Introduction to Communication Systems

Lecture 1: Review of Linear Systems

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Lecture 1 Objectives

- Review of basic functions (step, ramp, impulse)
- Classification and properties of linear systems
- Review of frequency techniques (Fourrier transform and series)

Some Important Properties of Signals

- DC Value:
  - Is the time average of a signal

$$x_{dc} = \langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

  - Over a finite interval $[t_1, t_2]$:

$$x_{dc} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x(t)dt$$

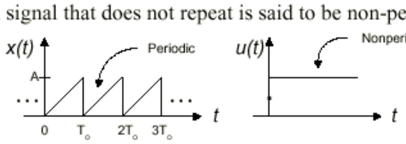
- Average Power:

$$P_{ave} = \langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt$$

- RMS Value:

$$x_{rms} = \sqrt{\langle x^2(t) \rangle} = \sqrt{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt}$$

- **Periodic and Nonperiodic Signals**
  - A periodic signal $x_p(t)$ repeats itself in a after time T called its period
  - That is, a periodic signal is one that has the following property
    $x_p(t) = x(t \pm kT)$ where k is an integer
  - That is, a periodic signal has no starting point or finishing time (eternal)
  - That is, a periodic signal repeats endlessly
  - A signal that does not repeat is said to be non-periodic



$$u(t) = \begin{cases} 1, & t>0 \\ \frac{1}{2}, & t=0 \\ 0, & t<0 \end{cases}$$

  - Some characteristics of period signals

$$Duty\ Ratio = \frac{Pulse\,Width}{Period}$$

$$Average\,Value\ x_{av} = \frac{1}{T}\int_{-T}^{T} x(t)dt$$

$$Power\ P = \frac{1}{T}\int_{-T}^{T} |x(t)|^2 dt$$

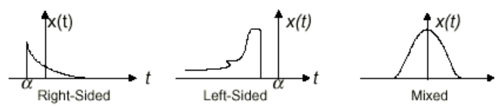
$$RMS\,Value\ x_{rms} = \sqrt{P}$$

Causal and Noncausal

- A signal *x(t)* is said to be causal (right sided at 0) if *x(t) = 0*, for all *t < 0*

$$x(t)=\begin{cases} A\cos(\omega_o t+\theta), & t\ge 0 \\ 0, & otherwise \end{cases}$$

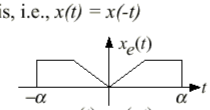
- Otherwise the signal *x(t)* is said to be noncausal (left sided at 0)
- Generally, a signal is right-sided (left-sided) if it is zero for t < α (t > α)
- A mixed signal (double sided) satisfies both conditions



- Right- or left-sided signals are said to have semi-infinite duration
- Signals with finite duration are said to be ***time-limited signal***
- A causal, finite duration or semi-finite duration signals can never be periodic

Even and Odd Signals

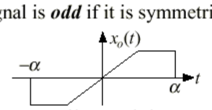
- The concept of even and odd is used to express symmetry that is present in signals covering a symmetric interval say (-α, α)
- A signal is said to be ***even*** if it is symmetric with respect to the vertical y-axis, i.e., *x(t) = x(-t)*



$x_e(t) = x_e(-t)$

$$\int_{-\alpha}^{\alpha} x_e(t)dt = 2\int_0^{\alpha} x_e(t)dt$$

- A signal is ***odd*** if it is symmetric with respect to the origin, i.e., *x(t) = -x(-t)*



$x_o(t) = -x_o(-t)$

$$\int_{-\alpha}^{\alpha} x_o(t)dt = 0$$

✍ ***Note: (also see page 14)***

- ❑ Sum of 2 even signals = EVEN
- ❑ Product of 2 even signals = EVEN
- ❑ Product of 2 odd signals = EVEN
- ❑ Sum of 2 odd signals = ODD
- ❑ Product of even & odd signals = ODD

- Sum of even and odd signals = No Symmetry
- Any signal can be written as the sum of its even and odd parts, i.e,

$$x(t) = x_e(t) + x_o(t)$$

- Conversely, any signal can be broken into odd and even parts

$$x_e(t) = \frac{x(t)+x(-t)}{2}, \quad x_o(t) = \frac{x(t)-x(-t)}{2}$$

- Even and Odd symmetry are mutually exclusive
  - If a signal is formed by summing its even and odd parts, then the signal has no symmetry
    - If $x(t)$ is even, then $x_o(t) = 0$, and if $x(t)$ is odd, then $x_e(t) = 0$

Example 1:

- $\boldsymbol{x(t) = A\cos(\omega t+\theta)}$ is neither even nor odd
- However, $x(t)$ is even when $\theta = 0$ and odd when $\theta = \pm\pi/2$
- Also $x(t)$ can be written in terms of even and odd parts

$$x_e(t) = \frac{A}{2}\cos\theta\cos\omega t; \qquad x_o(t) = -\frac{A}{2}\sin\theta\sin\omega t$$

- Since $\cos\omega t$ is even and $\sin\omega t$ is odd, then

$$x(t) = A\cos(\omega t+\theta) = \frac{A}{2}\cos\theta\cos\omega t - \frac{A}{2}\sin\theta\sin\omega t$$

Energy- and Power-Type Signals

- A signal is classified as *energy-type* if its energy *E,* is finite *(0 < E < ∞)*
- Energy may be computed in either time or frequency domain, whichever is easier

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$where\ |X(f)|^2 = \psi_X(f) \equiv Energy\ Spectral\ Density(ESD)$$

- The ESD gives the distribution of energy in frequency domain

$$E = \int_{-\infty}^{\infty} \psi_X(f)df = 2\int_{0}^{\infty} \psi_X(f)df$$

- All time-limited signals of finite amplitude are Energy signals
- Energy signals have zero power
- A signal is *power-type* if its power *P* is finite *(0 < P < ∞)*

$$P = \langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

- For periodic signals, E & P can be computed by integrating over one period
- Most periodic signals are power-type signals

- Similarly

$$G_X(f) = \lim_{n\to\infty} \frac{1}{T}|X_T(f)|^2 \equiv \textit{Power Spectral Density}(PSD)$$

$$P = \int_{-\infty}^{\infty} G_X(f)df = 2\int_{0}^{\infty} G_X(f)df$$

Example 3

- Find the energy of x(t) given by

$$x(t) = Ae^{-\alpha t}u(t),\ \alpha > 0$$

Example 4

- Find the power content of the signal x(t) given by

$$x(t) = A\cos(\omega_o t + \theta)$$

Some Important Signals/Functions

- **Sinusoidal Signal:**

$$x(t) = A\cos(w_o t + \theta)$$

Phase

Angular Frequency $w_o = 2\ f_o$

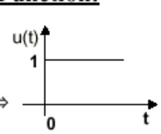
Amplitude

- **Complex Exponential Signal:**

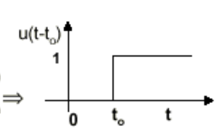
$$x(t) = Ae^{j(w_o t + \theta)}$$
$$= A\cos(w_o t + \theta) + jA\sin(w_o t + \theta)$$
$$\text{Re}\{x(t)\} = A\cos(w_o t + \theta)$$
$$\text{Im}\{x(t)\} = A\sin(w_o t + \theta)$$

- **Unit Step Function:**

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow$$

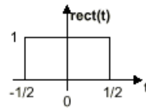


$$u(t) = \begin{cases} 1, & t - t_o \geq 0 \\ 0, & t - t_o < 0 \end{cases} \Rightarrow$$



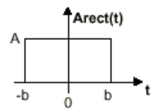
Rectangular Pulse Function

$$rect(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & |t|>\frac{1}{2}\end{cases} \Rightarrow$$

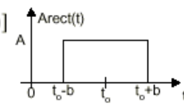


A rectangular function of unit amplitude and unit duration centered at $t = 0$

$$Arect(t)=A[u(t+b)-u(t-b)]$$
$$=\begin{cases}1, & |t|<b\\ 0, & |t|>b\end{cases} \Rightarrow$$



$$Arect(t-t_o)=A[u(t-t_o+b)-u(t-t_o-b)]$$
$$=\begin{cases}1, & |t-t_o|<b\\ 0, & |t-t_o|>b\end{cases} \Rightarrow$$



We may shift any signal on the time axis by simply replacing t by $t-t_o$. If $t_o>0$, the signal is shifted to the right and if $t_o<0$ it is shifted to the left.

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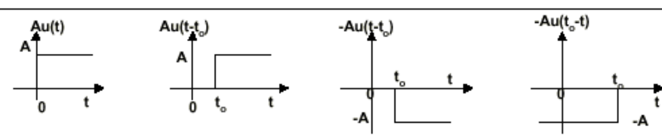
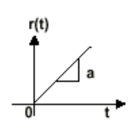
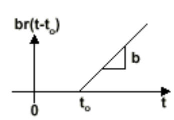


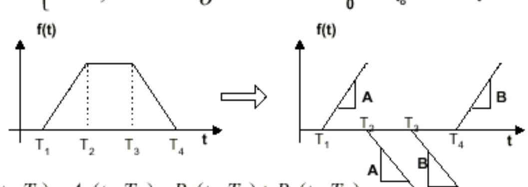
Ramp Function

$$r(at)=at(t)=\begin{cases}at, & t\geq 0\\ 0, & t<0\end{cases} \Rightarrow$$



$$br(t-t_o)=\begin{cases}br(t-t_o), & t\geq t_o\\ 0, & t<t_o\end{cases} \Rightarrow$$





$$f(t)=Ar(t-T_1)-Ar(t-T_2)-Br(t-T_3)+Br(t-T_4)$$

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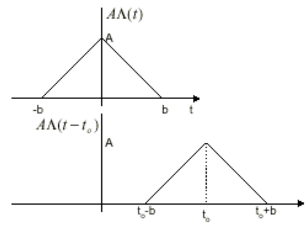
Triangular Pulse Function

$$\Lambda(t)=\begin{cases} A\left(1-\frac{|t|}{b}\right) & |t|<b \\ 0, & |t|>b \end{cases} \Rightarrow$$



$$\Lambda(t-t_o)=\begin{cases} A\left(1-\frac{|t-t_o|}{b}\right) & |t|\le b \\ 0, & |t|>b \end{cases} \Rightarrow$$

Sign (Signum) Function

$$\operatorname{sgn}(t)=\begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases} \Rightarrow$$

$$\operatorname{sgn}(t-t_o)=\begin{cases} 1, & t-t_o>0 \\ 0, & t-t_o=0 \\ -1, & t-t_o<0 \end{cases} \Rightarrow$$

$$\operatorname{sgn}(t)=-1+2u(t),\ |t|>0$$

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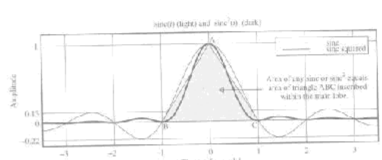
- The Sinc Function [denoted by sinc(t)]

$$\operatorname{sin}c(t)=\frac{\sin at}{at}, \quad -\infty<t<\infty$$

- Since the sine term oscillates and the 1/t term decreases with time, the sinc(t) shows decaying oscillation
- At t = 0, sinc(t) is not defined, but it is approximated using *l'Hopital's* rule

$$\lim_{t\to 0}\operatorname{sin}c(t)=\lim_{t\to 0}\frac{\sin at}{at}\approx\frac{at}{at}=1$$

$$\lim_{t\to 0}\operatorname{sin}c(t)=\lim_{t\to 0}\frac{\sin at}{at}=\lim_{t\to 0}\frac{a\cos(at)}{a}=1$$



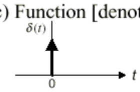
- sinc(0) = 1
- sinc(t) = 0 @ t = ±1, ±2, …

$$\operatorname{sin}c(t-t_o)=\frac{\sin a(t-t_o)}{a(t-t_o)}, \quad -\infty<t-t_o<\infty$$

Analog Signals 10

❍ The Impulse (Delta, Dirac) Function [denoted by δ(t)]

$$\delta(t)=\begin{cases}\infty, & t=0\\ 0, & t\neq 0\end{cases} \Rightarrow$$

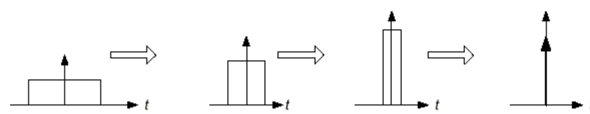


$$\int_{-\infty}^{\infty}\delta(t)dt = 1$$

$$\delta(t-t_o)=\begin{cases}\infty, & t=t_o\\ 0, & t\neq 0\end{cases} \Rightarrow$$

$$\int_a^b\delta(\tau-t_o)d\tau = \begin{cases}1, & a<t_o<b\\ 0, & else\end{cases}$$

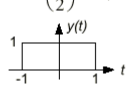
- δ(t) is very important in Engineering; many physical phenomena such as (a) point sources, (b) point charges, (c) concentrated load on structures (d) voltage or current sources acting for a very short time, etc., can be modeled as delta function
  - That is, any signal or behavior of a system that cannot be quantified by measuring instrument may be represented by the delta function
- Many functions behave like an impulse in their limiting form (see fig. 2.2)

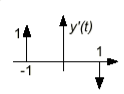


- If a signal has a sudden jump, its derivative gives rise to impulses

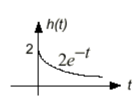
Example 7

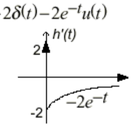
$y(t) = rect\left(\frac{t}{2}\right) = u(t+1)-u(t-1)$ $\quad \frac{d}{dt}y(t) = \delta(t+1)-\delta(t-1)$





$h(t) = 2e^{-t}u(t)$ $\quad \frac{d}{dt}h(t) = -2\delta(t)-2e^{-t}u(t)$





- Properties:

1) $\delta(t) = \frac{du(t)}{dt}$ $\quad$ 2) $u(t) = \int_{-\infty}^{t}\delta(t)dt$

3) *Scaling*: $\delta(\alpha t)=\frac{1}{|\alpha|}\delta(t)$

4) Pr*oduct*:

a) $x(t)\delta(t-\alpha)=x(\alpha)\delta(t-\alpha)$

b) $x(t)\delta(t)=x(0)\delta(t)$

5) $\int_{-\infty}^{\infty} x(t)\delta(t-\alpha)dt = x(\alpha) \quad \Rightarrow \quad \int_{a}^{b} x(t)\delta(t-\alpha)dt = \begin{cases} x(\alpha), & a<\alpha<b \\ \frac{1}{2}x(\alpha), & \alpha=a \;\; or \;\; \alpha=b \\ 0, & otherwise \end{cases}$

6) *Convolution*:

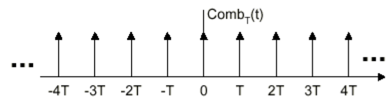
a) $x(t)*\delta(t) = x(t)$

b) $x(t)*\delta(t-\alpha) = x(t-\alpha)$

7) $\int_{-\infty}^{\infty} \delta^{(n)}(t-\alpha)x(t)dt = (-1)^n \frac{d^n}{dt^n}x(t)\Big|_{t=\alpha}$

- The Comb Function

$$Comb_T(t) = \sum_{n=1}^{\infty} \delta(t-nT)$$



Operations on Signals

- **Amplitude Scaling**

  $Kx(t)$, K is a constant

- **Amplitude Shifting**

  $K + x(t)$, K is a constant

- **Time Shifting**
  - Displaces a signal in time without changing its shape

    $y(t) = x(t \pm \alpha)$

    "+" shifts the signal left by α

    "-" shifts the signal right by α (delayed)

- **Time Scaling**
  - Slows down or speeds up time which results in signal compression or stretching

    $y(t) = x\left(\frac{t}{\alpha}\right)$

    - Implies α-fold expansion of x(t) since it slows down x(t) to t/α

    $y(t) = x(\alpha t)$

    - Implies α-fold compression of x(t) since it speeds up *x(t)* to αt

- **Reflection or Folding**
  - A scaling operation with $\alpha = -1 \Rightarrow x(t) = x(-t)$
  - The mirror image of $x(t)$ about the y-axis through t = 0
- **Operations in Combinations**
  - Shifting, scaling and folding operations can be combined and performed in succession

    $$y(t) = x(\alpha t \pm \beta)$$

    - x(t) $\Rightarrow$ delay (shift right) by $\beta \Rightarrow x(t-\beta) \Rightarrow$ compress by $\alpha \Rightarrow x(\alpha t-\beta)$
    - x(t) $\Rightarrow$ compress by $\alpha \Rightarrow x(\alpha t) \Rightarrow$ delay (shift right) by $\beta/\alpha \Rightarrow x(\alpha t-\beta)$

- Example 8:
  - Expressing signal by interval
- Example 9 (see example 2.2)

  $$x(t) = \begin{cases} \frac{3}{2}t, & 0 \le t \le 2 \\ 0, & else \end{cases}$$

  - Sketch $f(t) = 1 + x(t-1)$, $g(t) = x(t)$, $h(t) = x(0.5t+0.2)$, $w(t) = x(-2t+2)$
- Example 10 (see example 2.8)

  $$x(t) = 2r(t) - 2r(t-2) - 4u(t-3)$$

  - Sketch *x(t) and it derivatives*

Discrete-Time Signals

- Discrete-Time Signals (DTS) denoted by x[n] are signals defined at discrete times having discrete valued independent variable
- Independent variable ***n*** takes on integers values, e.g., x[n]=Acos[2πfn+θ], *n=1,2*
  - stem(cos(2*pi*0.07*(0:30)));
- That is DTS are represented as sequences of numbers

$$\Downarrow$$

$$x[n] = \{1, 2, 4, 8\cdots\}$$

- Digital Time Signals arises
  - Naturally (i.e. generated directly by discrete-time process)
  - Periodic sampling of a continuous-time signal
- A DTS is represented mathematically as sequences of numbers

$$x[n] = x(nt_s) = x(nT),\ n \text{ is an integer}$$

Classification

- **Periodic and Nonperiodic**
  - A discrete periodic signal repeats itself every *N* samples called its period

    $x[n] = x[n \pm kN]$ where k is an integer
  - The period N is the smallest number of samples that repeats
  - The period N is always an integer
  - For sums of DTS, the period N is the LCM of the individual periods
- **Causal and Noncausal (page 39)**
  - X[n] is causal if it is zero for $n < 0$
  - X[n] is noncausal if it is zero for $n \geq 0$
  - If the sequence is bounded by N, then
    - X[n] is left-sided if it is zero for $n > N$
    - X[n] is right-sided if it is zero for $n < N$
- **Even and Odd Parts**
  - The definition is similar to the analog counterpart
  - If x[n] = x[-n], the signal is said to be even
  - If x[n] = -x[-n], the signal is said to be odd

$$x[n] \Rightarrow \begin{cases} x_e[n] = x_e[-n], & even \\ x_o[n] = -x_o[-n], & odd \end{cases}$$

- **Note**

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n]$$

$$x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$$

$$x[-n] = x_e[n] - x_o[n] = x_e[-n] + x_o[-n]$$

$$\sum_{n=-M}^{M} x_o[n] = 0$$

$$x_o[0] = 0$$

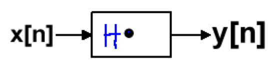
Classification of Systems

- A system is an interconnection of various elements that work as a whole
- In communication, a system is viewed as an entity that is excited by an input signal and produces an output signal
- That is, is a system is considered as a device, algorithm, process that, given an input *x(t)*, produces an output *y(t)*

- From a communications engineer's point of view, a system is a law that assigns output signal to various input signal such that $y(t) = O[x(t)]$
- Like signals, systems can be classified according to their behavior or properties as follows:

① ***Discrete-Time and Continuous-Time Systems***

- A discrete-time system accepts discrete-time signals as input and produces discrete-time signals at the output



- In continuous-time systems, both input and output signals are continuous-time signals

② ***Linear and Nonlinear Systems***

- Linear systems are systems for which superposition property is satisfied
  - the response of the system to linear inputs is a corresponding linear output
- Otherwise a system is said to be nonlinear

③ ***Causal and Noncausal Systems***

- A system is causal if its output at any time-instant t_o depends on the input at time prior to t_o
- In order words, output signal depends only on the values of the input up till time t_o, and does not depend on future input values

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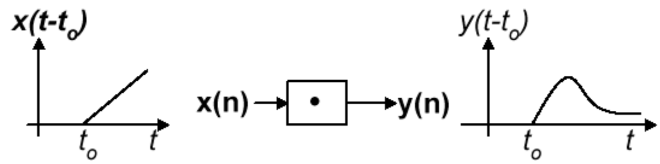
- The is the output is given by:

$$y(t) = f[x(t): t \le t_o]$$

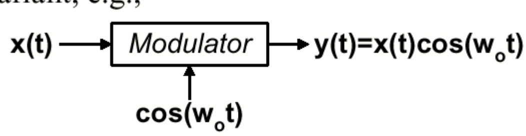
- Otherwise a system is said to be noncausal

④ ***Time-invariant and Time-varying Systems***

- A system is time-invariant if its input-output relation does not change with time
  - -- A time shift in the input results in a corresponding time shift in the output



- A system is time-varying if it is not time-invariant, e.g.,

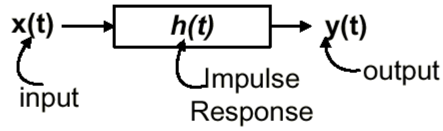


- The response to $x(t-t_o) = x(t-t_o)cos(w_ot)$ which is not equal to $y(t-t_o)$
- *Of particular interest is the group of linear systems called Linear Time Invariant (LTI) systems*
- *This is because most communication systems, -channels, transmitter/receiver sub-components, etc., can be modeled as a LTI system*

6

Linear Time Invariant Systems

- **Computing the output of a LTI system**



  - LTI systems are those linear systems that obey the law of superposition
  - The input-output (I/O) characteristics are similar; e.g., when a periodic signal *x(t)* is passed through a LTI system, the output signal *y(t)* is also periodic usually with the same period as the input