Confluence and Near-Confluence

James Propp, UMass Lowell

(based on work with Sam Hopkins and Thomas McConville, with help from Peter Winkler and Ander Holroyd)

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Slides at http://jamespropp.org/uw17a.pdf

I. Definitions and Warm-Up Examples

Given a finite acyclic directed graph and a starting vertex s, we say that confluence holds iff all maximal paths from s terminate at the same sink vertex t. ("Maximal" means "If you're not at a sink, don't stop".)

Directed graphs will often be called games, in which case vertices will be called states and directed edges will be called moves.

A state from which no moves can be made (corresponding to a sink) will be called stable.

Example 1

Each state is a finitely-supported function $c : \mathbb{Z} \to \mathbb{N}$, where we think of $\mathbb{Z}$ as the vertex set of an infinite path; $c(i)$ is the number of (indistinguishable) “chips” at vertex i.

When there are two or more chips at i, a legal move is to slide one of the chips to $i + 1$.

Claim: This game is confluent. In particular, if we start with n chips at 0, we end with a single chip at each of vertices 0 through $n - 1$, after exactly $n(n-1)/2$ moves.

Example 2

Each state is a function $\mathcal{C} : \{1, 2, \ldots, n\} \to \mathbb{Z}$, where we think of $\mathcal{C}(a)$ as the location of chip (a) $(1 \leq a \leq n)$.

When $\mathcal{C}(a) = i = \mathcal{C}(b)$ (that is, chips (a) and (b) are both at vertex i) with $a < b$, a legal move is to slide chip (b) to vertex $i + 1$.

Claim: This game is confluent. In particular, if we start with chips (1), (2), ..., (n) at 0, we end with chip (1) at 0, chip (2) at 1, ..., and chip (n) at $n - 1$, after exactly $n(n-1)/2$ moves.

Ethological "application"

A bunch of herd animals are released into a territory, all initially at the same rank (call it rank 0).

When two animals of unequal rank meet, they peacefully go their separate ways; when two animals of the same rank meet, they fight to establish dominance.

The stronger of the two animals always wins a fight (we assume that no two of the animals have the same strength).

The rank of an animal is equal to the number of fights it's won.

Fights happen until all animals have unequal rank.

The Claim says that eventually the ranking will agree with the animals' relative strengths (that is, A will have higher rank than B iff A is stronger than B).

Example 3

Each state is a finitely-supported function $c : \mathbb{Z} \to \mathbb{N}$.

When there are two or more chips at i, a legal move is to slide one chip to $i - 1$ and another to $i + 1$ (i.e., to "fire vertex i").

Claim (Bak, Tang, and Wiesenfeld) This game is confluent. In particular (Anderson, Lovász, Shor, Spencer, Tardos, and Winograd), if we start with $n = 2m + r$ chips at 0 (with $r = 0$ or 1), we end with isolated chips at $-m, \ldots, -1, 1, \ldots, m$ and r isolated chips at 0, after exactly $1^2 + 2^2 + \cdots + m^2$ moves.

In fact, the number of times vertex i fires is independent of the sequence of firings.

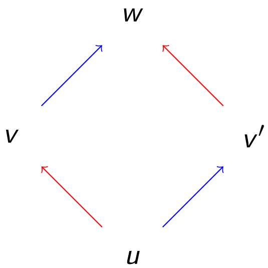
A helpful lemma for proving claims like this is...

The Diamond Lemma

Diamond (Jordan-Hölder, Church-Rosser, Newman) Lemma:

Suppose we have an acyclic directed graph G with colors associated with its edges satisfying the following "diamond property":

If u has outgoing edges (u, v) and (u, v'), then the two edges have different colors, and there exist edges (v, w) and (v', w) such that (v, w) has the same color as (u, v') and (v', w) has the same color as (u, v).



The Diamond Lemma, concluded

Then EITHER no maximal path from s leads to a sink OR every maximal path from s leads to the same sink t after the same number of moves. Moreover, in the latter case, every maximal path from s is related to every other maximal path from s by a finite sequence of "diamond moves" (trading $\cdots \to u \to v \to w \to \ldots$ for $\cdots \to u \to v' \to w \to \ldots$), from which it follows that the multiset of colors used is the same for all paths from s to t.

The Diamond Lemma implies the Bak-... Theorem:
Color each move according to which site in $\mathbb{Z}$ gets fired.

Example 4

The **sandpile model**: Every state is a finitely-supported function from $V(G)$ to $\mathbb{N}$, where the graph G is infinite and connected. When there are $\deg(v)$ or more chips at v, a legal move is to slide one chip to each neighbor of v.

Claim: This game is confluent.

Proof: Diamond Lemma.

But the Diamond Lemma is not the end of the confluence story.

Example 5

Each state is a finitely-supported function $c : \mathbb{Z} \to \mathbb{N}$. There are three kinds of moves:

(1) If there are two chips at i, you can send one to $i - 1$ and one to $i + 1$.

(2) If there are chips at i and $-i$, you can move them both to the right (to $i + 1$ and $-i + 1$ respectively).

(3) If there is a chip at 0, you can move it to 1.

Claim: If we start with n chips at 0, this game is confluent, and ends with isolated chips at $1, \ldots, n$.

The Diamond Lemma does not apply here (at least not with the obvious coloring). I'll motivate and prove the Claim below.

Our Main Result (Hopkins, McConville, and Propp)

$$
\begin{tikzcd}
 & ? & \\
\text{Example 2} \arrow[ur] & & \text{Example 3} \arrow[ul] \\
 & \text{Example 1} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Example 6

Each state is a function $\mathcal{C} : \{1, 2, \ldots, n\} \to \mathbb{Z}$. When $\mathcal{C}(a) = i = \mathcal{C}(b)$ (that is, chips (a) and (b) are both at vertex i) with $a < b$, a legal move is to slide chip (a) to vertex $i - 1$ and chip (b) to vertex $i + 1$. All chips start at 0.

It's not obvious that confluence holds, and indeed, it only holds for some n.

Claim (Theorem 14 in Hopkins-McConville-Propp): For $n > 1$, this game is confluent iff n is even.

Ethological “application”

A bunch of herd animals are released into a territory, all initially at rank 0.

As before, two animals fight when they have unequal rank, and the stronger one wins.

What’s different is that the rank of an animal is now equal to the number of fights it’s won minus the number of fights it’s lost.

Our theorem says that eventually the ranking will agree with the animals’ relative strengths, provided the number of animals in the herd is even (but not necessarily if the number is odd).

Remarks

It's amusing to note what happens when you start with $2m-1$ chips at 0, stabilize, add a chip at 0, and stabilize again. The first stabilization can give a nontrivial permutation of $1, 2, \ldots, 2m-1$, but adding another chip and stabilizing again gives the trivial (sorted) permutation of $1, 2, \ldots, 2m-1, 2m$.

It's also worth mentioning that the sorting process requires on the order of n^3 steps (or n^2 if processing is done in parallel), so it is not a competitive algorithm for sorting a list.

There are several ideas in the proof; the one I will mention here is Lemma 10, which is an independently interesting property of ordinary chip-firing on $\mathbb{Z}$.

Lemma 10

Let $\tilde{c}$ denote the stabilization of c.

Write $c \lessdot d$ iff d is obtained from c by moving one chip one step to the right.

Lemma 10: If $c \lessdot d$ then $\tilde{c} \lessdot \tilde{d}$.

E.g.: $(\ldots, 0, 2, 1, 0, \ldots) \lessdot (\ldots, 0, 1, 2, 0, \ldots)$; stabilizing both, we get $(\ldots, 1, 1, 0, 1, \ldots) \lessdot (\ldots, 1, 0, 1, 1, \ldots)$.

(Note: The location of the chip whose promotion turns c into d need not be the location of the chip whose promotion turns $\tilde{c}$ into $\tilde{d}$.)

For more proof details see our article (or our FPSAC 2017 poster).

III. The Odd Conjecture

When $n \geq 3$ is odd, there are many permutations of $1, \ldots, n$ that are reachable from the start state. One such permutation is

$$2, 3, \ldots, m+1, 1, m+2, \ldots, 2m+1,$$

with m inversions (never fire chip (1)!).

Conjecture: Permutations of $1, \ldots, 2m+1$ that are reachable from the initial state have at most m inversions.

But that is not the Odd Conjecture...

“You’re fired!”

Consider two possible protocols for firing in the odd case:

1. Hillary makes deliberate choices about which pairs of chips to fire, up until the last move; then she fires randomly.

2. Donald makes random moves up until the last move; then he fires deliberately (sending the smallest chip at i to the left and the largest chip at i to the right, where i is the sole remaining site containing more than one chip).

Only the last move matters?!

It is easy to show that the probability that Hillary "wins" (by sorting the chips correctly) is $1/3$ if she makes choices optimally.

The Odd Conjecture: The probability that Donald wins goes to 1 as the (odd) number of chips goes to infinity.

The conjecture appears to hold for several natural interpretations of what "random play" means, with exponential convergence.

A typical endgame

$$\ldots$$

$$\{1\}, \{2\}, \{3\}, \{\ \}, \{4,6\},\ \{5\}\ ,\ \{7,8\}\ ,\ \{\ \}\ , \{9\}$$

$$\{1\}, \{2\}, \{3\}, \{4\},\ \{\ \}\ , \{5,6\},\ \{7,8\}\ ,\ \{\ \}\ , \{9\}$$

$$\{1\}, \{2\}, \{3\}, \{4\},\ \{5\}\ ,\ \{\ \}\ , \{6,7,8\},\ \{\ \}\ , \{9\}$$

$$\{1\}, \{2\}, \{3\}, \{4\},\ \{5\}\ ,\ \{6\}\ ,\ \{7\}\ ,\ \{8\}\ , \{9\}$$

IV. Graph-theoretic Variations

We can add self-loops to $\mathbb{Z}$.

Example 7: If (a), (b), and (c) are all at i with $a < b < c$, then we can send (a) to $i - 1$ and (c) to $i + 1$.

Conjecture: For $n > 2$, this game is confluent iff $n \equiv 3 \pmod 4$.

We can add parallel edges to $\mathbb{Z}$.

Example 8: If (a), (b), (c), and (d) are all at i with $a < b < c < d$, then we can send (a) and (b) to $i - 1$ and (c) and (d) to $i + 1$.

Conjecture: For $n > 3$, this game is confluent iff $n \equiv 0 \pmod 4$.

For other conjectures, see our paper.

V. Root-system Variations

Inspired by our paper, Pavel Galashin came up with

"Vector firing": We are given a set A of vectors in $\mathbb{R}^n$. A state of the game is a vector $v \in \mathbb{R}^n$. We are allowed to move from v to $v + \alpha$ iff $\alpha \in A$ is orthogonal to v.

Example 9: $A = \{e_j - e_i \ : \ 1 \leq i < j \leq n\}$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$, with $n \geq 2$.

Claim: This game is confluent from $\vec{0}$ iff n is even.

Example 9 is Example 6 in disguise!

(Or vice versa!)

Specifically, associate to each $\mathcal{C} : \{1, 2, \ldots, n\} \to \mathbb{Z}$ the vector $v = (\mathcal{C}(1), \ldots, \mathcal{C}(n)) \in \mathbb{R}^n$. It is orthogonal to $e_b - e_a$ iff $\mathcal{C}(a) = \mathcal{C}(b)$, and sliding (a) to the left and (b) to the right is tantamount to adding $e_b - e_a$ to v.

The A from Example 9 is (one choice for) the set of positive roots of type A.

What if we try other types?

Example 10: "Type B" vector-firing

States are function $\mathcal{C} : \{-n, \ldots, -1, +1, \ldots, +n\} \to \mathbb{Z}$ that are "CP-invariant": if (a) is at i, $(-a)$ is at $-i$.

If $\mathcal{C}$ is CP-invariant, and it is possible to fire (a) and (b) from i (with $a < b$), then it is also possible to fire $(-b)$ and $(-a)$ from $-i$; we call these two (ordinary) moves dual to one another.

In Type B, we only allow self-dual moves and pairs of mutually dual moves. Type B moves preserve CP-invariance.

All chips start at 0.

Claim: This game is confluent for all n. This follows directly from our main theorem (Example 6) for $2n$ chips.

Where Example 5 came from

Confluence for Example 5 follows from confluence for Example 10: Just ignore all the chips with negative labels, and then erase the labels of the remaining chips! See Theorem 29 in HMP.

Is there a more direct proof?

For info on confluence in other types, and many interesting generalizations, see upcoming work by Galashin, Hopkins, McConville, and Postnikov.

VI. Something Completely Different

Example 11: Chips come in two colors (blue and red). You can fire two chips of the same color as long as there is a third chip present (of either color) at the same vertex.

If you ignore colors, this model is confluent (in fact, it's just chip-firing on $\mathbb{Z}$-with-self-loops).

Our initial state consists of m blue chips and n red chips at 0.

Conjecture: If m is odd OR n is odd, confluence occurs.

Conjecture: If m and n are even, with $m \geq n > 0$, then there are $\binom{n}{n/2}$ stable states accessible from the initial state.

There's a nice distributive lattice structure on this set of states. If anyone wants to help me figure out how to prove these conjectures, let me know!

VII. Random sorting

Jordan Ellenberg suggested a variant form of labelled chip-firing in which, every time we fire two chips from site i, we sort the two incorrectly with probability p and sort them correctly with probability $1-p$, with fixed $0 < p < 1/2$.

(This process is not "distribution-confluent"; that is, the distribution on permutations that you get depends on the firing protocol you follow.)

Empirically, it seems that, as $n \to \infty$, the probability of the chips ending up perfectly sorted goes to 0, but

Conjecture: The expected number of inversions is on the order of n (and in particular is negligible compared to $\binom{n}{2}$).

(This appears to hold for both n even and n odd.)

VIII. Motivation

I came up with labelled chip firing as an outgrowth of thinking about Dhar's "abelian" processors model; see e.g. numerous recent articles by Levine and others.

I was hoping it would serve as an example of a "nearly abelian processors model", governed by a non-abelian critical group, but that turned out not to be the case.

IX. Miscellaneous

Question: How many firing and unfiring moves are needed to sort the antisorted permutation $n, n-1, \ldots, 2, 1$ (with n odd)?

When $n = 9$, the minimum (26) is smaller than the number of moves needed to pile up the chips at the origin and then sort them.

I am developing a suite of electronic puzzles based on labelled chip-firing, allowing both firing moves and unfiring moves: see http://mathenchant.org/chipchip/.

ChipChip is being developed in coordination with the Global Math Project, which will roll out on October 10, 2017.

The theme of Global Math Week 2017 is chip-firing — or rather, Exploding Dots (a pedagogical outgrowth of chip-firing developed by James Tanton based on a talk I gave fifteen years ago).

Thank you!

The slides for this talk are at http://jamespropp.org/uw17a.pdf.