Trevor Bass
March 2003

# 1 Background

Let $T(m, n)$ be the number of domino tilings of the $2m$-by-$2n$ rectangle, and $T(n)$ be the number of domino tilings of the $2n$-by-$2n$ square.

For a matrix $A$, let $A(i, j)$ denote its $(i, j)$th entry. Suppose $A_n$ is an $n$-by-$n$ matrix and $A_{n+1}$ is an $(n+1)$-by-$(n+1)$ matrix. We define $A_k$ recursively in terms of $A_{k+1}$ and $A_{k+2}$ by

$$A_k(i,j)A_{k+2}(i+1,j+1) = A_{k+1}(i,j)A_{k+1}(i+1,j+1) + \lambda A_{k+1}(i+1,j)A_{k+1}(i,j+1)$$

if $A_{k+2}(i+1, j+1) \neq 0$ and $A_k(i, j) = 0$ otherwise, in which case we write $A_n \to A_{n-1} \to \cdots \to A_1$. The *$\lambda$-determinant of a pair* $(A_n, A_{n+1})$ is the sole entry of $A_1$. Let the *$\lambda$-determinant of a matrix* $A_n$ be the $\lambda$-determinant of the pair $(A_n, C)$, where $C$ is the $(n+1)$-by-$(n+1)$ matrix each of whose entries is 1. Note that the $(-1)$-determinant of a matrix is just its determinant. Let $\Lambda(A)$ denote the 1-determinant of $A$.

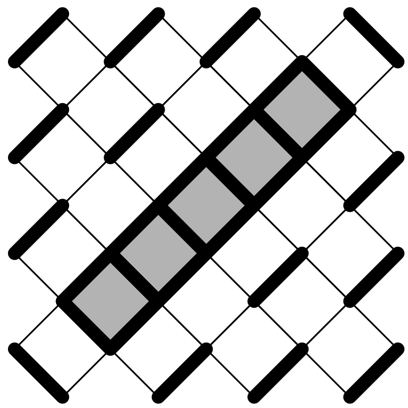
Let the *Aztec diamond graph* be the dual graph of an Aztec diamond, and a *weighted Aztec diamond graph (WAD)* be an Aztec diamond graph with associated edge weights. Designate by $W(F)$ the sum of the weighted perfect matchings of a WAD $F$. An $S$-WAD is a WAD whose entries are chosen from the set $S$; in this paper we will be using $\{0, 1\}$-WADs almost exclusively.

Consider the Aztec diamond graph as tilted 45 degrees. Call two faces $F, G$ of the graph *vertex adjacent* (in which case we write $F|G$) if they share a common vertex but not an edge, and *vertex connected* if either $F|G$ or there is a finite set $\{F_1, F_2, \ldots, F_n\}$ of faces in the graph such that $F_i|F_{i+1}$ for all $i$, $F|F_1$, and $F_n|G$. Call a face a *major face* if it is vertex connected to the upper left face. The major faces of an Aztec diamond $F$ form a finite square lattice; denote by $F(i, j)$ the one in column $i$, row $j$. If $F$ is a WAD, let $F(i, j)_{ne}, F(i, j)_{nw}, F(i, j)_{se}, F(i, j)_{sw}$ represent the weightings of the northeast, northwest, southeast, and southwest edges bordering $F(i, j)$, respectively.

Given a WAD $F$ of order $n$, let $F_{ne}, F_{nw}, F_{se}, F_{sw}$ be the northeast, northwest, southeast, and southwest order $n-1$ weighted Aztec subdiamonds, and let $F_m$ be the inner order $n-2$ weighted Aztec subdiamond. If we let the *edge multiplying factors* $n_e, n_w, s_e, s_w$ be the weights of the northeast, northwest, southeast, and southwest edges of $F$, then the number of weighted matchings of $F$ is given in [1] by the recurrence

$$W(F)W(F_m) = n_e s_w W(F_{nw})W(F_{se}) + n_w s_e W(F_{ne})W(F_{sw}). \qquad (1)$$

Given a $\{0, 1\}$-WAD $F$, create a square matrix $M$ whose $(i, j)$th entry is equal to $F(i, j)_{nw}F(i, j)_{se} + F(i, j)_{ne}F(i, j)_{sw}$ (the *edge factor* of $F(i, j)$). If $F$ is weighted all 1 within some (*embedded*) even-by-even rectangle and in a brickwork pattern outside this rectangle, the 1-determinant of $M$ is equal to the number of weighted matchings of $F$. This result comes almost immediately from (1), the only trick being that it works in general only if the possible weights are 0 and 1, in which case the edge multiplying factors drop out. Thus since the number of weighted perfect matchings of a WAD with the aforementioned weighting scheme is equal to the number of perfect matchings of the rectangle that is embedded in the WAD. [Another paper is currently being prepared by the current author with a more general and detailed proof of these statements.] For instance, when we embed the 2-by-6 rectangle in the order 4 Aztec diamond graph, we get the following WAD, where edges of weight 1 are represented by darkened lines and all other edges have weight 0.



This WAD has associated edge factor matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 8 \\ 8 & 1 \end{bmatrix} \to 13,$$

whose 1-determinant is 13, the number of perfect matchings of the 2-by-6 rectangle grid-graph.

## 2 Motivation

We will now consider the square case. Let us embed a $2n$-by-$2n$ square grid-graph in the center of weighted Aztec diamond graphs of orders $n+1$ and $n+2$. As above, with each of these weighted Aztec diamond graphs and respective embeddings is associated an matrix whose 1-determinant equals $T(n)$. First take $n = 2$. Using the order 3 Aztec diamond, we arrive at the matrix

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

By moving a step backward in the process of taking the 1-determinant of $M$, we get

$$M' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Note that $\Lambda(M) = \Lambda(M', M) = \Lambda(M')$. Using the order 4 Aztec diamond, we arrive at the matrix

$$M'' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The matrices $M', M''$ both have the general form

$$M_{2,2}(a) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & a & a & 1 \\ 1 & a & a & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, a = 1, 2.$$

For $n = 3$, the corresponding matrices are

$$M_{3,3}(a) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & a & a & 1 & 0 \\ 1 & a & a & a & a & 1 \\ 1 & a & a & a & a & 1 \\ 0 & 1 & a & a & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, a = 1, 2.$$

The definition of $M_{n,n}(a)$ extends easily to all positive integers $n$ and all nonzero real numbers $a$. Let $M_{n,n}(a)$ be the matrix associated with the $2n$-by-$2n$ square with parameter $a$, and let $m_{n,n}(a)$ be its 1-determinant.

**Conjecture 1.** *For all positive integral $n$, $m_{n,n}(a) = (2/a)^n p_n^2(a)$, where $p_n(a)$ is an integer polynomial of degree $n$ with $p_n(1)$ odd.*

This would strengthen Pachter's theorem that $T(n)/2^n$ is an odd square.

We can extend the above definitions canonically to yield $M_{m,n}(a)$ and $m_{k,n}(a)$ for any $2k$-by-$2n$ rectangle.

**Conjecture 2.** *For all positive integers $k, n$, we have the functional equation $m_{k,n}(a) = m_{k,n}(2/a)$.*

These two conjectures have been tested extensively, mostly in the square case.

# 3 The Case of 2-by-2n Rectangles

In this section, let $m_n(a) = m_{1,n}(a)$, so that $m_n(1) = m_n(2) = T(1, n)$, and let $M_n(a) = M_{1,n}(a)$.

For sake of clarity, $M_0(a) = 1$,

$$M_1(a) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to 2,$$

$$M_3(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & a+1 \\ a+1 & 1 \end{bmatrix} \to a + 2 + \frac{2}{a},$$

$$M_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & a & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & a+1 \\ 1 & a^2+1 & 1 \\ a+1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & a^2+a+1+\frac{2}{a} \\ a^2+a+1+\frac{2}{a} & 1 \end{bmatrix}$$

$$\to a^2 + 2a + 2 + \frac{4}{a} + \frac{4}{a^2}.$$

Let $S_n(a)$ be the matrix obtained from $M_n(a)$ by replacing its upper-right and lower-left entries with the indeterminant $a$, and let $s_n = s_n(a) = \Lambda(S_n(a))$. Let $T_n(a)$ be the matrix obtained from $S_n(a)$ by replacing its upper-right entry with 1, and let $t_n = t_n(a) = \Lambda(T_n(a))$. Note that if $T'_n(a)$ is the matrix obtained from $S_n(a)$ by replacing the lower-left entry with 1, $\Lambda(T'_n(a)) = t_n(a)$ as well; we will be using this fact without explicit reference.

**Lemma 3.** *The sequences $s_n, t_n, m_n, n \geq 1$ satisfy*

$$s_n s_{n-2} = s_{n-1}^2 + 1, \quad (2)$$
$$t_n s_{n-2} = t_{n-1} s_{n-1} + 1, \quad (3)$$
$$m_n s_{n-2} = t_{n-1}^2 + 1, \quad (4)$$

*with initial conditions*

$$s_1 = a, \qquad s_2 = a^2 + 1,$$
$$t_1 = 1, \qquad t_2 = a + 1,$$
$$m_1 = 1, \qquad m_2 = 2.$$

**Proof.** We have $S_1(a) = a = s_1, S_2(a) \to a^2 + 1 = s_2$,

$$S_3(a) = \begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & s_2 \\ s_2 & 1 \end{bmatrix} \to \frac{s_2^2 + 1}{s_1} = s_3,$$

and so forth, demonstrating Equation (5). Similarly, $T_1(a) = 1 = t_1, T_2(a) \to a + 1 = t_2$,

$$T_3(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & t_2 \\ s_2 & 1 \end{bmatrix} \to \frac{s_2 t_2 + 1}{s_1} = t_3,$$

and so forth, demonstrating Equation (6). Finally, $M_1(a) = 1 = m_1, M_2(a) \to 2 = m_2$,

$$M_3(a) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & t_2 \\ t_2 & 1 \end{bmatrix} \to \frac{t_2^2 + 1}{s_1} = s_3,$$

$$M_4(a) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & a & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & t_2 \\ 1 & s_2 & 1 \\ t_2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & t_3 \\ t_3 & 1 \end{bmatrix} \to \frac{t_3^2+1}{s_2} = t_4,$$

and so forth, demonstrating Equation (7). □

**Lemma 4.** *The sequences $s_n, t_n, m_n, n \geq 1$ satisfy the three linear recurrence relations*

$$\begin{aligned} s_n - \left(a + \frac{2}{a}\right) s_{n-1} + s_{n-2} &= 0, \\ s_n - t_n - (a-1)s_{n-1} &= 0, \\ m_n - t_n + (a-1)t_{n-1} &= 0. \end{aligned}$$

**Proof.** If we assume $s_n$ obeys a linear recurrence relation of order two, we can find it. Setting $s_3 = cs_2 + ds_1, s_4 = cs_3 + ds_2$ and solving the system, we find $c = a + \frac{2}{a}, d = -1$. The sequence $s_n$ can easily be shown to obey this order-two linear recurrence relation via straightforward induction or by viewing $s_n$ as a linear combination $a\alpha^n + b\beta^n$ of exponentials and straightforwardly finding constraints on $\alpha, \beta$. Now, if we take equation (5) minus equation (6), we obtain $s_{n-2}(s_n - t_n) = s_{n-1}(s_n - 1 - t_n - 1)$, which shows that the quantity $\frac{s_n - t_n}{s_{n-1}}$ has the same value for all $n$; this value is determined to be $a-1$ by plugging in initial conditions. Finally, taking equation (6) minus equation (7) yields $\frac{t_n - m_n}{t_{n-1}} = \frac{s_{n-1} - t_{n-1}}{s_{n-2}}$, which is again the same constant for all $n$. □

We can define generating functions $S(a,x) = s_1(a) + s_2(a)x + s_3(a)x^2 + \cdots$, and similarly $T(a,x)$ and $M(a,x)$. Then by the above linear recurrences and initial conditions,

$$\begin{aligned} S(x) - \left(a + \frac{2}{a}\right) xS(x) + x^2 S(x) &= a - x, \\ S(x) - T(x) - (a-1)xS(x) &= a - 1, \\ M(x) - T(x) + (a-1)xT(x) &= 0. \end{aligned}$$

Solving these equations for the generating functions yields $M(a,x) = \frac{(a-2a+ax)(1-x-ax)}{a-2x-a^2x+ax^2}$, which obeys the functional equation $M(a,x) = M(2/a, x)$. This proves Conjecture 2 for 2-by-even rectangles.

# 4 The Case of 4-by-2n Rectangles

For the purposes of this section, let $m_n(a) = m_{2,n}(a)$, since we are examining 4-by-$2n$ rectangles.

Similarly to above, we obtain the recurrence relations

$$\begin{aligned} b_n b_{n-2} &= b_{n-1}^2 + s_{n-1}, \\ s_n s_{n-2} &= b_{n-1}^2 + s_{n-1}^2, \\ c_n s_{n-2} &= d_{n-1}^2 + c_{n-1}s_{n-1}, \\ d_n b_{n-2} &= b_{n-1}d_{n-1} + s_{n-1}, \\ q_n b_{n-2} &= d_{n-1}^2 + s_{n-1}, \\ m_n s_{n-2} &= q_n^2 + c_n^2, \end{aligned}$$

and initial conditions

$$\begin{aligned} b_0 &= a, & b_1 &= a^2 + a, \\ s_0 &= a, & s_1 &= 2a^2, \\ c_1 &= 1, & c_2 &= 6a, \\ d_1 &= 2a. \end{aligned}$$

If we assume $m_n$ obeys a linear recurrence of order at most 10 (arbitrary and much too large), we can solve for this recurrence in the same way we solved for the recurrence of $s_n$ in Section 3. The result, along with conjectural linear recurrences for the other sequences, can be proved inductively. The sequence $m_n$ can thus be found to obey a linear recurrence relation of order 5, and thus we can solve for its generating function,

$$\frac{(1-x)(a^2-(6a+2a^2+3a^3)x+(4+4a+14a^2+2a^3+a^4)x^2-(6a+2a^2+3a^3)x^3+a^2x^4)}{x^2(36a^2-(94a+10a^2+47a^3)x+(40+22a+86a^2+11a^3+10a^4)x^2-(8+26a+10a^2+13a^3+2a^4)x^3+(2a+2a^2+a^3)x^4)}.$$

This obeys the desired functional equation, proving Conjecture 2 for 4-by-even rectangles.

# 5 The Future

For the two cases so far, we have obtained the systems

$$s_n s_{n-2} = s_{n-1}^2 + 1, \tag{5}$$

$$t_n s_{n-2} = t_{n-1} s_{n-1} + 1, \tag{6}$$

$$m_n s_{n-2} = t_{n-1}^2 + 1, \tag{7}$$

with initial conditions

$$\begin{aligned} s_1 &= a, & s_2 &= a^2 + 1, \\ t_1 &= 1, & t_2 &= a + 1, \\ m_1 &= 1, & m_2 &= 2 \end{aligned}$$

and

$$\begin{aligned} b_n b_{n-2} &= b_{n-1}^2 + s_{n-1}, \\ s_n s_{n-2} &= b_{n-1}^2 + s_{n-1}^2, \\ c_n s_{n-2} &= d_{n-1}^2 + c_{n-1} s_{n-1}, \\ d_n b_{n-2} &= b_{n-1} d_{n-1} + s_{n-1}, \\ q_n b_{n-2} &= d_{n-1}^2 + s_{n-1}, \\ m_n s_{n-2} &= q_n^2 + c_n^2, \end{aligned}$$

with initial conditions

$$\begin{aligned} b_0 &= a, & b_1 &= a^2 + a, \\ s_0 &= a, & s_1 &= 2a^2, \\ c_1 &= 1, & c_2 &= 6a, \\ d_1 &= 2a. \end{aligned}$$

A natural question to ask is: if you have a finite number $N$ of equations of the form

$$a_n b_{n-2} = c_{n-1} d_{n-1} + e_{n-1} f_{n-1},$$

where $a, b, c, d, e, f$ are not necessarily distict, do the involved sequences satisfy linear recurrences of order $\leq C(N)$, where $C$ is some bounded function? The answer to this question is No, as the recurrence $a_n a_{n-2} = a_{n-1} a_{n-1} + a_{n-1} a_{n-1}$ with appropriate initial conditions has $a_n = 2^{n(n-1)/2)}$ as a solution. However, there is probably something along these lines that is true. I am currently searching for that thing.

On the level of more generality: We must show first that a rational generating function $M_n(a, x)$ exists for each family of $2k$-by-$2n$ rectangles, holding $k$ fixed. Conjecture 1 probably follows easily from here. One way to proceed to prove Conjecture 2 is to show that these generating functions obey $M_n(a, x) = M_n(2/a, x)$ for all $n$. One possible way to do this is to obtain the generating function $G(a, x, y) = M_0(a, x) + M_1(a, x)y + M_2(a, x)y^2 + \cdots$ and show $G(a, x, y) = G(2/a, x, y)$, but $F$ is certainly not rational and is unlikely to have a nice closed form. This is because the problem of finding a closed form for $G$ is a direct generalization of the following problem (i.e. a solution to the former would immediately give a solution to the latter, by setting $a = 1$ or 2): if $q_{n,k}$ is the number of domino tilings of the $2n$-by-$2k$ rectangle, and $f_n(x) = q_{n,0} + q_{n,1}x + q_{n,2}x^2 + \cdots$ for each $n$, find a closed form for $F(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots$. The only known formula for domino tilings of rectangles involves large products of trigonometric functions and no one has so far been able to find a nice closed form for the generating function.

# References


[1] E. Kuo, *Applications of Graphical Condensation for Enumerating Matchings and Tilings*, preprint 2002