z-edges or place the unit just to the right of F and connect it to F and G using z-edges (see figure 3).

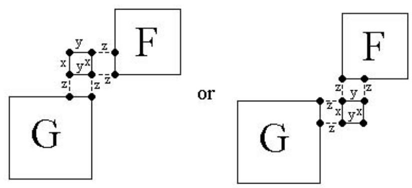

Figure 3: Possibilities for H.

The graph H', is defined as which one of these is symmetric with respect to 180 degree rotation. To see why this is always possible, we want that if (F,G,H) symmetric with respect to 180 rotation, then so is (F,G,H'). Our base cases W, VW, and UW check out. By induction, we can represent (F,G,H) by (F,FH,H) or (GH,G,H). FH in this example represents the graph F and H adjoined using the construction previously described (see figure 4). These cases are symmetric, so we just consider the case of constucting (F,FH, H') from (F,FH, H).

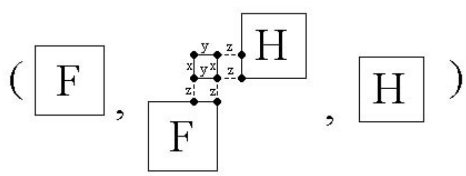

Figure 4: (F,FH,H)

Then choose the location of the unit between FH and F to be opposite in location to the unit connecting the F and H subgraphs in FH (see figure 5). H' = FHF as seen in figure 5(a) and since H and F are symmetric under rotation by 180, we have that H=FHF is symmetric as well. So our construction of snake graphs is valid. Finally, since FH is symmetric under 180 rotation, we may think of taking the FH subgraph in FHF and rotating it by 180 degrees to represent H' as HFF, the graph seen in figure 5(b).

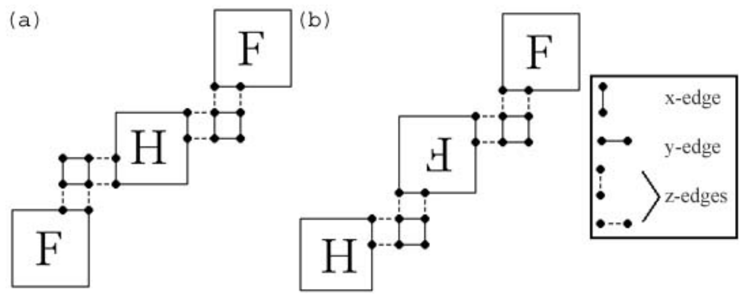

Figure 5: (a)FHF (b) HFF

By this construction, snake graphs of the form $X_n \cdots X_2 W$ are all units consisting of vertical x-edges, horizontal y-edges that are linked by z-edges that connect units together both horizontally or vertically.

To construct Snake graphs of the form $X_n \cdots X_2 U$, we use the exact same construction as above except that we relabel x with z, y with x, and z with y, and we replace the base cases W, VW, and UW with the base cases V, UV, and WV respectively. The snake graphs of W, WV, and UW will look the same as those for V, UV, and WV respectively save a change in variables and that the graphs will be in different positions of the triple (see figure 6). Similarly for Snake graphs of the form $X_n \cdots X_2 V$, we replace x with y, y with z, and z with x, and our base cases W, VW, and UW with base cases U, WU, and VU.

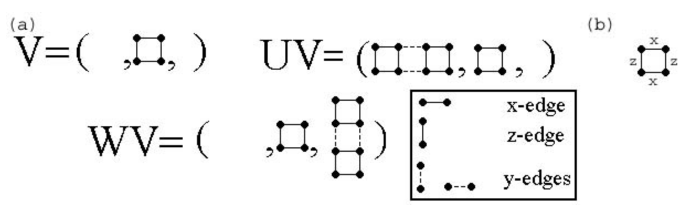

Figure 6: (a) Base cases V, UV, and WV (b)a unit for these cases.

3 Snake Matching Polynomials

We associate to a snake graph F a polynomial M(F). To construct this polynomial, we take a matching of F to represent a monomial term $x^a y^b z^c$ where a,b, and c correspond to the number of x, y, and z-edges used respectively. Then M(F) is the sum over all matchings of these monomials.

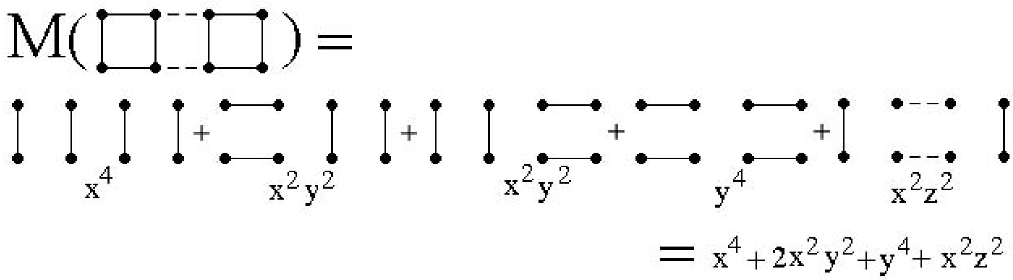

Figure 7: A graph, its matchings, and its matching polynomial.

4 Snake Markov Polynomials

Given M(F), we construct its Snake Markov Polynomial P(F) using certain rules. For graphs of the form $X_n \cdots W$, divide M(F) by $x^{\frac{h-2}{2}} y^{\frac{w-2}{2}} z^{\frac{n}{4}}$ where n is the number of vertices in F, h is the height of F, and w is the width of F. Here the height is the number of vertices high the graph of F is so a square of four vertices would have a height of 2 while a single vertex has a height of 1. Similarly, the width of is the number of vertices wide F is, so an array 2 vertices high and 4 vertices wide has a width of 4 (see figure 8).

For graphs of the form $X_n \cdots U$, do the same as in the case with $X_n \cdots W$ except in our dividing monomial, replace z by y, y by x, and x by z. For graphs of the form $X_n \cdots V$, do the same as in the case with $X_n \cdots W$ except in our dividing monomial, replace z by x, y by z, and x by z. As a convention, we give the triple of empty graphs (, ,) the Snake Markov polynomials (x,y,z).

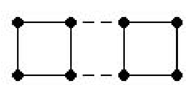

Figure 8: A graph of eight vertices, height 2, and width 4.

Theorem 4.1 $x_n \cdots x_1 = P(X_n \cdots X_1)$, *where* $x_i = X_i$.

That is that the triple of polynomials represented by $x_n \cdots x_1$ is the same as the triple of Snake Matching Polynomials of $X_n \cdots X1$ where $x_i = X_i$ represent the analogous moves of u and U, v and V, and w and W acting on a triple of polynomials or a triple of Snake graphs.

5 Proofs

There are many different ways of proving the above theorem. The most direct proofs are based on graphical condensation (as described in [Ek]???). A formal proof using this technique is presented here, as well as the sketch of a second proof. Furthermore, we present an important property of cuts on the Snake graphs that leads to two other proofs. One of these proofs is formally presented, whereas the second proof is only sketched.

5.1 Equations

All proofs here presented will be based on mathematical induction and on the fact that Theorem 4.1 is in fact true for many base cases. All the work will be concentrated on proving that the operators u, v, w are equivalent to U, V, W. If this is so, the structural induction step will be proved, and, therefore Theorem 4.1 will be true. But to prove that u, v, w are equivalent to U, V, W is to prove equations (1) and (2) for the operators U, V, W. In other words, given a triple of graphs (F, G, H) and $W(F, G, H) = (F, G, H')$ we have to prove that:

$$P(H)P(H') = P(F)^2 + P(G)^2 \tag{3}$$

$$P(H) + P(H') = K(x, y, z)P(F)P(G) \tag{4}$$

Let $G = FH$, i.e. $(F, G, H) = V(F, G', H)$. We can manipulate the equations above to obtain simpler equations that depend more on the matchings of the graphs F, G and H, and no so much on their dimensions.

$$M(H)M(H') = z^{\frac{n_H}{2}+2} y^{w_H} x^{h_H} M(F)^2 + M(G)^2 \tag{5}$$

$$z^{\frac{n_F}{2}+2} y^{w_F} x^{h_F} M(H) + M(H') = (x^2 + y^2 + z^2)M(F)M(G) \tag{6}$$

Here n_H is the number of vertices in H, w_H is the width of H and h_H is the height of H. Similar definitions apply to n_F, w_F and h_F.

5.2 Graphical Condensation

Graphical condensation consists of merging two graphs into a double graph, and then splitting this double graph into two other graphs. The idea is finding a relation between the matchings of the inital graphs and matchings of the resulting ones. In fact, Eric Kuo, in his article about graphical condensation proved the following theorems:

Theorem 5.1 *Let $G = (V_1, V_2, E)$ be a planar bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c and d appear on G as in figure 9(a). If $a, c \in V_1$ and $b, d \in V_2$, then*

$$W(G)W(G - \{a, b, c, d\}) =$$
$$W(G - \{a, b\})W(G - \{c, d\}) + W(G - \{a, d\})W(G - \{b, c\})$$

Theorem 5.2 *Let $G = (V_1, V_2, E)$ be a planar bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c and d appear on G, as in figure 9(b). If $a, b \in V_1$ and $c, d \in V_2$, then*

$$W(G - \{a, d\})W(G - \{b, c\}) =$$
$$W(G)W(G - \{a, b, c, d\}) + W(G - \{a, c\})W(G - \{b, d\})$$

Here $W(G)$ means the weighted sum of the matchings of graph G, and $G - U$, where U is a set of vertices, is the graphs obtained by removing all the vertices in U from G, and also all edges connected to those vertices.

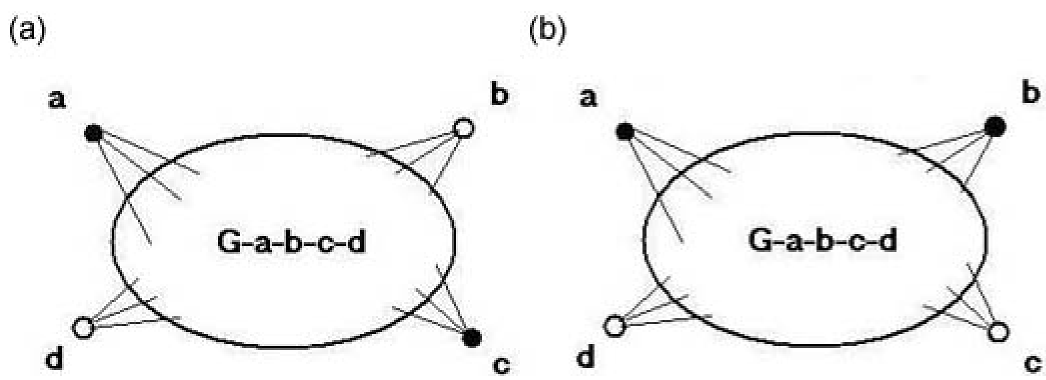

Figure 9: (a) Set up for Theorem 5.1. (b) Set up for Theorem 5.2.

5.3 Using Graphical Condensation I

The first proof will show that equations (6) holds by an application of Theorem 5.2.

Since $(F, G, H') = W(V(F, G', H)$ (by induction??), we can represent H' as a combination of $F's, H's$ and square units as in figure 10.

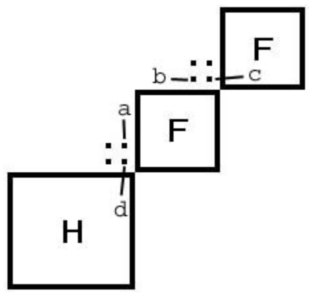

Figure 10: HFF and labeled vertices for the first graphical condensation proof.

Note that we label the vertices a, b, c, d and that if our snakes are thought of as bipartite graphs then a and b belong to the same half of the graph, and c and d to a different half. Hence, we can apply theorem 5.2.

If we remove the vertices a and d from H' we obtain a disconneted graph with three components. The first component is an X-edge alone, the second is a copy of H and the third one is two copies of F connected by a unit square. The unit square can be matched using two X-edges or two Y-edges, in which case we obtain $M(H)M(F)^2x^2$ or $M(H)M(F)^2y^2$. If, instead, we use Z-edges to connect the unit square to one of the copies of F, then in total we obtain $M(H)M(F)^2z^2$ (MORE EXPLANATION????), since F is symmetric. Therefore,

$$M(H' - \{a, d\}) \;=\; (x^2 + y^2 + z^2)M(H)M(F)^2 \tag{7}$$

If we remove b and c from H' we obtain HF = G, F and an y edge. Hence,

$$M(H' - \{b, c\}) = yM(G)MF) \tag{8}$$

If we remove a and c, or if we remove b and d then we force a matching of the central F part of H'. We can simply count the edges to obtain,

$$M(H' - \{a, c\}) \;=\; z^{\frac{n_F}{4}+1}y^{\frac{w_F}{2}}x^{\frac{h_F}{2}+1}M(F)M(H) \tag{9}$$

$$M(H' - \{b,d\}) \quad = \quad z^{\frac{n_F}{4}+1} y^{\frac{w_F}{2}+1} x^{\frac{h_F}{2}} M(F)M(H) \qquad (10)$$

Using Theorem 5.2 we can put equations (7)-(10) together, and by cancelling a factor of $xyM(H)M(F)^2$ we obtain

$$z^{\frac{n_F}{2}+2} y^{w_F} x^{h_F} M(H) + M(H') \quad = \quad (x^2+y^2+z^2)M(F)M(G) \qquad (11)$$

5.4 Using Graphical Condensation II

We can also apply theorem 5.1 to the snake graphs by reassigning the vertices a, b, c, d (see figure 11), but the new proof is very similar to the previous one and does not provide us with any further information.

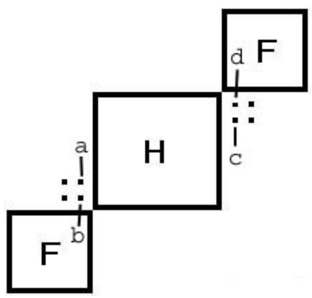

Figure 11: FHF and labeled vertices for the second graphical condensation proof.

5.5 Cuts of Snakes

Snake graphs have the property that their width along the "spinal column" of the snake is always two. This fact can be used to prove a useful property relating cuts a matchings of snakes. Consider two different matchings of the same snake. If we "superimpose" them into a single snake, in general, we can find a cut through that "double-snake" that sperates the corners of the snake. But there are particular matchings for which no cuts of the "double-snake" can be found. Those matchings must, together, use all the edges that consitute the border of the graph.

There are, then, only two matchings of the same snake that do not share a common cut.. One of those matchings has only X-edges and horizontal Z-edges and the other one has only Y-edges and vertical Z-edges.

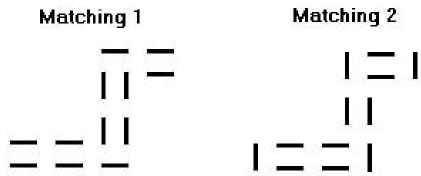

Figure 12: The two matchings of a particular snake graph with no common cut.

5.6 Using Cuts I

This special property of snakes leads to two other proofs of Theorem 4.1. First we proof equation (5), and then we sketch the proof of equation (6). Equation (5) represents a bijection between the matchings of the graphs $H \cup H'$ and $F \cup F$ plus $G \cup G$.

First note that $G \cup G$ can be represented by the union of a copy of G with a rotated copy of G.

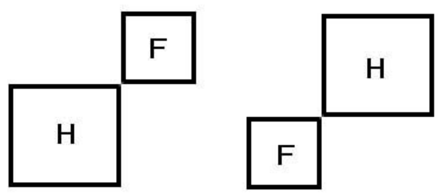

Figure 13: Using Cuts I.

Note that if one of the pairs of Z-edges connecting to H is not being used then we can rearrange the parts of this graph to obtain a matching of $H \cup H'$. If both pairs of Z-edges are being used, then the H parts will have a common cut (since they both have the ending vertices of a Y-edge being used by "exterior" graphs), and we can rearrange the parts of the graph to get a matching of $H \cup H'$

This is where figure 14 goes.

Now there is only one way to match the H part of H' so H' and H do not have a common cut. In this case, we get $z^{\frac{n_H}{2}+2}y^{w_H}x^{h_H}M(F)^2$ matchings. Therefore,

$$M(H)M(H') = z^{\frac{n_H}{2}+2}y^{w_H}x^{h_H}M(F)^2 + M(G)^2$$

5.7 Using Cuts II

Note: This proof is tricky. Including it would either lend itself to a good amount of space as seen below, or a brief mention with perhaps an appendix of the details in the back.

This proof shows that equation (6) is true using an argument with cuts. To do this, we use represent H' as in the graph of figure 5(b) to look like HFF. Then equation 6 may be rewritten as

$$z^{\frac{n_F}{2}+2}y^{w_F}x^{h_F}M(H) + M(HFF) = (x^2 + y^2 + z^2)M(F)M(HF)$$

Case 1: Start by considering the matchings of HFF. Look at the unit that adjoins the two F subgraphs together. If the unit is not connected by z-edges to either of the F subgraphs of HFF, then we may break this up into a matching of F, a matching of HF and either an x^2 or y^2 term (figure 15).

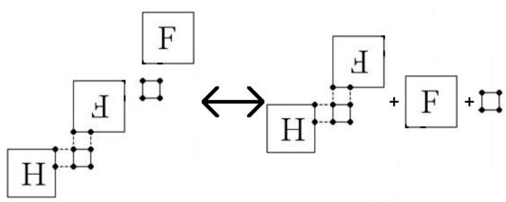

Figure 14: A bijection between matchings of HFF as in Case 1 and $(x^2 + y^2)M(F)M(HF)$

So this takes matchings with no z-edges between the two F subgraphs of HFF on the left-hand side of the equation to $(x^2 + y^2)M(F)M(HF)$ on the right-hand side.

The remaining two possibilities are that the unit between the two F subgraphs is connected to HF or it is connected to F.

Case 2: A matching of HFF in which the unit is connected to HF has an equivalent snake matching monomial as a z^2 term, a matching of HF subject to the condition that the edge between vertices a and b is used, and a matching of F (see figure 16). So the remaining matchings of the right hand side we need to pair with the left hand side are matchings of HF in which the edge between b and c is used union a matching of F union a z^2 term.

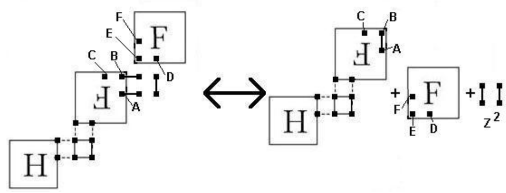

Figure 15: Decomposition of Case 2 matchings.

Case 3: A matching of HFF in which the unit is connected to F has an equivalent snake matching monomial as a z^2 term, a matching of HF, and a matching of F subject to the condition that the edge between d and e is used (see figure 17).

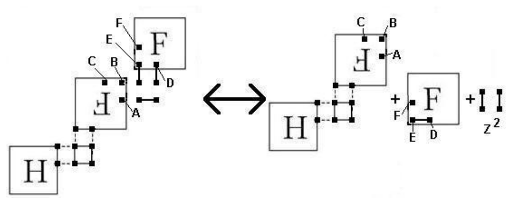

Figure 16: Decomposition of Case 3 matchings.

Now we may partially cancel on both sides of our equation. We may cancel out terms in both matchings in which the bc and de edges are used. On the left-hand side, this leaves us with matchings in which the ab-edge is used and the de-edge is used. On the right hand side, this leaves us with matchings in which the bc-edge is used and the ef-edge is used (see figure 18).

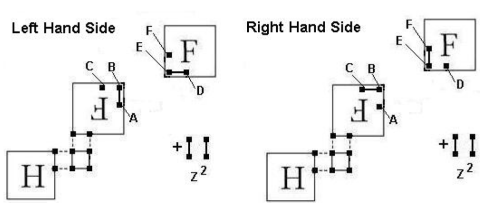

Figure 17: Remaining terms to be canceled.

Now the trick is that the left hand side always contains a common cut between the two F subgraphs. To see this, rotate the upper F subgraph by

180 degrees. By symmetry, we are then comparing two matchings of F. The worst case scenario is if we have the one pair of matchings of F that share no common cut. But in this case, it follows that the F subgraph in the HF subgraph is not by z-edges to the unit between H and F (see figure 19).

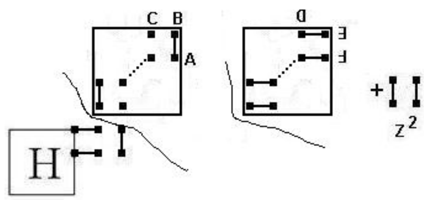

Figure 18: Worst case scenario for finding a common cut.

So in there is always a common cut.In this case, we swap the cut pieces and rotate the new F matching by 180 degrees to obtain a matching in which the new bc-edge is used and the new ef-edge is used (see figure 20). This matches up to all remaining matchings of the RHS except for those in which the bc-edge is used, the ef-edge is used, and in which there does not exist a common cut between the two F graphs. These then have snake matching polynomials that are exactly $z^{\frac{n_F}{2}+2}y^{w_F}x^{h_F}M(H)$. Thus we have created a bijection between the two sides of the equation.

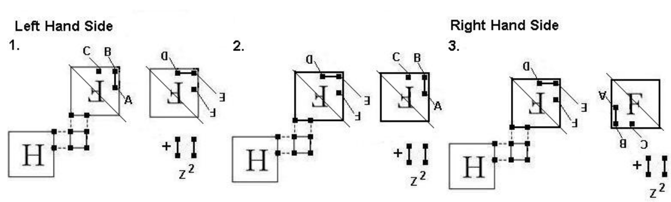

Figure 19: 1. Starting from the left hand side, rotate the F subgraph and find a common cut. 2. Swap pieces of the F subgraphs. 3. The result when the new F subgraph is rotated back yields a graph of the right hand side.

6 A conjectured generalization

There is an obvious generalization of the Markov numbers. If we consider n-tuples instead of triples of numbers, we can re-define inductive steps and operators.

Definition: A n-tuple of polynomials $(X_1, X_2, \ldots, X_n)$ is a Markov n-tuple if, and only if, $\sum X_i = K(X_1, X_2, \ldots, X_n) \prod X_i$, where $K(X_1, X_2, \ldots, X_n) = \frac{\sum X_i}{\prod X_i}$

Just as in the case with three polynomials, we can reach all of these polynomials by applying operators similar to:

$$O_n(X_1, X_2, \ldots, X_n) = (X_1, X_2, \ldots, X'_n)$$

, where $X'_n = \frac{\sum_{i=1}^{n-1} X_i}{X_n}$.

6.1 Laurent polynomials

There is a simple proof that all the polynomials generated by the above operators will be Laurent polynomials. We will prove that $K(X_1, X_2, \ldots, X_n) = K(X_1, X_2, \ldots, X'_n)$, and if so X_i is just the product and sum of Laurent polynomials, and therefore it is also a Laurent polynomail.

$X_n {X'_n}^2 = X'_n \sum_{i=1}^{n-1} X_i^2$

and

$X_n \sum_{i=1}^{n-1} X_i^2 = X'_n X_n^2$

Thus,

$X_n(\sum_{i=1}^{n-1} X_i^2 + {X'_n}^2) = X'_n \sum_{i=1}^{n} X_i^2 \leftrightarrow \frac{\sum_{i=1}^{n-1} X_i^2 + {X'_n}^2}{X'_n} = \frac{\sum_{i=1}^{n} X_i^2}{X_n} \leftrightarrow$
$\leftrightarrow K(X_1, X_2, \ldots, X_n) = K(X_1, X_2, \ldots, X'_n)$

6.2 A Guess

The Laurentness of n-tuples of polynomials strongly suggests that there exists a combinatorial interpretation for these polynomials. Furthermore, we should expect that the snakes graphs are particular cases of a general interpretation. Our best guess for a general intepretation so far is to use $(n-1)$-dimensional units instead of square units in the construction of our graphs. These new units will have $n-1$ matchings. The hard problem is to find out which edges connect to each edges in inter-unit connections (equivalente to our Z-edges).

Should I put the neat "Fibonacci" example in the case with 4 variables???