Richard Stanley and Combinatorial Reciprocity

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For our purposes, *combinatorial reciprocity* will denote the relationship between

(1) a combinatorially-defined sequence

$$a_0, a_1, a_2, a_3, \ldots$$

given by some formula or recurrence relation, and

(2) the extrapolated sequence

$$a_{-1}, a_{-2}, a_{-3}, \ldots,$$

especially when the latter also has combinatorial meaning.

See EC I (chapter 4).

Colorings and orientations

Let G be a finite graph with n vertices.

A *k-coloring* of G is a way to assign the colors 1,2,...,k to the vertices of G such that no two adjacent vertices have the same color.

There exists a polynomial $\chi_G(x)$ (the *chromatic polynomial* of G) such that for all $k \geq 1$, $\chi_G(k)$ is the number of k-colorings of G.

An acylic orientation of G is a way to orient the edges of G so that no cycle is created.

Stanley (1973): $\chi_G(-1)$ is equal to $(-1)^n$ times the number of acylic orientations of G.

E.g., let G be the complete graph K_3.
$\chi_G(k) = k(k-1)(k-2)$.
So $\chi_G(-1) = (-1)(-2)(-3) = -6 = (-1)^3 \times 6$.

Newer perspective

Question: What set has -1 elements?
Answer (according to Schanuel): the open interval $(0, 1)$ (or any other open interval)!

Euler's alternating sum: For an open interval, $V - E + F = 0 - 1 + 0 = -1$.

A *polyhedral set* is a subset of $\mathbf{R}^n$ that can be defined using linear equations and inequalities.

Theorem (Hadwiger-Lenz): There is a unique valuation (or finitely additive measure) on polyhedral sets, called *Euler measure*, that assigns measure 1 to every singleton in $\mathbf{R}^n$ and measure $(-1)^d$ to every polyhedral set in $\mathbf{R}^n$ that is homeomorphic to the open d-dimensional ball.

By finite additivity, Euler measure assigns measure $(V)(+1) + (E)(-1) + (F)(+1) + \ldots = V - E + F - \ldots$ to every polyhedral set with V vertices, E edges, F faces, ...

Edges, faces, etc. need not be bounded.

E.g., $\mathbf{R}$ itself it homeomorphic to an open 1-dimensional ball and so has Euler measure -1.

The set of $\mathbf{R}$-colorings of a graph G is just the complement of the hyperplane arrangement associated with G.

The hyperplane arrangement associated with the graph G is the union of the hyperplanes $\{(x_1, x_2, ..., x_n) \in \mathbf{R}^n : x_i = x_j\}$, where the union is taken over all i, j such that the ith and jth vertices of G are adjacent.

Example (continued): $G =$ the complete graph K_3 with vertices X, Y, and Z. The set of $\mathbf{R}$-colorings of G is $\{(x, y, z) \in \mathbf{R}^3 : x \neq y,\ x \neq z,\ y \neq z\}$. This is the complement of the hyperplane arrangement associated with G.

```
      z=0      y=0
        \     /
         \   /             (a 2-D slice)
  ______\/______  x=0
         /\
        /   \
       /     \
```

Each cell has Euler measure $(-1)^3$, and the cells correspond bijectively to the acylic orientations of G; e.g., the cell $\{(x, y, z) \in \mathbf{R}^3 : x < y < z\}$ corresponds to the acylic orientation $X \to Y \to Z$.

This point of view is due to Greene and Zaslavsky.

More generally, suppose S is an subset of $\mathbf{R}$ with Euler measure k for any integer k (positive or negative or zero). If we intersect the complement of the hyperplane arrangement associated with the graph G with the subset $S^n = S \times S \times ... \times S$ of $\mathbf{R}^n$, we get a set whose Euler measure is $\chi_G(k)$.

Exercise: Apply this to the case where S is a union of disjoint open intervals. Use this to rederive Stanley's combinatorial interpretation of $\chi_G(k)$ where k is any negative integer.

Upshot: We can think of the polyhedral subsets of $\mathbf{R}$ as modelling elements of $\mathbf{Z}$, in much the same way that the finite subsets of $\mathbf{R}$ model elements of $\mathbf{N}$.

Tilings

A *domino* is a 1×2 (or $2{\times}1$) rectangle.

For any fixed m, let $T(m, n)$ be the number of domino-tilings of the m-by-n rectangle. We can give a formula for $T(m, n)$ that makes sense for *all* integers n, not just non-negative integers.

Stanley (1985), paraphrased: The left half of the doubly-infinite sequence

$$..., T(m, -1), T(m, 0), T(m, 1), ...$$

looks just like the right half, up to sign; e.g.,

$m = 2$: $..., -8, 5, -3, 2, -1, 1, 0, \mathbf{1}, 1, 2, 3, 5, 8, ...$

$m = 3$: $..., 41, 0, 11, 0, 3, 0, 1, 0, \mathbf{1}, 0, 3, 0, 11, 0, 41, ...$

(where the boldfaced **1**'s correspond to $n = 0$).

That is,

$$T(m, -n) = \pm T(m, n - 2).$$

(The sign is $-$ if $m \equiv 2 \pmod 4$ and n is odd; otherwise, the sign is $+$.)

Newer perspective

$T(2,3) = 3$:

Let $S = \{1, 2, 3\}$. Represent each cell of the 2-by-3 board by a point, so that the cells correspond to the points of the product set $S \times \{1, 2\}$.

We represent each tiling of the board by a different way of marking some subset of $\{1, 2, 3\} \times \{1, 2\}$ with $*$'s.

Specifically, we mark a point with a $*$ if and only if the corresponding cell in the board is covered by a vertical domino. The allowed markings are

```
*  o  o

*  o  o
```

and

```
o  o  *

o  o  *
```

and

```
*  *  *

*  *  *
```

$T(2,-5) = -3$:

Let S be the union of five disjoint open intervals. Mark the set $S \times \{1,2\}$ with $*$'s. The allowed markings are of the form

```
(---) (-*-) (---) (---) (---)
(---) (-*-) (---) (---) (---)
```

and

```
(---) (---) (---) (-*-) (---)
(---) (---) (---) (-*-) (---)
```

and

```
(---) (-*-) (-*-) (-*-) (---)
(---) (-*-) (-*-) (-*-) (---)
```

where `(-*-)` denotes an open interval with exactly one marked point. Marks that are shown one above another are required to be aligned.

The "number" of ways to mark $S \times \{1,2\}$ (that is, the Euler measure of the set of allowed markings) is $(-1)^1 + (-1)^1 + (-1)^3 = -3$.

What markings are legal?

1. The number of $*$'s is finite.

2. In each column, the starred points must consist of disjoint consecutive pairs. That is: Imagine each column begins and ends with an extra non-starred point. Then, in each column, between any two non-starred points, the number of starred points must be even.

3. Imagine each row begins and ends with an extra $*$. Then, in each row, between any two starred points, the set of non-starred points must be a set whose Euler measure is even.

$T(3,2) = 3$:

```
o  o

o  o

o  o
```

and

```
*  *

*  *

o  o
```

and

```
o  o

*  *

*  *
```

The number of ways to mark is +3.

$T(3, -4) = +3$:

```
(---) (---) (---) (---)
(---) (---) (---) (---)
(---) (---) (---) (---)
```

and

```
(---) (-*-) (-*-) (---)
(---) (-*-) (-*-) (---)
(---) (---) (---) (---)
```

and

```
(---) (---) (---) (---)
(---) (-*-) (-*-) (---)
(---) (-*-) (-*-) (---)
```

The "number" of ways to mark is $(-1)^0 + (-1)^2 + (-1)^2 = +3$.

In each case, there is a direct way to relate $T(m, n)$ and $T(m, -n-2)$.

For a purely combinatorial version of this, extended to the case where we can use 1×1 tiles as well, see `math.CO/0304359`: A reciprocity theorem for monomer-dimer coverings, by N. Anzalone, J. Baldwin, I. Bronshtein, T.K. Petersen (written under the auspices of Research Experiences in Algebraic Combinatorics at Harvard).

Other work (unpublished) on this topic has been done by David Speyer (also under the auspices of REACH).

Challenge: Can we use an approach like this to interpret $T(m, n)$ when m and n are *both* negative?