AN ARCTIC CIRCLE THEOREM FOR GROVES

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ABSTRACT. In [4], Jockusch, Propp, and Shor proved a theorem descibing the limiting shape of the boundary between the uniformly tiled corners of a random tiling of an Aztec diamond and the more unpredictable 'temperate zone' in the interior of the region. The so-called arctic circle theorem made precise a phenomenon observed in random tilings of large Aztec diamonds.

Here we examine a related combinatorial model called groves. Created by Carrol and Speyer [1] as combinatorial interpretations for Laurent polynomials given by the cube recurrence, groves have observable frozen regions which we describe precisely here via asymptotic analysis of generating functions borrowed from Pemantle [6].

1. INTRODUCTION

Groves came into existence as combinatorial interpretations of rational functions generated by the cube recurrence:

$$f_{i,j,k}f_{i-1,j-1,k-1} = f_{i-1,j,k}f_{i,j-1,k-1} + f_{i,j-1,k}f_{i-1,j,k-1} + f_{i,j,k-1}f_{i-1,j-1,k},$$

where some initial functions are specified. Typically, $f_{i,j,k} := x_{i,j,k}$ for a certain choice of $(i, j, k) \in \mathbf{Z}^3$ called the *initial conditions*. Fomin and Zelevinsky [3] were able to show that for arbitrary initial conditions the rational functions generated by the cube recurrence were in fact Laurent polynomials in the initial conditions. However, they were unable to prove positivity of all coefficients with their cluster algebra methods [OR DID THEY PROVE POSITIVITY BUT NOT THAT THE COEFFICIENTS WERE ALL EQUAL TO ONE?]. The advent of groves by Carrol and Speyer [1] solved the problem however, by showing that each term of a Laurent polynomial generated by the cube recurrence encodes a unique grove (depending on the initial conditions). In this paper we will only examine the family of groves on *standard initial conditions* as described below.[1]

Before getting into the details of groves, let us first describe the motivation for this paper: random domino tilings of large Aztec diamonds. An Aztec diamond of order n consists of the union of all unit squares with integer vertices contained in the region $\{(x, y) : |x + y| \leq n + 1$. A random domino tiling of a large Aztec diamond consists of two qualitatively different regions. As seen in Figure1, the dominos in the corners of the diamond are *frozen* in a brickwork pattern, whereas the dominoes in the interior have a more random, *temperate* behavior. It was shown in [4], [2], and [5] that asymptotically, the boundary between the frozen and temperate regions in a random tiling is given by the circle inscribed in the Aztec diamond. Since everything outside the circle is expected to be frozen, it is referred to as the *arctic* circle.

[1]Herein we will invoke some of the basic properties of groves without proof. For such arguments, as well as a general treatment of groves and the cube recurrence, the reader is referred to [1].

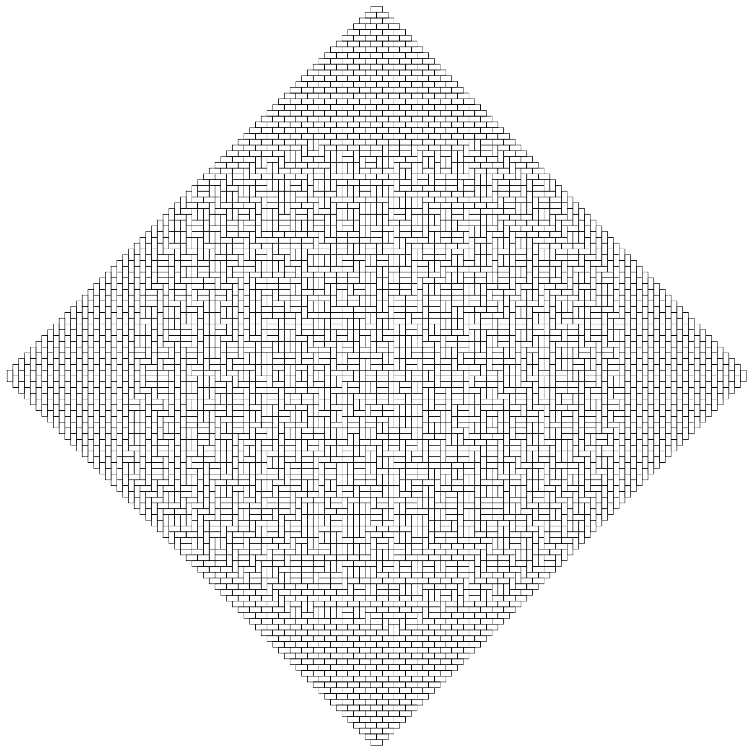

FIGURE 1. A random domino tiling of an Aztec diamond of order 64

In this paper we shall see that groves on standard initial conditions exhibit a very similar behavior. A grove, however, is not a type of tiling. In fact, as the name may suggest, a grove is a collection of trees. As we shall view them, groves are forests on a triangular lattice satisfying certain connectivity conditions on the boundary. We will show that outside of the circle inscribed in the triangle, the trees of a large random grove line up uniformly.

Despite their superficial differences, groves and random domino tilings of Aztec diamonds are linked in a way other than by their asymptotic behavior. Recall that groves are encoded in terms of a Laurent polynomial given by the cube recurrence. There is a more general form of the cube recurrence:

$$f_{i,j,k}f_{i-1,j-1,k-1} = \alpha f_{i-1,j,k}f_{i,j-1,k-1} + \beta f_{i,j-1,k}f_{i-1,j,k-1} + \gamma f_{i,j,k-1}f_{i-1,j-1,k}$$

where α, β, γ are constants. If $\alpha = \beta = \gamma = 1$ we have the original form of the cube recurrence from whence come groves. If $\alpha = \beta = 1$ and $\gamma = 0$, we have (after re-indexing), the octahedron recurrence:

$$f_{i,j,k+1}f_{i,j,k-1} = f_{i-1,j,k}f_{i+1,j,k} + f_{i,j-1,k}f_{i,j+1,k},$$

with which we may encode tilings of Aztec diamonds. How the cube recurrence and the octahedren recurrence affect asymptotic behavior will be seen later.

1.1. **Groves on standard initial conditions.** The standard initial conditions of order n specify a vertex set $I(n) = C(n) \cup B(n)$ where $C(n) = \{(i,j,k) \in \mathbf{Z}^3 | -n-1 \leq i+j+k \leq -n+1, i,j,k \leq 0\}$ and $B(n) = \{(i,j,k) \in \mathbf{Z}^3 | i +$

$j+k < -n-1, i,j,k \leq 0$ and $i, j,$ or $k = 0\}$. We draw its projection as shown in Figure **??** for the case $n = 4$. One way to generate all groves of order n is to set $f_{i,j,k} := x_{i,j,k}$ for all $(i,j,k) \in I(n)$, and compute $f_{0,0,0}$. Each term in the resulting Laurent polynomial defines a grove as follows. Let $G(n)$ be the the graph on the vertex set $I(n)$ where vertex (i,j,k) has as its neighbors the vertices $I(n) \cap \{(i \pm 1, j \pm 1, k), (i \pm 1, j, k \pm 1), (i, j \pm 1, k \pm 1)\}$.

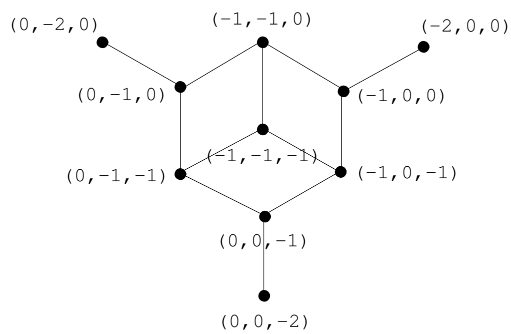

FIGURE 2. The standard initial conditions of order 1

The terms in $f_{0,0,0}$ are Laurent monomials of the form

$$m(g) = \prod_{(i,j,k) \in I(n)} x_{i,j,k}^{\deg(i,j,k)-2}.$$

Then the grove defined by $m(g)$ is the unique graph where each vertex (i,j,k) in $I(n)$ has degree $\deg(i,j,k)$. For example, $f_{0,0,0}$ on $I(1)$ is

$$\frac{x_{-1,-1,0}x_{0,0,-1}}{x_{-1,-1,-1}} + \frac{x_{-1,0,-1}x_{0,-1,0}}{x_{-1,-1,-1}} + \frac{x_{0,-1,-1}x_{-1,0,0}}{x_{-1,-1,-1}},$$

and the respective groves are shown in Figure 3.

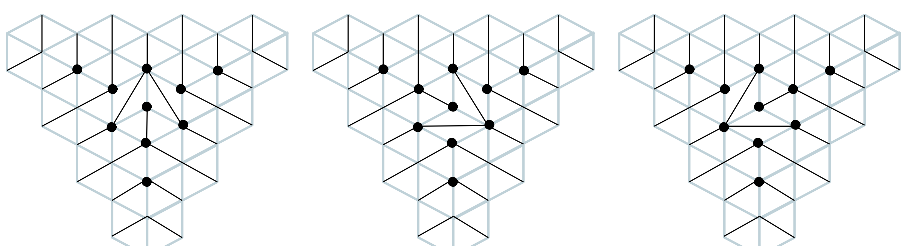

FIGURE 3. The three groves of order 1.

For a more interesting example, one term of the polynomial $f_{0,0,0}$ on $I(4)$ is

$$\frac{x_{-3,0,-2}x_{-2,-1,-1}x_{-1,-3,0}x_{0,-2,-2}}{x_{-3,-1,-2}x_{-2,-3,-1}x_{-1,-2,-2}};$$

its corresponding grove, g, is shown in Figure 4. The grove has interesting connectivity properties; in fact they are the determining characteristics of groves. Every vertex on the boundary of $C(n)$ (where cubes have been pushed down) is connected to another vertex on the boundary of $C(n)$ if and only if those vertices are equidistant to the nearest corner of the grove. Groves are acyclic - every connected component of a grove is a tree.

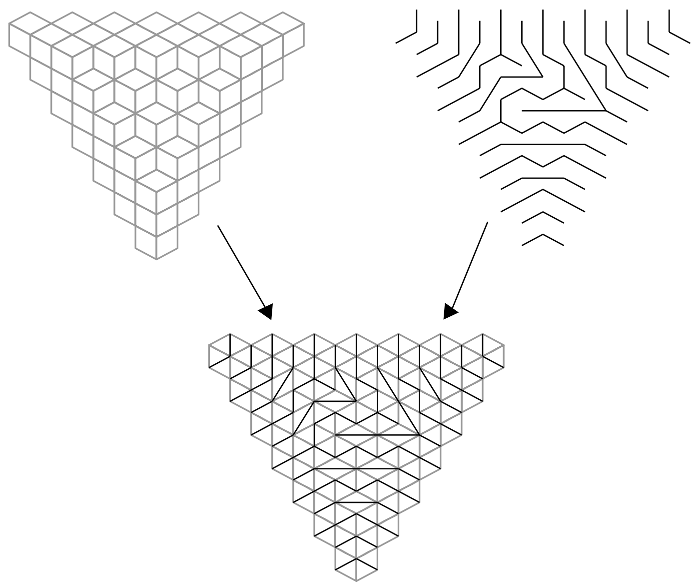

FIGURE 4. A grove g of order 4, superimposed on $I(4)$

Notice that there are two types of edges: *long* edges and *short* edges. It is shown in [1] that every vertex in $B(n)$ (the boring set) has degree 2 and only uses its short edges. As a result, there are only finitely many long edges, and these determine the grove. This observation leads to a more convenient way of looking at groves.

1.2. **Simplified groves.** We begin by constructing a modified form of the cube recurrence. Let $a_{i,j}$, $b_{k,j}$, $c_{i,k}$ be *long edge variables.* The variable $a_{i,j}$ is the label for the edge between vertices $(i, j-1, k+1)$ and $(i-1, j, k+1)$, $b_{k,j}$ for the edge between $(i-1, j, k+1)$ and (i, j, k), and $c_{i,k}$ for the edge between (i, j, k) and $(i, j-1, k+1)$. We rewrite the cube recurrence as follows:

$$\begin{aligned} f_{i,j,k}f_{i-1,j-1,k-1} &= b_{i,k}c_{i,j}f_{i-1,j,k}f_{i,j-1,k-1} + c_{i,j}a_{j,k}f_{i,j-1,k}f_{i-1,j,k-1} \\ &\quad + a_{j,k}b_{i,k}f_{i,j,k-1}f_{i-1,j-1,k} \end{aligned}$$

As we said, the long edges determine the grove, so rather than setting $f_{i,j,k} := x_{i,j,k}$ for $(i, j, k) \in I(n)$, we set $f_{i,j,k} := 1$ for $(i, j, k) \in I(n)$. Then $f_{0,0,0}$ is simply a polynomial in the edge variables a, b, c. Each term describes a unique grove, and we still produce every grove. This form of the cube recurrence is called the *edge variables version.* We can draw a simpler picture of our groves by ignoring all short edges and all of the vertices incident with them. In other words, specify a subset of the standard initial conditions of order n, called the *simplified initial conditions*: $I'(n) = \{(i, j, k) \in \mathbf{Z}^3 | i + j + k = -n, i, j, k \leq 0\} \subset I(n)$. We now represent our groves as graphs on this vertex set a triangular lattice shown in Figure 5. Also in Figure 5 we see the same grove from Figure 4, but with only the long edges included. In terms of edge variables, this grove is given by

$$a_{0,0}a_{0,1}a_{0,2}a_{1,0}a_{1,1}a_{2,1}b_{0,0}b_{0,1}c_{0,0}c_{0,1}c_{1,0}c_{2,0}.$$

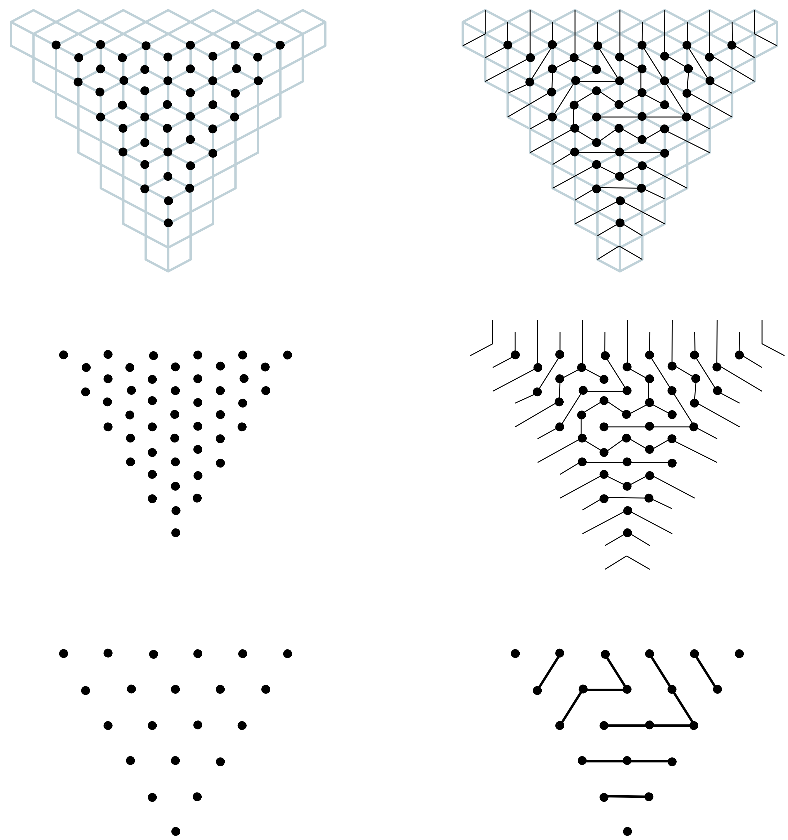

FIGURE 5. $I'(4)$, the simplified initial conditions of order 4, and a simplified grove.

Another modification of the cube recurrence that we shall like to use is the *edge-and-face variables version*. In the original version of the cube recurrence, the variables $x_{i,j,k}$ such that $i+j+k=-n+1$ were vertex variables. In the simplified picture, we call them the *face variables* of order n. Rather than setting $f_{i,j,k} := 1$ for all (i,j,k) in $I(n)$, we give the face variables their formal weights. That is, we set $f_{i,j,k} := 1$ for $(i,j,k) \in \{(i,j,k) \in \mathbf{Z}^3 | -n-1 \leq i+j+k \leq n, i,j,k \leq 0\}$ and $f_{i,j,k} := x_{i,j,k}$ for $(i,j,k) \in \{(i,j,k) \in \mathbf{Z}^3 | i+j+k = -n+1, i,j,k \leq 0\}$. Generating $f_{0,0,0}$ using these initial conditions, we get a Laurent polynomial in the edge and face variables. The vertices of the simplified initial conditions can be seen as forming $n(n+1)/2$ downward-pointing equilateral triangles, each with top-left vertex $(i,j-1,k+1)$, top-right vertex $(i-1,j,k+1)$, and bottom vertex (i,j,k). The face variables then correspond to each of these downward-pointing triangles. The triangle with (i,j,k) as its bottom vertex has face variable $x_{i,j,k+1}$. The exponent of the face variable is $-1,0,1$, corresponding to whether the downward-pointing triangle has, respectively, two, one, or zero edges present (recall that groves are acyclic). Of course, the face variables don't tell us anything new about a particular grove, but they will be useful later in deriving probabilities of edges being present in random groves.

1.3. **Grove shuffling and frozen regions.** We have given one definition for what a grove is, and how they may be generated. The methods and notation introduced in the previous section will be very helpful for later proofs. However, there is another tool we will like to use; an algorithm called grove shuffling (or cube-popping – see [1]). Grove shuffling not only gives a purely combinatorial definition of a grove, but also a method for generating groves of order n uniformly at random. For proof that grove shuffling does indeed give rise to the same objects as the terms of the Laurent polynomials given by the cube recurrence, see Carrol and Speyer [1]. Here we will only include a description of the algorithm and proof that the generation of groves of order n is uniform.

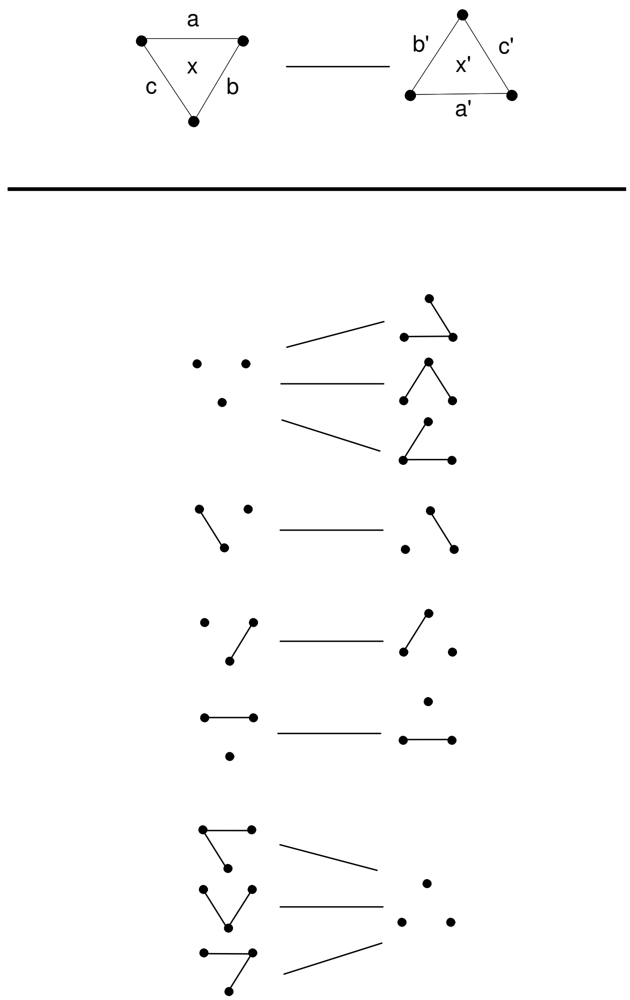

FIGURE 6. Grove shuffling.

Grove shuffling can be thought of as a local move on the downward-pointing triangles according to whether they have zero, one, or two edges present. See Figure 6. Let x be a generic downward-pointing triangle with possible edges a, b, c as shown, and let x' be a generic upward-pointing triangle with possible edges a', b', c'

as shown. There are three configurations of x with two edges: ab, ac, bc. Grove shuffling takes each of these triangles and replaces them with an upward-pointing triangle x' having none of its possible edges present (x and x' are concentric). There are three configurations of x with exactly one edge: a, b, c. Each of these is replaced by the upward-pointing triangle x' with only the parallel edge: a', b', c', respectively present. Lastly, there is one configuration of x with none of its possible edges present. This triangle is replaced with the upward-pointing triangle x' containing any two of its three possible edges: $a'b', a'c', b'c'$, chosen randomly with probability 1/3. After we have turned every downward-pointing triangle into an upward-pointing triangle, we add three new vertices to the corners of the grove so that we may shuffle again.

There is a unique grove of order 1: it has one downward-pointing triangle with zero edges. It is a basic fact that there are $3^{\lfloor \frac{k^2}{4} \rfloor}$ groves of order k. With this information we can prove a theorem about the uniformity of grove shuffling.

Theorem 1. *Beginning with the unique grove of order one, any given grove of order k will be generated after $k-1$ iterations of grove shuffling with probability $1/3^{\lfloor \frac{k^2}{4} \rfloor}$. That is, grove shuffling can be used to generate groves uniformly at random.*

Proof. Clearly the statement holds for k=2. Suppose that the claim holds for some $k \geq 1$. We would like to know the probability of an arbitrary grove of order $k+1$ being generated. Fix such a grove and call it $G(k+1)$. Only a certain subset of the groves of order k can be shuffled to become $G(k+1)$. Call this set the shuffling pre-image of $G(k+1)$, denoted $S^{-1}(G(k+1))$. Let $G(k) \in S^{-1}(G(k+1))$. Let a be the number of downward-pointing triangles in $G(k)$ with zero edges, let b be the number with exactly one edge, and c be the number of downward-pointing triangles with two edges.

The order of $S^{-1}(G(k+1))$ is 3^c. Each pre-image is gotten by making different choices of the the two edges appearing in each of the c downward-pointing triangles of $G(k)$. So since we have supposed the probability of generating a particular grove of order k to be uniform, the probability is

$$\frac{3^c}{3^{\lfloor \frac{k^2}{4} \rfloor}}$$

that after k shuffles we produce a grove in $S^{-1}(G(k+1))$.

Let $S(G(k)) = S(S^{-1}(G(k+1)))$ be the set of groves of order $k+1$ that can be obtained by shuffling a grove in $S^{-1}(G(k+1))$. The order of $S(G(k))$ is 3^a. This is because in each of the pre-images there are a downward-pointing triangles with no edges present, and every such triangle can be shuffled to any of three upward-pointing triangles. Furthermore, the only edges where the groves of $S^{-1}(G(k+1))$ differ will be annihilated upon shuffling. So there is a $1/3^a$ chance that one of the pre-images of $G(k+1)$ will actually shuffle into $G(k+1)$. Therefore the probability that $k+1$ iterations of grove shuffling yields $G(k+1)$ is

$$\frac{1}{3^{\lfloor \frac{k^2}{4} \rfloor}} \cdot \frac{1}{3^{a-c}}.$$

Now we claim that $a - c = \lfloor \frac{k+1}{2} \rfloor$. If so, then the probability computed above is equal to

$$\frac{1}{3^{\lfloor \frac{(k+1)^2}{4} \rfloor}}$$

as desired.

Let us make some basic observations from [1] or by easy induction. First, $a + b + c = k(k+1)/2$; the total number of downward-pointing triangles in any grove of order k. Secondly, $b + 2c = \lfloor \frac{k^2}{2} \rfloor$; the total number of edges in any grove of order k. Then $a - c = k(k+1)/2 - \lfloor \frac{k^2}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor$, and the theorem is proved. □

We now give a definition that will allow us to describe the phenomenon that we hope to make precise in section 2. It may not be obvious, but when a downward-pointing triangle with one edge is shuffled, the edge in the corresponding upward-pointing triangle has the same name. That is, edges are indexed relative to the corners perpendicular to them. Horizontal edges are indexed relative to the bottom corner, and the diagonal edges are indexed relative to the top-right and top-left corners. In this way we can think of grove-shuffling as more akin to domino shuffling [7]. Rather than replacing edges with parallel edges, we "slide" edges toward the corners along perpendicular lines. When a downward-pointing triangle has two edges, we remove both of those edges because they "annihilate" each other. When a downward-pointing triangle has no edges, we create two new ones randomly.

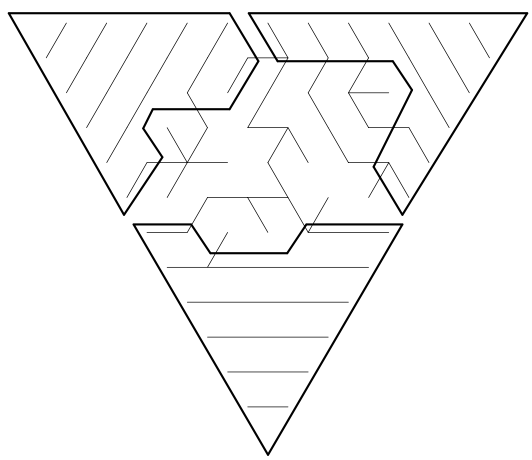

FIGURE 7. Frozen regions of a random grove of order 12

With this viewpoint, we define an edge to be *frozen* if cannot be annihilated under any further iterations of grove shuffling. Equivalently, an edge (horizontal without loss of generality) is frozen if and only if all of the below[MAKE 'BELOW' PRECISE IN TERMS OF COORDINATES] it are frozen. In figure() all the highlighted edges are frozen. We conclude this section by examining a picture of a large random grove generated by grove shuffling. In Figure 8, we see that outside of a certain region, all of the edges are parallel. Moreover, the boundary between the less uniform interior and the frozen regions in the corners seems to approximate a circle. Proving that this boundary approaches a circle in the limit is the main goal of this paper. We will speculate on variance of the boundary shape as well as interior statistics in section().

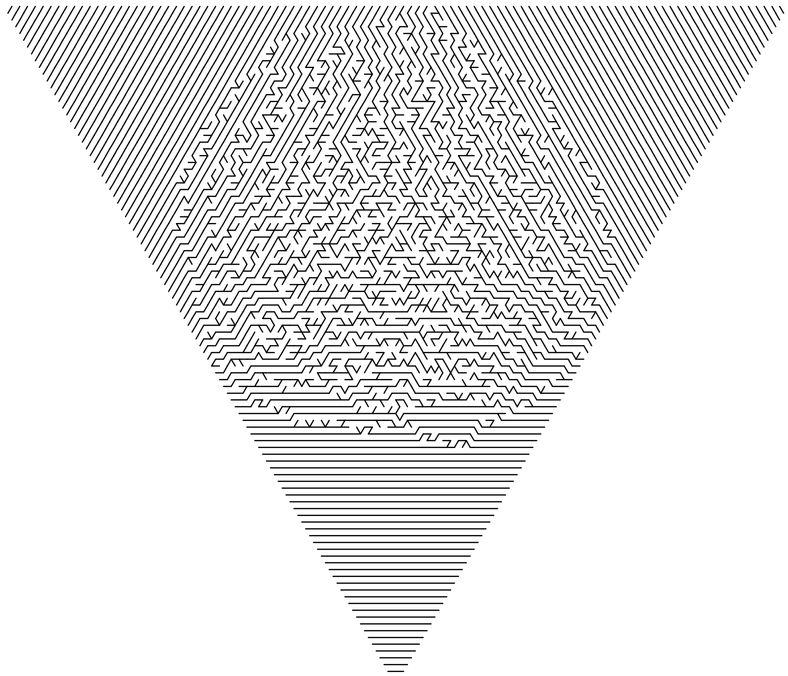

FIGURE 8. A grove on standard initial conditions of order 100

2. The arctic circle theorem

For any n, we can scale the initial conditions so that they resemble an equilateral triangle with sides of length $\sqrt{2}$. We will show that outside of the circle inscribed in this triangle, there is homogeneity of the edges in an appropriately scaled random grove of order n, with probability approaching 1 as $n \to \infty$. Specifically, we will examine the limiting probability of finding a particular type of edge in a given location outside of the inscribed circle.

2.1. **Edge probabilities.** Let $p_n(i,j) = p(i,j,k), k = -n-i-j-1$, be the probability that $a_n(i,j) = a(i,j,k)$, the horizontal edge on triangle $x_{i,j,k+1}$, is present in a random grove of order n. Define $E_n(i,j) = E(i,j,k+1)$ to be the expectation of the exponent of the face variable $x_{i,j,k}$.[2] We will prove the following formula for finding the edge probability $p_n(i,j)$.

Theorem 2. *$p_n(i,j) = p_{n-1}(i,j) + \frac{2}{3}E_{n-1}(i,j)$. Inductively,* $p_n(i,j) = \dfrac{2}{3}\displaystyle\sum_{l=1}^{n-1} E_l(i,j)$.

Proof. We will first derive a relation between $p_n(i,j)$ and $p_{n-1}(i,j)$. Define $q_n(k,i)$, $r_n(k,j)$, $k = -n-i-j-1$, to be the respective probabilities of the diagonal edges $b_n(k,i)$ and $c_n(k,j)$ being present in triangle $x_{i,j,k+1}$; see figure ().

[2] If $k > 0$ then define $E(i,j,k) = 0$.

In order to simplify notation, we let [MAKE THIS A TABLE]

$$\begin{array}{ccc} p = p_{n-1}(i,j) & q = q_{n-1}(k,i) & r = r_{n-1}(k,j) \\ a = a_{n-1}(i,j) & b = b_{n-1}(k,i) & c = c_{n-1}(k,j) \\ P = p_n(i,j) & Q = q_n(k,i) & R = r_n(k,j) \\ A = a_n(i,j) & B = b_n(k,i) & C = c_n(k,j) \end{array}$$

Let $pr(*)$, where $*$ is a subset of $\{a, b, c\}$, be the probability that a random grove contains that set of edges and not its compliment. Define $Pr(*)$ similarly. Some observations that come directly from grove shuffling: [MAKE THIS A TABLE]

- $pr(ab) = pr(ac) = pr(bc)$
- $pr(abc) = 0$
- $pr(\emptyset) + pr(a) + pr(b) + pr(c) + pr(ab) + pr(ac) + pr(bc) = 1$
- $p = pr(a) + pr(ab) + pr(ac)$
- $q = pr(b) + pr(ab) + pr(bc)$
- $r = pr(c) + pr(ac) + pr(bc)$
- $Pr(A) = pr(a)$
- $Pr(B) = pr(b)$
- $Pr(C) = pr(c)$
- $Pr(AB) = Pr(AC) = Pr(BC) = 1/3pr(\emptyset)$

We will now deduce $P = p_n(i, j)$.

$$\begin{aligned} P &= Pr(A) + Pr(AB) + Pr(AC) \\ &= pr(a) + 2/3pr(\emptyset) \\ &= pr(a) + 2/3(1 - pr(a) - pr(b) - pr(c) - pr(ab) - pr(ac) - pr(bc)) \\ &= pr(a) + 2/3(1 - p - q - r + pr(ab) + pr(ac) + pr(bc)) \\ &= pr(a) + 2/3(pr(ab) + pr(ac) + pr(bc)) + 2/3(1 - p - q - r) \\ &= pr(a) + pr(ab) + pr(ac) + 2/3(1 - p - q - r) \\ &= p + 2/3(1 - p - q - r) \end{aligned}$$

Let $x = x_{i,j,k+1}$ be the face variable of the downward-pointing triangle in question. Notice that

$$\begin{aligned} E_n(i,j) &= E(x) \\ &= 1 \cdot pr(\emptyset) + 0 \cdot (pr(a) + pr(b) + pr(c)) - 1 \cdot (pr(ab) + pr(ac) + pr(bc)) \\ &= 1 - pr(a) - pr(b) - pr(c) - 2pr(ab) - 2pr(ac) - 2pr(bc) \\ &= 1 - p - q - r \end{aligned}$$

Therefore, $P = p + 2/3E(x)$. In the coordinate system, we have

$$p_n(i,j) = p_{n-1}(i,j) + 2/3E_{n-1}(i,j) = \frac{2}{3}\sum_{l=1}^{n-1} E_l(i,j)$$

and the theorem is proved. □

2.2. **A generating function.** We now know that to compute the probability of a particular edge being present in a random grove, it will be enough to compute the expectations $E_l(i, j)$. In this section we derive a generating function for computing these numbers. We will also derive the generating function for the horizontal edge probabilities.

Let $F(x,y,z) = \sum_{i,j,k\geq 0} E(-i,-j,-k)x^i y^j z^k$ be the generating function for the expectations. We can write $E(-i,-j,-k+1) = E_n(-i,-j)$, for $n = i+j+k$ when convenient. First consider the uniformly weighted version of the cube recurrence:

$$f_{i,j,k}f_{i-1,j-1,k-1} = \frac{1}{3}\left(f_{i-1,j,k}f_{i,j-1,k-1} + f_{i,j-1,k}f_{i-1,j,k-1} + f_{i,j,k-1}f_{i-1,j-1,k}\right)$$

Using this recurrence to calculate $f_{i,j,k}$ we will get each monomial weighted uniformly, so that if we set all the initial conditions equal to 1, $f_{i,j,k} = 1$. If we want the expectation of the exponent of the face variable $x = x_{i_0,j_0,k_0}$, we need only calculate the derivative of $f_{0,0,0}$ with respect to this variable, then set all variables equal to one. In other words,

$$E(i_0,j_0,k_0) = \frac{\partial}{\partial x} f_{0,0,0}|_{x_{i,j,k}=1}$$

Furthermore, we can calculate the intermediate expectations for $(i',j',k') \in I(n')$ with $n' < n$ by

$$E(i',j',k') = \frac{\partial}{\partial x} f_{i',j',k'}|_{x_{i,j,k}=1}$$

(the proof of this only requires a re-labeling of vertices). With this in mind, let us differentiate the weighted cube recurrence with respect to x:

$$\begin{aligned} f'_{i,j,k}f_{i-1,j-1,k-1} + f_{i,j,k}f'_{i-1,j-1,k-1} &= \frac{1}{3}\left(f'_{i-1,j,k}f_{i,j-1,k-1} + f_{i-1,j,k}f'_{i,j-1,k-1}\right) + \\ &\quad \frac{1}{3}\left(f'_{i,j-1,k}f_{i-1,j,k-1} + f_{i,j-1,k}f'_{i-1,j,k-1}\right) + \\ &\quad \frac{1}{3}\left(f'_{i,j,k-1}f_{i-1,j-1,k} + f_{i,j,k-1}f'_{i-1,j-1,k}\right) \end{aligned}$$

Now by setting $x_{i,j,k} = 1$ for all (i,j,k), we get a linear recurrence for the expectations in question:

$$\begin{aligned} E(i,j,k) + E(i-1,j-1,k-1) &= \frac{1}{3}\left(E(i-1,j,k) + E(i,j-1,k-1)\right) + \\ &\quad \frac{1}{3}\left(E(i,j-1,k) + E(i-1,j,k-1)\right) + \\ &\quad \frac{1}{3}\left(E(i,j,k-1) + E(i-1,j-1,k)\right) \end{aligned}$$

By making $E(i,j,k)$ the coefficient of $x^{-i}y^{-j}z^{-k}$, we can form the rational generating function in the variables x, y, z:

$$\begin{aligned} F(x,y,z) &= \sum_{i,j,k\geq 0} E(-i,-j,-k)x^i y^j z^k \\ &= \frac{1}{1 + xyz - \frac{1}{3}(x+y+z+xy+xz+yz)} \end{aligned}$$

Now using the fact that $p(i,j,k) = p(i,j,k+1) + 2/3E(i,j,k)$, we can derive the formula for $G(x,y,z) = \sum_{i,j,k\geq 0} p(-i,-j,-k)x^i y^j z^k$. If we define $p(0,0,1) = 0$, we have

$$\begin{aligned} G(x,y,z) &= \sum_{i,j,k\geq 0} (p(-i,-j,-k+1) + 2/3E(i,j,k))x^i y^j z^k \\ &= \sum_{i,j,k\geq 0} p(-i,-j,-k+1)x^i y^j z^k + \frac{2}{3}F(x,y,z) \\ &= zG(x,y,z) + \frac{2}{3}F(x,y,z) \\ &= \frac{2F(x,y,z)}{3(1-z)} \end{aligned}$$

Now we have two rational functions with which to calculate edge probabilities.

2.3. **Asymptotic analysis.** With our generating functions in hand, we can prove our main theorem. First let us embed a triangle in three-space by $T := \{(x,y,z) \in \mathbf{R}^3 | x,y,z \leq 0, x+y+z = -1\}$. This is the triangle that we will scale $I(n)$ to fit. A point $(x,y,z) \in T$ is outside of the inscribed circle (what will show is the arctic circle) if and only if the angle between the vector (x,y,z) and vector $(-1,-1,-1)$ is greater than $\cos^{-1}(\sqrt{2/3})$.

Notice that for any point (x,y,z) outside of the inscribed circle, we can safely increase two of the coordinates without wandering inside the circle, but increasing the third coordinate moves us closer to the circle. Call any coordinate with the property that it can be increased without moving into the arctic circle a *small* coordinate.

Theorem 3 (Weak Arctic Circle). *Let (x_0,y_0,z_0) be a point in T outside of the inscribed circle for which z_0 is a small coordinate. Let (i_n,j_n,k_n), $i_n+j_n+k_n = -n-1$, be a sequence with nonpositive integer entries such that*

$$\lim_{n\to\infty} \frac{1}{n+1}(i_n,j_n,k_n) = (x_0,y_0,z_0)$$

Then $\lim_{n\to\infty} p(i_n,j_n,k_n) = p_n(i_n,j_n) = 0.$

In other words, the theorem states that in the upper two regions of T outside of the arctic circle, the probability of finding a horizontal edge goes to zero as the order of a (scaled) random grove goes to infinity. By symmetry, we have that only horizontal edges will be found in the lower region outside of the arctic circle, and only the respective diagonal edges in the upper-left and upper-right regions outside the circle.

The following lemma is the heart of the proof.

Lemma 1. *Fix a point (x_0,y_0,z_0) in T outside of the inscribed circle. Then there are constants A,B,C such that*

$$p(-i,-j,-k) = O(e^{-(Ai+Bj+Ck)})$$

and $Ax_0+By_0+Cz_0 < 0.$

Let us suppose the lemma is true and present the proof of the theorem.

Proof of Theorem 3. By the lemma, $p(i_n, j_n, k_n) = O(e^{Ai_n+Bj_n+Ck_n})$, so we will have that $p(i_n, j_n, k_n) \to 0$ if $Ai_n + Bj_n + Ck_n \to -\infty$. But

$$\begin{aligned}\lim_{n\to\infty} Ai_n + Bj_n + Ck_n &= \lim_{n\to\infty}(n+1)\left(A\frac{i_n}{n+1} + B\frac{j_n}{n+1} + C\frac{k_n}{n+1}\right)\\ &= \lim_{n\to\infty}(n+1)(Ax_0 + By_0 + Cz_0)\end{aligned}$$

And since the lemma gives that $Ax_0 + By_0 + Cz_0 < 0$, $p(i_n, j_n, k_n) \to 0$ as $n \to \infty$. □

To facilitate the proof of the lemma, we will use the following claim.

Claim 1. *Let $f(x, y, z)$ be an analytic function. Let r, s, t be positive real numbers such that $f(x, y, z) \neq 0$ for $|x| \leq r$, $|y| \leq s$, $|z| \leq t$. If*

$$G(x, y, z) = \frac{1}{f(x, y, z)} = \sum a_{i,j,k} x^i y^j z^k,$$

then $a_{i,j,k} = O(r^{-i}s^{-j}t^{-k})$.

Proof of Claim. We have

$$\begin{aligned}a_{i,j,k} &= \frac{1}{i!j!k!}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\frac{\partial}{\partial z^k}F(x, y, z)|_{(0,0,0)}\\ &= \frac{1}{(2\pi i)^3}\int_\gamma\int_{\gamma'}\int_{\gamma''}\frac{F(x, y, z)}{x^{i+1}y^{j+1}z^{k+1}}dxdydz\\ &\quad (\gamma = \{|z| = t\}, \gamma' = \{|y| = s\}, \gamma'' = \{|x| = r\})\\ &\leq \frac{M}{(2\pi i)^3}\int_\gamma\int_{\gamma'}\int_{\gamma''}\frac{1}{x^{i+1}y^{j+1}z^{k+1}}dxdydz\\ &\quad (\text{Since } F(x, y, z) \text{ is bounded on the compact set } \gamma \times \gamma' \times \gamma''.)\\ &= M\frac{1}{r^{i+1}s^{j+1}t^{k+1}}\end{aligned}$$

In other words, $a_{i,j,k} = O(r^{-i}s^{-j}t^{-k})$ and the claim is proved. □

Proof of Lemma. We now apply the claim to the edge probability generating function:

$$\begin{aligned}G(x, y, z) &= \sum_{i,j,k\geq 0} p(-i, -j, -k)x^i y^j z^k\\ &= \frac{1}{\frac{3}{2}(1-z)(1 + xyz - \frac{1}{3}(x + y + z + xy + xz + yz))}\\ &= \frac{1}{f(x, y, z)}\end{aligned}$$

We need to show that we can choose real numbers A, B, C so that

- $Ax_0 + By_0 + Cz_0 < 0$
- $f(x, y, z) \neq 0$ for any $(x, y, z) \in \{(x, y, z) \in \mathbb{C}^3 : |x| \leq e^A, |y| \leq e^B, |z| \leq e^C\}$

Let K be the set of all triples (A, B, C) satisfying the first criterion, and let L be the set of all triples (A, B, C) satisfying the second. We need to show that if (x_0, y_0, z_0) lies outside of the circle inscribed in the triangle T, then K and L have a nonempty intersection.

If $|x|, |y|, |z|$ are less than one, then f is nonzero. So if A, B, C are all less than zero, then $(A, B, C) \in L$. Unfortunately, since x_0, y_0, z_0 are all less than or equal to zero, $Ax_0 + By_0 + Cz_0$ cannot be negative. Somehow we must determine how far we can stretch A, B, C while keeping f from vanishing. Because of the factor $(1 - z)$ we will keep $C < 0$, and hope to push A or B greater than zero. We hope to boil the problem down to a geometry exercise; that is, we will realize the sets K and L as geometric objects.

Think of (x_0, y_0, z_0) as a vector and let N be the normal plane to (x_0, y_0, z_0) passing through the origin. Then the set K is just the half space on the opposite side of N. Describing the geometry of L is not as simple.

By forcing $C < 0$, we've avoided the zeros of f in the factor $(1 - z)$, so our main concern is with the zeros of $g = f/(1 - z)$. Rewrite $g(x, y, z)$ as $g(e^A, e^B, e^C)$, and consider the Taylor expansion of g near $(A, B, C) = (0, 0, 0)$. Here, we get $g = AB + AC + BC+$ higher powers of A, B, C. Then near $(0, 0, 0)$, the set L looks like the region inside the cone Co given by $AB + AC + BC = 0$ containing $(-1, -1, -1)$ and with $C < 0$.

If the angle between (x_0, y_0, z_0) and $(-1, -1, -1)$ is small, then K is far from L as it is described above. As (x_0, y_0, z_0) moves farther from $(-1, -1, -1)$, the normal plane N tilts closer to intersecting the half-cone containing L. We must determine when it has tilted far enough that K and L might intersect.

Let θ be the angle between a vector (A, B, C) and $(-1, -1, -1)$. If (A, B, C) is on the cone Co, then we have the identity

$$(A + B + C)^2 = A^2 + B^2 + C^2.$$

Therefore,

$$\begin{aligned} (\cos\theta)^2 &= \frac{\langle (A, B, C), (-1, -1, -1)\rangle^2}{\|(A, B, C)\|^2 \|(-1, -1, -1)\|^2} \\ &= \frac{(-A - B - C)^2}{3(A^2 + B^2 + C^2)} \\ &= \frac{1}{3} \end{aligned}$$

So we need to determine when the angle between the normal plane and $(-1, -1, -1)$ is less than $\cos^{-1}(1/\sqrt{3})$. If θ is the angle between (x_0, y_0, z_0) and $(-1, -1, -1)$, then this condition means that $\theta > \frac{\pi}{2} - \cos^{-1}(1/\sqrt{3}) = \cos^{-1}(\sqrt{2/3})$. All the vectors in the triangular region T with this property lie outside of the inscribed circle. Hence, the lemma is proved. □

3. Further speculation on statistics of groves

We hope to apply more sophisticated asymptotic methods to determine statistics for the region inside the arctic circle, the so called 'temperate region' where behavior is no longer uniform, but is not perfectly random either.

Another future examination is to apply the methods of growth models and statistical mechanics to groves. In terms of its effect on the frozen region, grove shuffling seems to be nearly isomorphic to some well-known growth models, most likely a randomly-growing Young diagram.

Both these methods were applied the study of asymptotic behavior of random tilings of Aztec diamonds, and we hope that similar techniques will yield similar results for groves, as they are in a sense cousins of Aztec diamond tilings.

4. Acknowledgments

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