Combinatorial Interpretations for the Markoff Numbers

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Abstract

In this paper we generalize the concept of Markoff numbers with the construction of Markoff polynomials. We then present a combinatorial model based on a family of recursively-defined graphs which we call snakes, such that the Markoff polynomials and Markoff numbers are respectively weighted and unweighted enumerators of the perfect matchings of snakes. We have found four proofs of the central combinatorial theorem; we present two of them formally and sketch the other two. At the end we conjecture a generalization of this model.

1 Markoff Polynomials

1.1 Introduction

Markoff numbers are positive integers a, b, c that occur in triples (a, b, c) ("Markoff triples") satisfying $a^2 + b^2 + c^2 = 3abc$ (e.g., $a = 2$, $b = 5$, $c = 29$). Such triples have beens studied in detail for more than a century, and play a role in both Diophantine approximation and the study of lengths of geodesics on hyperbolic surfaces; see pages 187–189 of [?] and the web-page [?] for short expositions on the basic facts about Markoff triples and the web-page [?] for a detailed bibliography.

Perhaps the most important basic fact about Markoff triples is that if (a, b, c) is a Markoff triple then so is (a, b, c') where $c' = (a^2 + b^2)/c$ (and likewise for similar operations that replace a or b). To see this, note that the equation $a^2 + b^2 + c^2 = 3abc$ is quadratic in c; writing it in the form

$c^2 - (3ab)c + (a^2 + b^2) = 0$, we see that, for fixed a, b, the two roots c, c' satisfy $c + c' = 3ab$ and $cc' = a^2 + b^2$. The first equation tells us that c' is an integer (since that a, b, and c are), and the second equation tells us that c' is positive (since that a, b, and c are).

Almost as important is the fact that every Markoff triple can be obtained from the initial Markoff triple $(1, 1, 1)$ by repeatedly applying the replacement operation $(a, b, c) \mapsto (a, b, c')$ and its two siblings. This was first proved by Markoff himself [**?**], and is easily proved by a descent argument.

Fomin and Zelevinsky recognized that these three replacement operations generate a cluster algebra (see [**?**] for background on cluster algebras) and studied triples of rational functions obtained from the triple (x, y, z) of formal indeterminates by repeated application of the three replacement operations. Following this idea, we study triples of rational functions $(f(x, y, z), g(x, y, z), h(x, y, z))$ satisfying the relation

$$f^2 + g^2 + h^2 = \frac{x^2 + y^2 + z^2}{xyz} fgh$$

Equivalently, if we let

$$K(a, b, c) := \frac{a^2 + b^2 + c^2}{abc}$$

we may write the above relation as

$$K(f, g, h) = K(x, y, z)$$

Clearly the triple (x, y, z) satisfies this relation. We define replacement operations u, v and w:

$$u(f, g, h) = (\frac{g^2 + h^2}{f}, g, h) = (f', g, h),$$

$$v(f, g, h) = (f, \frac{f^2 + h^2}{g}, h) = (f, g', h),$$

$$w(f, g, h) = (f, g, \frac{f^2 + g^2}{h}) = (f, g, h').$$

If (f, g, h) satisfies the defining relation, then so do $u(f, g, h)$, $v(f, g, h)$ and $w(f, g, h)$. Note that u, v, and w are involutions (i.e., $u(u(f, g, h)) = (f, g, h)$, and likewise for v and w); so in considering triples obtained by repeated application of our basic substitutions, we may limit ourselves to compositions of u, v, and w in which none of the three operations occurs twice in succession.

Fomin and Zelevinsky observed that the rational functions that arise from (x, y, z) by repeating application of u, v, and w are *Laurent polynomials* (i.e., elements of $\mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$), and they proved that this holds in a quite general setting (see Theorem 1.10 in [**?**]). We can prove Laurentness directly for our problem by viewing the defining relation

$$f^2 + g^2 + h^2 = \frac{x^2 + y^2 + z^2}{xyz} fgh$$

in turn as a quadratic in f, g, or h; this tells us that f', g', and h', in addition to satisfying the multiplicative relations

$$hh' = f^2 + g^2 \tag{1}$$

etc., also satisfy the additive relations

$$h + h' = K(x, y, z) fg \tag{2}$$

etc. If f, g, and h are Laurent polynomials, then so is $h' = h - K(x, y, z)fg$. Thus, it is easy to see that if (f, g, h) is a triple of Laurent polynomials then so are $u(f, g, h)$, $v(f, g, h)$, and $w(f, g, h)$. We refer to Laurent polynomials that arise in this fashion as *Markoff polynomials*.

Fomin and Zelevinsky also observed that the coefficients of Markoff polynomials seem to be positive, but they did not prove that this holds in general, as such positivity assertions were not within the scope of the methods they used. One might at first suppose that the relation (1) gives a proof of positivity by induction, but it does not; there are many circumstances in which a ratio of two polynomials with positive coefficients is a polynomial with some coefficients negative (e.g., $(x^3 + y^3)/(x + y)$).

In this article, we will prove the positivity conjecture of Fomin and Zelevinsky by giving a combinatorial interpretation of all the coefficients. Specifically, we will show that each Markoff polynomial $p(x, y, z)$ is associated with a finite graph G whose edges are labeled x, x^{-1}, y, y^{-1}, z, and z^{-1}. A *perfect matching* of such a graph G is a collection of edges of G such that each vertex of G belongs to one and only one edge in the collection. If we assign to each perfect matching a *weight* equal to the product of the labels associated with its edges then the numerator of $p(x, y, z)$ (when $p(x, y, z)$ is written in reduced form as a polynomial divided by a monomial) turns out to be the sum of the weights of all the perfect matchings of G, which by its very definition can be seen to be a Laurent polynomial with positive coefficients.

Two important special cases of Markoff numbers arise from applying the replacement operations in the sequence $u, v, u, v, u, v, \ldots$ and in the sequence $u, v, w, u, v, w, \ldots$. In the former case, the numbers grow as slowly as possible, and are just the alternate Fibonacci numbers $1, 2, 5, 13, 34, 89, \ldots$; our general combinatorial interpretation Markoff numbers specializes in this case to a well-known combinatorial interpretation of the Fibonacci numbers in terms of perfect matchings of the 2-by-n grid (see e.g. [**?**]). The case of the replacement-sequence $u, v, w, u, v, w, \ldots$, whose terms grow as rapidly as possible, is less well-known; the resulting sequence $1, 2, 5, 29, 433, 37666, \ldots$ was most recently reinvented by Dana Scott [**?**]. We show that these successive numbers count the perfect matchings in an infinite sequence of graphs, each nested in the next. Moreover, the union of these graphs is of interest in its own right. [ADD SOMETHING ABOUT THIS? MAYBE LATER?]

An important and long-standing conjecture about Markoff numbers concerns their purported "uniqueness". That is, each Markoff number arises from repeated substitution operations in essentially only one way. To make this precise, we define a graph whose vertices are Markoff triples, with an edge joining two vertices if the associated Markoff triples are related by an elementary substitution operation. Markoff showed that this graph is connected. It is fairly simple to show that this connected graph must in fact be a 3-regular tree (this only requires that one show that if (a, b, c) is a Markoff triple with $a < b < c$, then $a' > a$ and $b' > b$, so that the triples (a', b, c) and (a, b', c) are both more "complicated" than the triple (a, b, c)). Thus the Markoff triples are in bijection with *reduced words* consisting of the symbols u, v, and w such that no symbol appears twice in succession.

The uniqueness claim may be stated as follows: for every Markoff number m, the Markoff triples of the form (a, b, m) form a connected 2-regular subtree (i.e., a bi-infinite path) in the tree, related by substitution operations that replace a or b but not m.

Our combinatorial interpretation of Markoff numbers does not shed any direct light on the uniqueness conjecture. However, we will show in a separate article that the Markoff polynomials are unique in the way that Markoff numbers are conjectured to be. This result has an application to the study of hyperbolic surfaces; in particular, it implies that almost all hyperbolic structures on the once-punctured torus have the property that all simple geodesics that are not related by a symmetry of the surface have distinct hyperbolic length.

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1.2 Snake graphs

1.3 Definition

Snakes are (undirected) weighted graphs that will serve as our combinatorial model for the Markoff polynomials. More specifically, snakes are finite subgraphs of the square grid formed by taking a union of *cells*, where a cell consists of four cyclically connected vertices of the grid and the edges that join them, and where the cells that form a snake can be ordered in a linear fashion, so that each cell is either immediately to the right of or immediately above the preceding cell.

Like Markoff polynomials, snakes come in triples and are constructed with the use of three operations that act on these triples. The starting point is simply the triple of empty graphs. We will call the graph operations U, V and W, and as the notation suggests they will have effects similar to those of the operations u, v and w. A snake will be represented by a string of the letters U, V and W, where we do not allow a letter to appear consecutively (corresponding to the fact that the symbols u, v, and w cannot be repeated in the reduced word associated with a triple of Markoff numbers).

The operation W will replace the third component of the triple (F, G, H) by a new graph H' obtained by joining together a copy of F and a copy of G with an intervening extra unit containing four vertices and four edges, two of weight x and two of weight y (a *square unit*). The square unit will be connected to the ends of the other two components by four edges of weight z, as shown in Figure 1.

Note that the square unit could be connected to the snakes in two different

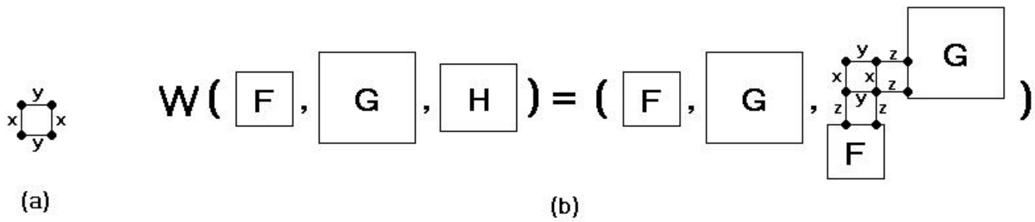

Figure 1: (a) The square unit. (b) The operation W in action.

ways: with the square unit appearing above F and to the left of G, as shown, or appearing to the right of F and below G. As will be shown later, exactly one of these possible positions will form a graph that is invariant under 180 degree rotation, and that will be the correct position of the square unit. We will informally write H as FG to signify that H results from joining F to G by way of a square unit, with F to the left of, and below, G; we say G is on top of F. (Note that because of the aforementioned rotational symmetry, we could also obtain the graph H' as GF, with F on top.)

The operations U and V are defined analogously, with U acting on the first component of the triple and V acting on the second component. A small drawback of our model is that this assignment of weights to the edges will only generate a direct correspondence with Markoff polynomials if the operation W is the first one applied. Otherwise, a different assignment of weights to edges is required. However, in the study of Markoff numbers we lose no generality in assuming that w is applied first (and indeed, we lose no generality in further assuming that v is applied second, although we will not make use of this fact here). Hence in succeeding sections we will only consider snakes constructed through a sequence of operations of which the first is W.

Figure 2 shows an example of a possible sequence of application of the operations U, V and W to the initial triple (whose three components are all the empty graph).

Note that X-edges (i.e., edges labelled x) are always vertical, Y-edges always horizontal, and Z-edges can have either orientation.

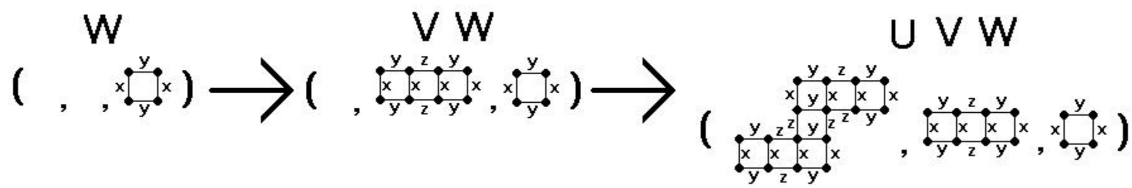

Figure 2: An example of snakes.

1.4 Symmetry

To complete the definition of snakes we must prove that one of the possible positions of the square unit (shown in Figure 3) forms a symmetric snake.

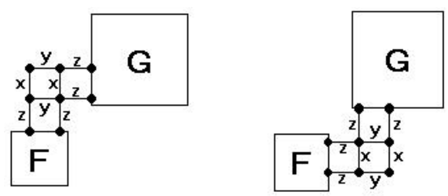

Figure 3: The two positions for the square unit.

The proof is a simple induction. Consider a triple (F, G, H) obtained by repeated applications of the operations U, V, and W; without loss of generality, we may assume that the operation W was the operation applied most recently, so that $H = FG$, as shown in Figure 4(a). Making use of the induction assumption, we may assume that F, G, and FG are all symmetric. Then, if we apply (say) V to this triple, the new snake we construct can have either F or FG on top. For each of these cases there is a unique position of the square unit that makes the resulting FGF (or FFG) snake symmetric, shown in Figure 4. It is visually clear that snake (b) is symmetric if F and G are symmetric. The symmetry of snake (c) may be less evident, but it is in fact the same graph as (b), since FG has been assumed to be symmetric. That is, we can rotate the top FG in snake (c); on the one hand this converts (c) into (b), but on the other hand it has no effect since FG is invariant under rotation. Therefore, snake (c) is also symmetric. Later on, we will make use of both representations shown here in order to derive important facts about snakes.

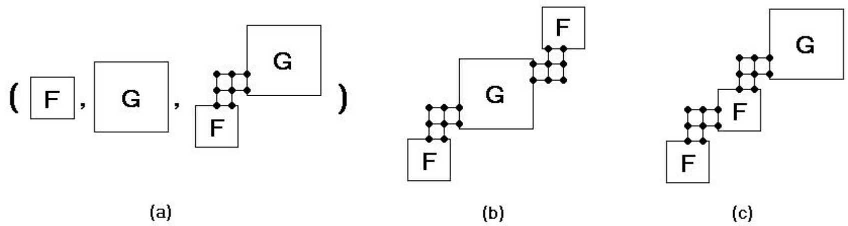

Figure 4: (a) The initial triple. (b) F on top. (c) FG on top.

2 Snake Markoff Polynomials

2.1 Definition

We will represent the weighted sum of all (perfect) matchings of a snake F by $M(F)$, and we will call it the snake matching polynomial. (Hereafter, we will drop the modifier "perfect" and use the term "matching" to signify a perfect matching of a graph.) The snake Markoff Polynomial $P(F)$ is defined to be $M(F)$ divided by a monomial on x, y and z; the exponents of this monomial will depend on the snake and the first operation applied. If W is the first operation applied to the base case, as we have been assuming so far, then the monomial is $x^{(h-2)/2}y^{(w-2)/2}z^{n/4}$, where h is the height of the snake, w its width and n the number of vertices. E.g., the snake represented in Figure 5 has height 2, width 4 and 8 vertices. Its snake matching polynomial is $M(F) = x^4 + 2x^2y^2 + y^4 + x^2z^2$ and its snake Markoff polynomial is $M(F)$ divided by yz^2.

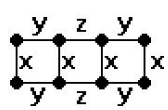

Figure 5: (a) The VW snake.

2.2 A main theorem

Each non-empty reduced word in the symbols u, v, and w whose rightmost (i.e. "first") symbol is w corresponds to a Markoff number-triple with a unique largest element m; we may call this reduced word the "name" of the associated Markoff polynomial $p(x,y,z)$. E.g., the word vw is associated with the (Laurent) polynomial

$$\frac{x^2+\left(\frac{x^2+y^2}{z}\right)^2}{y} = \frac{x^4+2x^2y^2+y^4+x^2z^2}{yz^2}.$$

Likewise, each non-empty reduced word in the symbols U, V, and W whose rightmost symbol is W corresponds to a triple of snakes whose largest (i.e. last-created) snake is some snake G. We may call the reduced word the "name" of the graph G. E.g., the word VW is the name of the snake shown in Figure 5.

Theorem 2.1 *Let $x_n \cdots x_1$ be a reduced word representing a Markoff polynomial $p(x,y,z)$, and $X_n \cdots X_1$ be a reduced word representing a snake G, where X_i is the upper case version of x_i for $1 \le i \le n$. Then $p(x,y,z) = P(G)$.*

In other words, the Markoff polynomial represented by $x_n \cdots x_1$ is the snake Markoff polynomial of the snake represented by $X_n \cdots X_1$, where X_i is U, V, or W according to whether x_i is u, v, or w.

The reader may check directly that the theorem holds for the words VW and vw.

3 Proofs

There are many different ways to prove the above theorem. The most direct proofs are based on a technique called graphical condensation (as described in [?]). A formal proof that makes uses of this technique is presented here, as well as the sketch of a second proof. Furthermore, we present an important property of cuts on the snake graphs that leads to two other proofs of the same theorem. One of these proofs is formally presented, whereas the second proof is only sketched.

3.1 Equations

All proofs here presented will be based on mathematical induction and on the fact that Theorem 3.1 is in fact true for many base cases, which we will not check here. We will be concentrate our efforts on proving that the operations u, v and w are equivalent to U, V and W. If this is so, the structural induction step will be proved, and, therefore Theorem 3.1 will be true. But to prove that u, v and w are equivalent to U, V and W is to prove either (1) or (2) for the the snake Markoff Polynomials of triples of snakes generated by U, V and W. In other words, given a triple of graphs (F, G, H) and $W(F, G, H) = (F, G, H')$ we have to prove that:

$$P(H)P(H') = P(F)^2 + P(G)^2 \quad (3)$$

$$P(H) + P(H') = K(x, y, z)P(F)P(G) \quad (4)$$

(The claims for $U(F, G, H)$ and $V(F, G, H)$ are equivalent to the claim for $W(F, G, H)$; the difference amount to relabeling the elements of the triple of graphs.)

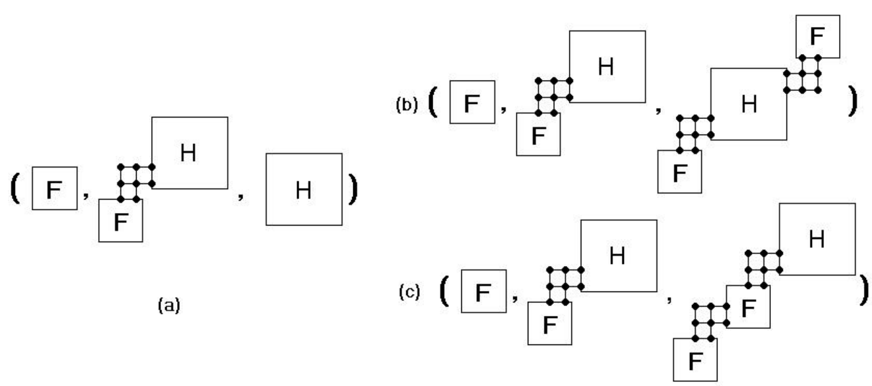

Figure 6: (a) (F, G, H) (b) $W(F, G, H)$ corresponding to (3). (c) $W(F, G, H)$ corresponding to (4).

Let $G = FH$, i.e. $(F, G, H) = V(F, G', H)$. We can "simplify" the above equations by canceling out some of the terms in the denominators. Here we explicitly simplify equation (3). By inspection, we have that the powers of x,y, and z in the denominators of $H' = FHF$, H, F, and $G = FH$ are as in Table 1 where n_H is the number of vertices in H, w_H is the width of H, and h_H is the height of H, and similar definitions apply to n_F, w_F and h_F.

Applying this to equation (3), note that the sum of respective sums of the powers of x, y, and z in H and H' are equal to twice the powers of G. What this means is if we multiply equation (3) by the denominator of $P(H)P(H')$, it cancels out the denominator of both $P(H)P(H')$ and $P(G)^2$

Graph	Power of x	Power of y	Power of z
$H' = FHF$	$\frac{2h_F+h_H-2}{2}$	$\frac{2w_F+w_H-2}{2}$	$\frac{2n_F+n_H+8}{4}$
H	$\frac{h_H-2}{2}$	$\frac{w_H}{2}$	$\frac{n_H}{4}$
F	$\frac{h_F-2}{2}$	$\frac{w_F-2}{2}$	$\frac{n_F}{4}$
$G = FH$	$\frac{h_F+h_H-2}{2}$	$\frac{w_F+w_H-2}{2}$	$\frac{n_F+n_H+4}{4}$

Table 1: The powers of x,y, and z in the denominator for terms in equation (3).

to yield the terms $M(H)M(H')$ and $M(G)^2$. Finally, if we multiply through by the denominator of $P(H)P(H')$ and consider the $P(F)^2$ term, we get

$$\begin{aligned} x^{\frac{2h_F+2h_H-4}{2}} y^{\frac{2w_F+2w_H-4}{2}} z^{\frac{2n_F+2n_H+8}{4}} P(F)^2 &= \frac{x^{\frac{2h_F+2h_H-4}{2}} y^{\frac{2w_F+2w_H-4}{2}} z^{\frac{2n_F+2n_H+8}{4}}}{x^{\frac{2h_F-4}{2}} y^{\frac{2w_F-4}{2}} z^{\frac{2n_F}{4}}} M(F)^2 \\ &= x^{h_H} y^{w_H} z^{(n_H)/2+2} M(F)^2 \end{aligned}$$

The final result is that equation (3) may be rewritten as

$$M(H)M(H') = x^{h_H} y^{w_H} z^{(n_H)/2+2} M(F)^2 + M(G)^2 \quad (5)$$

A similar calculation simplifies equation (4).

$$z^{(n_F)/2+2} y^{w_F} x^{h_F} M(H) + M(H') = (x^2 + y^2 + z^2) M(F) M(G) \ , \quad (6)$$

3.2 Graphical Condensation

From graph theory, a bipartite graph is one in which the vertex set may be partitioned into two sets such that no two vertices in within the same set are adjacent to one another (i.e. no single edge connects the two vertices). The bipartition of the vertices is equivalent to coloring every vertex one of two colors to satisfy the above condition. Applied to snake graphs, placing the opposite corners of a square unit in the same vertex set (making them the same color) demonstrates that it is a bipartite graph. Subsequent snake graphs by construction are also bipartite (see figure 7).

To prove theorem 2.1, we use graphical condensation as outlined in by Kuo in [?]. We restate the relevant results of [?] below.

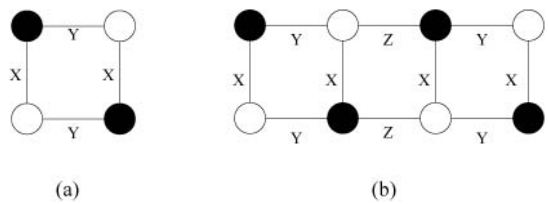

Figure 7: (a)The square unit as a bipartite graph. (b)VW as a bipartite graph constructed from square units.

Theorem 3.1 *Let $G = (V_1, V_2, E)$ be a planar bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c and d appear on G as in Figure 8(a). If $a, c \in V_1$ and $b, d \in V_2$, then*

$$R(G)R(G - \{a, b, c, d\}) = \\ R(G - \{a, b\})R(G - \{c, d\}) + R(G - \{a, d\})R(G - \{b, c\})$$

Theorem 3.2 *Let $G = (V_1, V_2, E)$ be a planar bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c and d appear on G, as in Figure 8(b). If $a, b \in V_1$ and $c, d \in V_2$, then*

$$R(G - \{a, d\})R(G - \{b, c\}) = \\ R(G)R(G - \{a, b, c, d\}) + R(G - \{a, c\})R(G - \{b, d\})$$

Here $R(G)$ represents the weighted sum of the matchings of graph G, and $G - S$, where S is a set of vertices, is the graph obtained by removing all the vertices in S from G, and also all edges connected to those vertices.

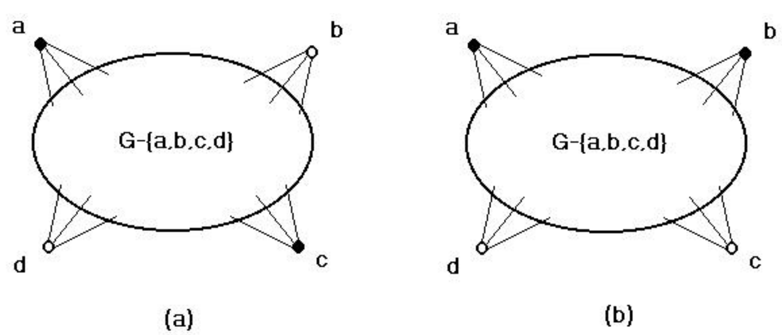

Figure 8: (a) Set-up for Theorem 3.1. (b) Set-up for Theorem 3.2.

3.3 Using Graphical Condensation I

The first proof will show that equation (6) holds by an application of Theorem 3.2 to the graph H' (Figure 7).

As defined before $(F, G, H') = W(V(F, G', H))$, thus we can represent H' as a combination of Fs, Hs and square units as in Figure 9.

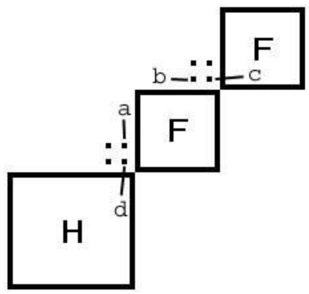

Figure 9: $H' = HFF$ and its labeled vertices for the first graphical condensation proof.

Note that if we label the vertices a, b, c, d as in Figure 9 and our snakes are thought of as bipartite graphs, then a and b belong to the same half of the graph, and c and d to a different half. Hence, we can apply theorem 3.2.

Let us count the weighted matchings of each of the graphs necessary for the application of theorem 3.2:

- $H' - \{a, d\}$
  We obtain three disconnected components:
  - An X-edge;
  - A copy of H;
  - Two copies of F connected by a unit square. The unit square has two matchings. One uses two X-edges, in which case we obtain $M(F)^2x^2$ matchings for FF, and the other one uses two Y-edges, in which case we obtain $M(F)^2y^2$ matchings for FF. There is also a third possibility in which we use the Z-edges connected to the unit square. By the symmetry of F, it is not hard to see that this case generates $M(F)^2z^2$ new matchings of FF.

  Therefore,

$$M(H' - \{a, d\}) \quad = \quad x(x^2 + y^2 + z^2)M(H)M(F)^2 \tag{7}$$

- $H' - \{b, c\}$
  Again, we obtain three disconnected components:
  - A Y-edge;
  - A copy of F;
  - A copy of $HF = G$.

  Therefore,

$$M(H' - \{b, c\}) \; = \; yM(G)M(F) \tag{8}$$

- $H' - \{a, c\}$ or $H' - \{b, d\}$
  The central F part, as well as the two square units of H' have forced matchings, and the other F and H parts of the graph are "free". We can simply count the forced matchings to obtain

$$M(H' - \{a, c\}) \; = \; z^{(n_F)/4+1} y^{(w_F)/2} x^{(h_F)/2+1} M(F)M(H) \tag{9}$$

$$M(H' - \{b, d\}) \; = \; z^{(n_F)/4+1} y^{(w_F)/2+1} x^{(h_F)/2} M(F)M(H) \tag{10}$$

- $H' - \{a, b, c, d\}$

$$M(H' - \{a, b, c, d\}) \; = \; xyM(F)^2 M(H) \tag{11}$$

Using Theorem 3.2 we can put equations (7)-(11) together to obtain

$$z^{(n_F)/2+2} y^{w_F} x^{h_F} M(H) + M(H') \; = \; (x^2 + y^2 + z^2)M(F)M(G) \; , \tag{12}$$

which is exactly the same as equation (6).

3.4 Using Graphical Condensation II

By a similar application of theorem 3.1 to H' with a different assignment of the vertices a, b, c, d (see Figure 8) we can prove equation (5), but the new proof does not provide us with any further information, and it is not shown here.

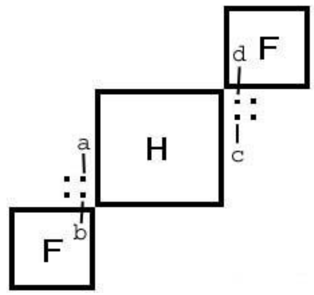

Figure 10: $H' = FHF$ and its labeled vertices for the second graphical condensation proof.

3.5 Cuts of Snakes

Snake graphs have the property that their width along the "spinal column" of the snake is always two. This fact can be used to prove a useful property relating cuts to matchings of snakes. Consider two different matchings of the same snake. If we "superimpose" them into a single snake, in general, we can find a cut through the resulting "double-snake" that separates the corners of the snake. But there are particular matchings for which no cuts of the "double-snake" can be found. Those matchings must, together, use all the edges that constitute the border of the graph.

As in Figure (9) there are only two matchings of the same snake that do not share a common cut when superimposed on top of each other. One of those matchings has only X-edges and horizontal Z-edges (matching 1) and the other one has only Y-edges and vertical Z-edges (matching 2).

3.6 Using Cuts I

This special property of snakes leads to two other proofs of Theorem 3.1. First we prove equation (5), and then we sketch the proof of equation (6).

Equation (5) represents a bijection between the matchings of the graphs $H \cup H'$ and $F \cup F$ plus $G \cup G$.

First note that $G \cup G$ can be represented by the union of a copy of G with a rotated copy of G, since snakes are symmetric.

If one of the pairs of Z-edges connecting to the H parts of $G \cup G$ is not used in a matching, then we can rearrange the parts of this graph to obtain a matching of $H \cup H'$ (Figure 11 - (a) and (b)). If both pairs of Z-edges are being

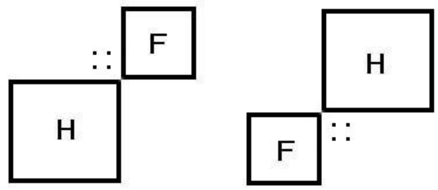

Figure 11: Using Cuts I.

used, then the H parts will have a common cut when superimposed (since they both have the ending vertices of an Y-edge being used by "exterior" graphs, which is equivalent to them both having a Y-edge), and we can rearrange the parts of the graph using a cut to get a matching of $H \cup H'$ (Figure 11 - (c) and (d)).

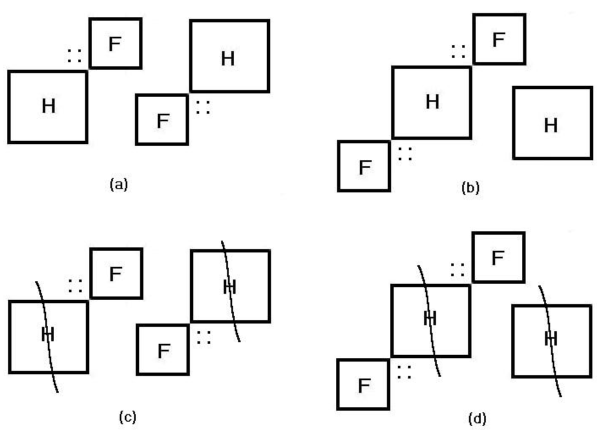

Figure 12: (a) We rearrange the parts of $G \cup G \cdots$ (b) to obtain $H \cup H'$. (c) We use a cut of H to rearrange the parts of $G \cup G \cdots$ (d) to obtain $H \cup H'$.

Now, to complete the bijection, we look at the matchings of the H part in H' that do not share a common cut with the matchings of H in the graph $H \cup H'$. These matchings are unique, as shown in the previous section , and to include them we only have to add $z^{(n_H)/2+2} y^{w_H} x^{h_H} M(F)^2$ matchings to the equation. Therefore,

$$M(H)M(H') = z^{(n_H)/2+2} y^{w_H} x^{h_H} M(F)^2 + M(G)^2 \ ,$$

which is equation (5).

3.7 Using Cuts II

There is still a second proof based on cut arguments, but it will not be shown here, as it is a fairly complicated proof with many unimportant details. The new argument would prove equation (6) by analyzing the different possibilities for matchings of the bottom connecting square unit in Figure 4(c).

4 Snakes: Two Special Cases

We note two special cases of the Markoff numbers and their associated snake graphs. In [?], it was noted that the alternate Fibonacci numbers $1, 2, 5, 13, 34, 89, \ldots$ correspond to matchings of a 2-by-n grid. It is also possible to generate these numbers by applying the replacement operations on $u, v, u, v, u, v, \cdots$ on the initial Markoff triple (1,1,1). Applying the corresponding operations $U, V, U, V, \cdots$ to the initial empty triple of graphs yields snake graphs in which the next triple is created by adjoining the longer of the two 2-by-n snake graphs to an empty graph creating a longer 2-by-n snake graph (see Figure 13). Thus the Markoff combinatorial interpretation specializes to an earlier known interpretation for the Fibonacci numbers. [?]

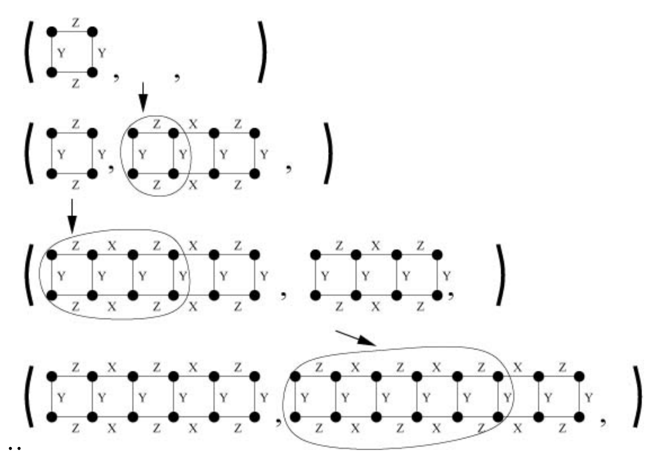

Figure 13: From top to bottom, the snakes corresponding to V,UV,VUV, and $UVUV$, respectively with 2, 5, 13, and 34 matchings.

The second special case we consider are the Markoff numbers generated by the replacement operations $u, v, w, u, v, w, \ldots$. Starting from the initial

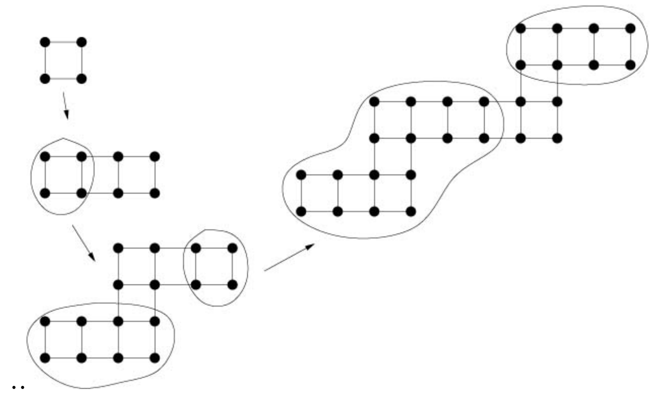

Figure 14: Following the arrows, the graph of U, VU, WVU, and UWVU. Components of the previous graphs that have been adjoined together are circled.

triple of empty graphs, the corresponding operations $U, V, W, U, V, W, \ldots$ correspond to adjoining the two largest snake graphs of a triple to generate a new triple. Thus, this is the fastest growing sequence of Markoff numbers using the replacement operations that can be generated. The largest graphs in the triples U, VU, WVU, and UWVU are shown in Figure 14. Note that each successive snake graph is consists of the previous two snake graphs adjoined together.

As noted by Gabriel Carroll and David Speyer that the snakes generated in this manner thus have slopes approaching the golden ratio or equivalently the inverse of the golden ratio (see Figure 15). The issue of snake graphs and their slopes will be discussed in a separate paper.

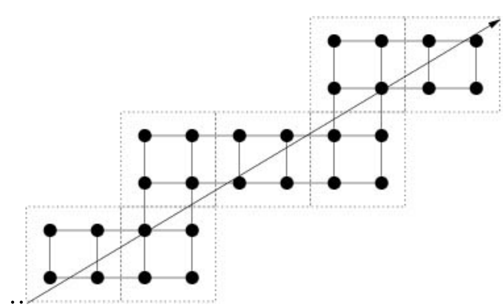

Figure 15: The largest snake graph in the triple UWVU. Embedding it into a lattice shows that this snake in the sequence has slope approximately $\frac{3}{5}$

5 A conjectured generalization

One direct generalization of the Markoff numbers is if we consider n-tuples, instead of triples of numbers, and redefine all operations to act on n-tuples.

5.1 Definition

Let

$$K(X_1, X_2, \ldots, X_n) = \frac{\sum_1^n X_i^2}{\prod_1^n X_i}.$$

If $(x_1, x_2, \ldots, x_n)$ is our starting n-tuple, then $(Y_1, Y_2, \ldots, Y_n)$ is a Markoff n-tuple if and only if

$$K(Y_1, Y_2, \ldots, Y_n) = K(x_1, x_2, \ldots, x_n).$$

Define the operations O_i, similar to O_n as below:

$$O_n(Y_1, Y_2, \ldots, Y_n) = (Y_1, Y_2, \ldots, \frac{\sum_{i=1}^{n-1} Y_i^2}{Y_n}) = (Y_1, Y_2, \ldots, Y_n')$$

It can be algebraically verified that these operations will take Markoff n-tuples to Markoff n-tuples.

5.2 Laurent polynomials

As before, there is a simple proof that all the polynomials generated by the operations O_i will be Laurent polynomials.

$$Y_n + Y_n' = Y_n + \frac{\sum_1^{n-1} Y_i^2}{Y_n} = K(Y_1, Y_2, \ldots, Y_n) \prod_1^{n-1} Y_i = K(x_1, x_2, \ldots, x_n) \prod_1^{n-1} Y_i$$

Therefore, if $Y_1, Y_2, \ldots, Y_n$ are Laurent polynomials then Y_n' is the sum and product of Laurent polynomials and is then a Laurent polynomial itslef.

By induction, all polynomials generated by the operations O_i from $(x_1, x_2, \ldots, x_n)$ will have the Laurentness property.

5.3 A Guess

The Laurentness of n-tuples of polynomials strongly suggests that there exists a combinatorial interpretation for these polynomials. Furthermore, we should expect that snakes are particular cases of a general interpretation. We have not yet been able to generate such model. Our best guess for a general interpretation so far is to use $(n-1)$-dimensional units instead of square units in the construction of our graphs. These new units will have $n-1$ matchings. The hard part of the problem is to find a way of making the "inter-unit" connections (the equivalent of Z-edges in the case $n = 3$).

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