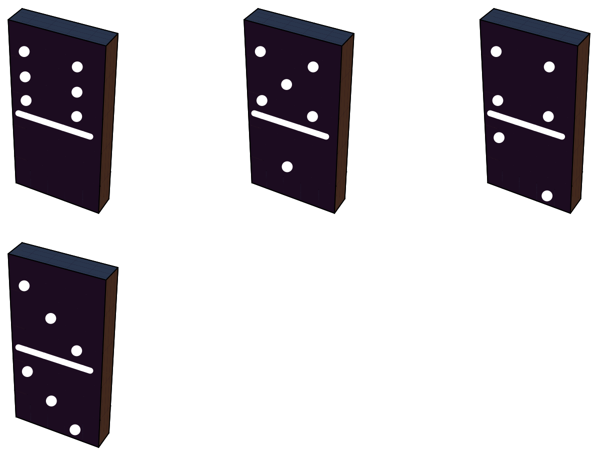
# chapter 6

# RELATIONS AND GRAPHS



### GOALS

One understands a set of objects completely only if the structure of that set is made clear by the interrelationships between its elements. For example, the individuals in a crowd can be compared by height, by age, or through any number of other criteria. In mathematics, such comparisons are called relations. The goal of this chapter is to develop the language, tools, and concepts of relations.

## 6.1 Basic Definitions

In Chapter 1 we introduced the concept of the Cartesian product of sets. Let's assume that a person owns three shirts and two pairs of slacks. More precisely, let $A = \{\text{blue shirt}, \text{tan shirt}, \text{mint green shirt}\}$ and $B = \{\text{grey slacks}, \text{tan slacks}\}$. Then certainly A × B is the set of all possible combinations (six) of shirts and slacks that the individual can wear. However, the individual may wish to restrict himself or herself to combinations which are color coordinated, or "related." This may not be all possible pairs in $A \times B$ but will certainly be a subset of $A \times B$. For example, one such subset may be $\{(\text{blue shirt}, \text{grey slacks}), (\text{blue shirt}, \text{tan slacks}), (\text{mint green shirt}, \text{tan slacks})\}$.

***Definition: Relation.*** *Let A and B be sets. A relation from A into B is any subset of $A \times B$.*

**Example 6.1.1**. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Then $\{(1, 4), (2, 4), (3, 5)\}$ is a relation from $A$ into $B$. Of course, there are many others we could describe; 64, to be exact.

**Example 6.1.2.** Let $A = \{2, 3, 5, 6\}$ and define a relation $r$ from $A$ into $A$ by $(a, b) \in r$ if and only if $a$ divides evenly into $b$. The set of pairs that qualify for membership is $r = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$.

***Definition: Relation on a Set.*** *A relation from a set A into itself is called a relation on A.*

The relation "divides" in Example 6.1.2 is will appear throughout the book. Here is a general definition on the whole set of integers.

***Definition: Divides.*** *Let $a, b \in \mathbb{Z}$.*
*$a \mid b$ if and only if there exists an integer k such that $a\,k = b$.*

Be very careful in writing about “divides.” The vertical line symbol use for this relation, if written carelessly, can look like division. While $a \mid b$ is either true or false, $a / b$ is a number.

Based on the equation $a\,k = b$, we can say that $a \mid b$ is equivalent to $k = \frac{b}{a}$, or $a$ divides evenly into $b$. In fact the "divides" is short for "divides evenly into." You might find the equation $k = \frac{b}{a}$ initially easier to understand, but in the long run we will find the equation $a\,k = b$ more convenient.

Sometimes it is helpful to illustrate a relation with a graph. Consider Example 6.1.1. A graph of $r$ can be drawn as in Figure 6.1.1. The arrows indicate that 1 is related to 4 under $r$. Also, 2 is related to 4 under $r$, and 3 is related to 5, while the upper arrow denotes that $r$ is a relation from the whole set $A$ into the set $B$.

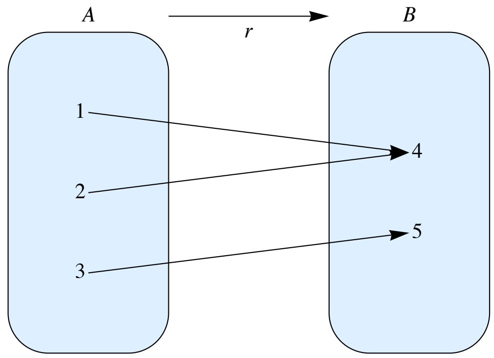



FIGURE 6.1.1 A graph of a relation

A typical element in a relation $r$ is an ordered pair $(x, y)$. In some cases, $r$ can be described by actually listing the pairs which are in $r$, as in the previous examples. This may not be convenient if $r$ is relatively large. Other notations are used with certain well-known relations. Consider the "less than on equal" relation on the real numbers. We could define it as a set of ordered pairs this way:

$$s = \{(x, y) \mid x \leq y\}.$$

The notation $x \leq y$ is clear and self-explanatory; it is a more natural, and hence preferred, notation to use than $(x, y) \in s$.

Many of the relations we will work with "resemble" the relation $\leq$, so $x\,s\,y$ is a common way to express the fact that $x$ is related to $y$ through the relation $s$.

***Relation Notion.*** *Let s be a relation from a set A into a set B. Then the fact that* $(x, y) \in s$ *is frequently written* $x\,s\,y$.

Let $A = \{2, 3, 5, 8\}$, $B = \{4, 6, 16\}$, and $C = \{1, 4, 5, 7\}$; let $r$ be the relation "divides," denoted by $|$, from $A$ into $B$; and let $s$ be the relation $\leq$ from $B$ into $C$. So $r = \{(2, 4), (2, 6), (2, 16), (3, 6), (8, 16)\}$ and $s = \{(4, 4), (4, 5), (4, 7), (6, 7)\}$.

Notice from Figure 6.1.2 that we can, for certain elements of $A$, go through elements in $B$ to results in $C$. That is:

$$2 \mid 4 \text{ and } 4 \leq 4$$

$$2 \mid 4 \text{ and } 4 \leq 5$$

$$2 \mid 4 \text{ and } 4 \leq 7$$

$$2 \mid 6 \text{ and } 6 \leq 7$$

$$3 \mid 6 \text{ and } 6 \leq 7$$

Based on this observation, we can define a new relation, call it $rs$, from $A$ into $C$. In order for $(a, c)$ to be in $rs$, it must be possible to travel along a path in Figure 6.1.2 from $a$ to $c$. In other words, $(a, c) \in rs$ if and only if $(\exists\, b)_B\,(a\,r\,b$ and $b\,s\,c)$. The name $rs$ was chosen because it reminds us that this new relation was formed by the two previous relations $r$ and $s$. The complete listing of all elements in $rs$ is $\{(2, 4), (2, 5), (2, 7), (3, 7)\}$. We summarize in a definition.

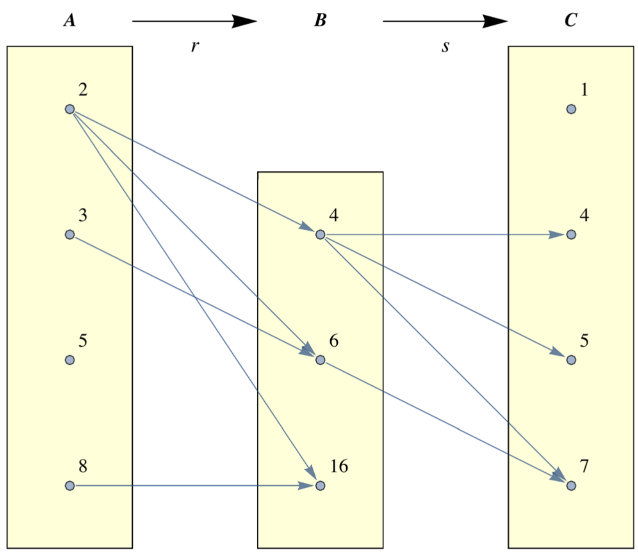



FIGURE 6.1.2 Graphical representation of composition of relations

***Definition: Composition of Relations.*** *Let $r$ be a relation from a set $A$ into a set $B$, and let $s$ be a relation from $B$ into a set $C$. The composition of $r$ with $s$, written $rs$, is the set of pairs of the form $(a, c) \in A\times C$, where $(a, c) \in rs$ if and only if there exists $b \in B$ such that $(a, b) \in r$ and $(b, c) \in s$.*

Remark: A word of warning to those readers familiar with composition of functions. (For those who are not, disregard this remark. It will be repeated at an appropriate place in Chapter 7.) As indicated above, the traditional way of describing a composition of two relations is $rs$ where $r$ is the first relation and $s$ the second. However, function composition is traditionally expressed "backwards"; that is, $s\circ r$, where $r$ is the first function and $s$ is the second.

## EXERCISES FOR SECTION 6.1

### A Exercises

1. For each of the following relations $r$ defined on $\mathbb{P}$, determine which of the given ordered pairs belong to $r$.

(a) $x\,r\,y$ iff $x \mid y$; $(2, 3), (2, 4), (2, 8), (2, 17)$

(b) $x\,r\,y$ iff $x \leq y$; $(2, 3), (3, 2), (2, 4), (5, 8)$

(c) $x\,r\,y$ iff $y = x^2$ ; $(1,1), (2, 3), (2, 4), (2, 6)$

2. The following relations are on $\{1, 3, 5\}$. Let $r$ be the relation $x\,r\,y$ iff $y = x + 2$ and $s$ the relation $x\,s\,y$ iff $x \leq y$.

(a) Find $rs$.

(b) Find $sr$.

(c) Illustrate $rs$ and $sr$ via a diagram.

(d) Is the relation (set) $rs$ equal to the relation $sr$? Why?

3. Let $A = \{1, 2, 3, 4, 5\}$ and define $r$ on A by $x\,r\,y$ iff $x + 1 = y$. We

define $r^2 = rr$ and $r^3 = r^2 r$. Find:

(a) $r$

(b) $r^2$

(c) $r^3$

4. Given $s$ and $t$, relations on $\mathbb{Z}$, $s = \{(1, n) : n \in \mathbb{Z}\}$ and $t = \{(n, 1) : n \in \mathbb{Z}\}$, what are $st$ and $ts$? Hint: Even when a relation involves infinite sets, you can often get insights into them by drawing partial graphs.

### B Exercises

5. Let $\rho$ be the relation on the power set, $\mathcal{P}(S)$, of a finite set S of cardinality $n$. Define $\rho$ by $(A, B) \in \rho$ iff $A \cap B = \emptyset$.

(a) Consider the specific case $n = 3$, and determine the cardinality of the set $\rho$.

(b) What is the cardinality of $\rho$ for an arbitrary $n$? Express your answer in terms of $n$. (Hint: There are three places that each element of S can go in building an element of $\rho$.)

6. Let $r_1$, $r_2$, and $r_3$ be relations on any set $A$. Prove that if $r_1 \subseteq r_2$ then $r_1 r_3 \subseteq r_2 r_3$.

## 6.2 Graphs of Relations on a Set

In this section we introduce directed graphs as a way to visualize relations on a set.

**Example 6.2.1**, Let $A = \{0, 1, 2, 3\}$, and let

$$r = \{(0, 0), (0, 3), (1, 2), (2, 1), (3, 2), (2, 0)\}.$$

The elements of $A$ are called the vertices of the graph. They are represented by labeled points or occasionally by small circles. We connect vertex $a$ to vertex $b$ with an arrow, called an edge, going from vertex $a$ to vertex $b$ if and only if $a\,r\,b$. This type of graph of a relation $r$ is called a *directed graph* or *digraph*. Figure 6.2.1 is a digraph for $r$. Notice that since $1\,r\,2$ and $2\,r\,1$, we draw a single edge between 1 and 2 with arrows in both directions. Since 0 is related to itself, we draw a "self-loop" at 0.

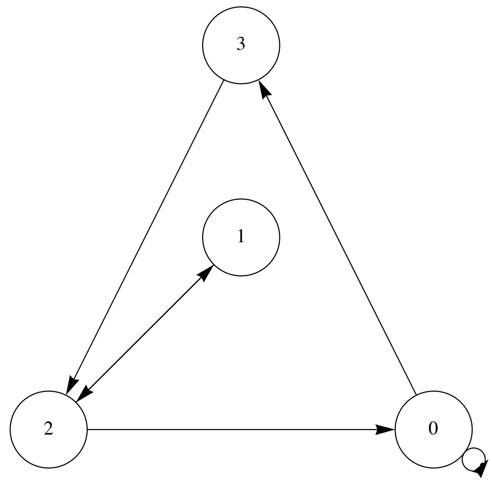



**FIGURE 6.2.1**

The actual location of the vertices is immaterial. The main idea is to place the vertices in such a way that the graph is easy to read. Obviously, after a rough-draft graph of a relation, we may decide to relocate the vertices so that the final result will be neater. Figure 6.2.1 could also be presented as in Figure 6.2.2.

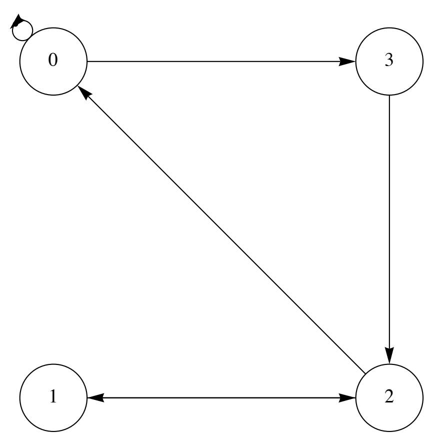



**FIGURE 6.2.2**

A vertex of a graph is also called a node, point, or a junction. An edge of a graph is also referred to as an arc, a line, or a branch. Do not be concerned if two graphs of a given relation look different.

**Example 6.2.2.** Consider the relation $s$ whose digraph is Figure 6.2.3. What information does this give us? The graph tells us that $s$ is a relation on $A = \{1, 2, 3\}$ and that

$$s = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 3)\},$$

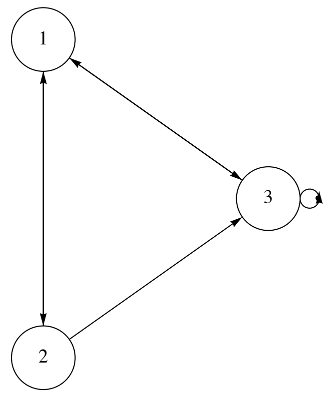



FIGURE 6.2.3

**Example 6.2.3.** Let $B = \{1, 2\}$, and let $A = \mathcal{P}(B) = \{0, \{1\}, \{2\}, \{1, 2\}\}$. Then $\subseteq$ is a relation on $A$ whose digraph is Figure 6.2.4.

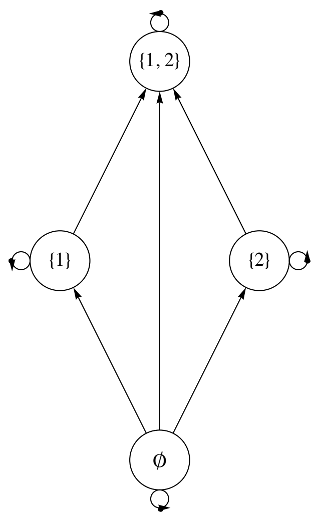



**FIGURE 6.2.4**

We will see in the next section that since $\subseteq$ has certain structural properties that describe "partial orderings." We will be able to draw a much simpler type graph than this one, but for now the graph above serves our purposes.

## EXERCISES FOR SECTION 6.2

### A Exercises

1. Let $A = \{1, 2, 3, 4\}$, and let $r$ be the relation $\leq$ on $A$. Draw a digraph for $r$.

2. Let $B = \{1, 2, 3, 4, 6, 8, 12, 24\}$, and let $s$ be the relation "divides," on $B$. Draw a digraph for $s$.

3. Let $A = \{1, 2, 3, 4, 5\}$. Define $t$ on $A$ by $a\, t\, b$ if and only if $b - a$ is even. Draw a digraph for $t$.

4. (a) Let $A$ be the set of strings of 0's and 1's of length 3 or less. Define the relation of $d$ on A by $x\, d\, y$ if $x$ is contained within $y$. For example, 01 $d$ 101. Draw a digraph for this relation.

   (b) Do the same for the relation $p$ defined by $x\, p\, y$ if $x$ is a prefix of $y$. For example, 10 $p$ 101, but 01 $p$ 101 is false.

### B Exercises

5. Recall the relation in Exercise 5 of Section 6.1, $\rho$ defined on the power set, $\mathcal{P}(S)$, of a set $S$. The definition was $(A, B) \in \rho$ iff $A \cap B = \emptyset$. Draw the digraph for $\rho$ where $S = \{a, b\}$.

6. Let $C = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and define $t$ on $C$ by

   $a\, t\, b$ if and only if $a$ and $b$ share a common divisor greater than 1.

   Draw a digraph for $t$.

## 6.3 Properties of Relations

Consider the set $B = \{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ 36,\ 48\}$ and the relations "divides" and $\leq$ on $B$. We notice that these two relations on $B$ have three properties in common:

(1) Every element in $B$ divides itself and is less than or equal to itself. This is called the reflexive property.

(2) If we search for two elements from $B$ where the first divides the second and the second divides the first, then we are forced to choose the the two numbers to be the same. In other words, no two *different* numbers are related in both directions. The reader can verify that a similar fact is true for the relation $\leq$ on $B$. This is called the antisymmetric property,

(3) Next if we choose three numbers from $B$ such that the first divides the second and the second divides the third, then we always find that the first number to divides the third. Again, the same is true if we replace "divides" with "is less than or equal to." This is called the transitive property.

Relations that satisfy these properties are of special interest to us. Formal definitions of the properties follow.

***Definition: Reflexive Relation.*** *Let A be a set and let r be a relation on A.*
*r is **reflexive** if and only if a r a for all $a \in A$.*

***Definition: Antisymmetric Relation.*** *Let A be a set and let r be a relation on A.*

*r is **antisymmetric** if and only if whenever a r b and $a \neq b$ then b r a is false.*

An equivalent condition for antisymmetry is that whenever $a\,r\,b$ and $b\,r\,a$ then $a = b$. You are encouraged to convince yourself that this is the case.

A word of warning about antisymmetry: Students frequently find it difficult to understand this definition. Keep in mind that this term is defined through an "If . .. then . . ." statement. The question that you must ask is: Is it true that whenever there are elements $a$ and $b$ from $A$ where $a\,r\,b$ and $a \neq b$, it follows that $b$ is not related to $a$? If so, then the relation $r$ is antisymmetric.

Another way to determine whether a relation is antisymmetric is to examine its digraph. The relation is *not* antisymmetric if there exists a pair of vertices that are connected by edges in both directions.

***Definition: Transitive Relation.*** *Let A be a set and let r be a relation on A.*

*r is **transitive** if and only if whenever a r b and b r c then a r c.*

### Partial Orderings

Not all relations have all three of the properties discussed above. Those that do, are a special type of relation.

***Definition: Partial Ordering, Poset.*** *A relation on a set A that is reflexive, antisymmetric, and transitive is called a partial ordering on A. A set on which there is a partial ordering relation defined is called a partially ordered set or poset.*

**Example 6.3.1.** Let $A$ be a set. Then $\mathcal{P}(A)$ together with the relation $\subseteq$ (set containment) is a poset. To prove this we observe that the three properties hold, as discussed in Chapter 4.

(1) Let $B \in \mathcal{P}(A)$. The fact that $B \subseteq B$ follows from the definition of subset. Hence, set containment is reflexive.

(2) Let $B_1,\ B_2 \in \mathcal{P}(A)$ and assume that $B_1 \subseteq B_2$ and $B_1 \neq B_2$ . Could it be that $B_2 \subseteq B_1$? No. There must be some element $a \in A$ such that $a \notin B_1$, but $a \in B_2$. This is exactly what we need to conclude that $B_2$ is not contained in $B_1$. Hence, set containment is antisymmetric

(3) Let $B_1,\ B_2, B_3 \in \mathcal{P}(A)$ and assume that $B_1 \subseteq B_2$ and $B_2 \subseteq B_3$ . Does it follow that $B_1 \subseteq B_3$ ? Yes, if $a \in B_1$, then $a \in B_2$ because $B_1 \subseteq B_2$. Now that we have $a \in B_2$ and we have assumed $B_2 \subseteq B_3$, we conclude that $a \in B_3$. Therefore, $B_1 \subseteq B_3$ and so set containment is transitive.

Figure 6.3.1 is the graph for the "set containment" relation on {1, 2}.

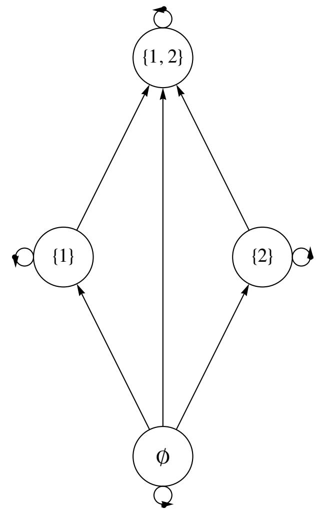



**FIGURE 6.3.1**

This graph is helpful insofar as it reminds us that each set is a subset of itself (How?) and shows us at a glance the relationship between the various subsets in $\mathcal{P}(\{1, 2\})$. However, when a relation is a partial ordering, we can streamline a graph like this one. The streamlined form of a graph is called a *Hasse diagram* or *ordering diagram*. A Hasse diagram takes into account the following facts.

(1) By the reflexive property, each vertex must be related to itself, so the arrows from a vertex to itself (called "self-loops") are not drawn in a Hasse diagram. They are simply assumed.

(2) By the antisymmetry property, connections between two distinct elements in a directed graph can only go one way, if at all. When there is a connection, we agree to always place the second element above the first (as we do above with the connection from $\{1\}$ to $\{1, 2\}$). For this reason, we can just draw a connection without an arrow, just a line.

(3) By the transitive property, if there are edges connecting one element up to a second element and the second element up to a third element, then there will be a direct connection from the first to the third. We see this in Figure 6.3.1 with $\emptyset$ connected to $\{1\}$ and then $\{1\}$ connected to $\{1, 2\}$. Notice the edge connecting $\emptyset$ to $\{1, 2\}$. Whenever we identify this situation, remove the connection from the first to the third in a Hasse diagram and simply observe that an upward path of any length implies that the lower element is related to the upper one.

Using these observations as a guide, we can draw a Hasse diagram for $\subseteq$ on $\{1, 2\}$ as in Figure 6.3.2.

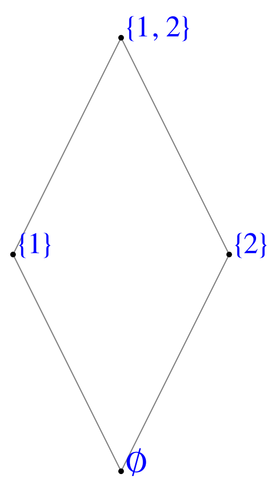



FIGURE 6.3.2

**Example 6.3.2.** Consider the partial ordering relation $s$ whose Hasse diagram is Figure 6.3.3.

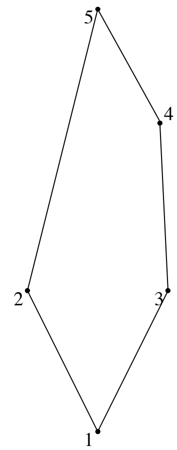



FIGURE 6.3.3

How do we read this diagram? What is $A$? What is $s$? What does the digraph of s look like? Certainly $A = \{1, 2, 3, 4, 5\}$ and $1\,s\,2$, $3\,s\,4$, $1\,s\,4$, $1\,s\,5$, etc., Notice that $1\,s\,5$ is implied by the fact that there is a path of length three upward from 1 to 5. This follows from the edges that are shown and the transitive property that is presumed in a poset. Since $1\,s\,3$ and $3\,s\,4$, we know that $1\,s\,4$. We then combine $1\,s\,4$ with $4\,s\,5$ to infer $1\,s\,5$. Without going into details why, here is a complete list of pairs defined by $s$.

$$s = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (1, 4), (1, 5), (1, 2), (3, 4), (3, 5), (4, 5), (2, 5)\}$$

A digraph for $s$ is Figure 6.3.4. It is certainly more complicated to read and difficult to draw than the Hasse diagram.

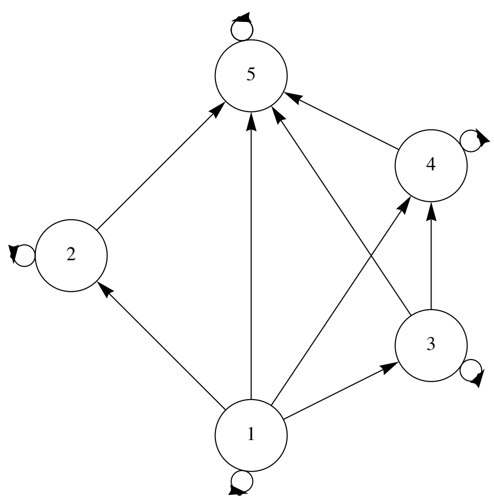



FIGURE 6.3.4 Digraph of Example 6,3.2

A classic example of a partial ordering relation is $\leq$ on the real numbers, $\mathbb{R}$. Indeed, when graphing partial ordering relations, it is natural to "plot" the elements from the given poset starting with the "least" element to the "greatest" and to use terms like "least," "greatest," etc. Because of this the reader should be forewarned that some texts use the symbol $\leq$ for arbitrary partial orderings. This can be quite confusing for the novice, so we continue to use generic letters $r$, $s$, etc.

## Equivalence Relations

Another common property of relations is symmetry.

***Definition: Symmetry.*** *Let r be a relation on a set A. r is* ***symmetric*** *if and only if whenever a r b, it follows that b r a.*

Consider the relation of equality (=) defined on any set $A$. Certainly $a = b$ implies that $b = a$ so equality is a symmetric relation on $A$.

Surprisingly, equality is also an antisymmetric relation on $A$. This is due to the fact that the condition that defines the antisymmetry property, $a = b$ and $a \neq b$, is a contradiction. Remember, a conditional proposition is always true when the condition is false. So a relation can be both symmetric and antisymmetric on a set! Again recall that these terms are *not* negatives of one other. That said, there are very few relations that are both symmetric and antisymmetric.

***Definition: Equivalence Relation.*** *A relation r on a set A is called an equivalence relation if and only if it is reflexive, symmetric, and transitive.*

The classic example of an equivalence relation is equality on a set $A$. In fact, the term equivalence relation is used because those relations which satisfy the definition behave quite like the equality relation. Here is another important equivalence relation.

**Example 6.3.3.** Let $\mathbb{Z}^*$ be the set of nonzero integers. One of the most basic equivalence relations in mathematics is the relation $q$ on $\mathbb{Z}\times\mathbb{Z}^*$ defined by $(a, b)\, q\, (c, d)$ if and only if $a d = b c$. We will leave it to the reader to, verify that $q$ is indeed an equivalence relation. Be aware that since the elements of $\mathbb{Z}\times\mathbb{Z}^*$are ordered pairs, proving symmetry involves four numbers and transitivity involves six numbers. Two ordered pairs, $(a, b)$ and $(c, d)$, are related if the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are numerically equal.

FIGURE 6.3.5.

**Example 6.3.4**. Let $m$ be a positive integer, $m \geq 2$. We define *congruence modulo m* to be the relation $\equiv_m$ defined on the integers by

$$a \equiv_m b \text{ if and only if } m \mid (a-b)$$

(1) This relation is reflexive, for if $a \in \mathbb{Z}$, $m \mid (a-a) \Rightarrow a \equiv_m a$.

(2) This relation is symmetric. We can prove this through the following chain of implications.

$$\begin{aligned} a \equiv_m b \quad &\Rightarrow \quad m \mid (a-b) \\ &\Rightarrow a-b = m k \quad \text{for some } k \in \mathbb{Z} \\ &\Rightarrow b-a = m(-k) \\ &\Rightarrow m \mid (b-a) \\ &\Rightarrow b \equiv_m a \end{aligned}$$

(3) Finally, this relation is transitive. We leave it to the reader to prove that if $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

On occasion, you will see the equivalent notation $a \equiv b \pmod m$ for this relation.

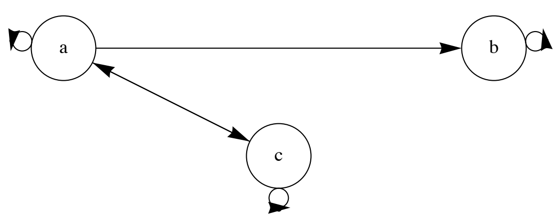
**Example 6.3.5**. Consider the relation s described by the digraph in Figure 6.3.5.

This relation is reflexive (Why?)

It is not symmetric (Why?)

It is not transitive (Why?)

Is $s$ an equivalence relation or a partial ordering? It is neither, and among the valid reasons why is that $s$ is not transitive.




### EXERCISES FOR SECTION 6.3

## A Exercises

1. (a) Let $B = \{a, b\}$ and $U = \mathcal{P}(B)$. Draw a Hasse diagram for $\subseteq$ on $U$.

   (b) Let $A = \{1, 2, 3, 6\}$. Show that divides, $\mid$, is a partial ordering on $A$.

   (c) Draw a Hasse diagram for divides on $A$.

   (d) Compare the graphs of parts a and c.

2. Repeat Exercise 1 with $B = \{a, b, c\}$ and $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

3. (a) Consider the relations defined by the digraphs in Figure 6.3.6. Determine whether the given relations are reflexive, symmetric, antisymmetric, or transitive. Try to develop procedures for determining the validity of these properties from the graphs,

   (b) Which of the graphs in Figure 6.3.6 are of equivalence relations or of partial orderings?

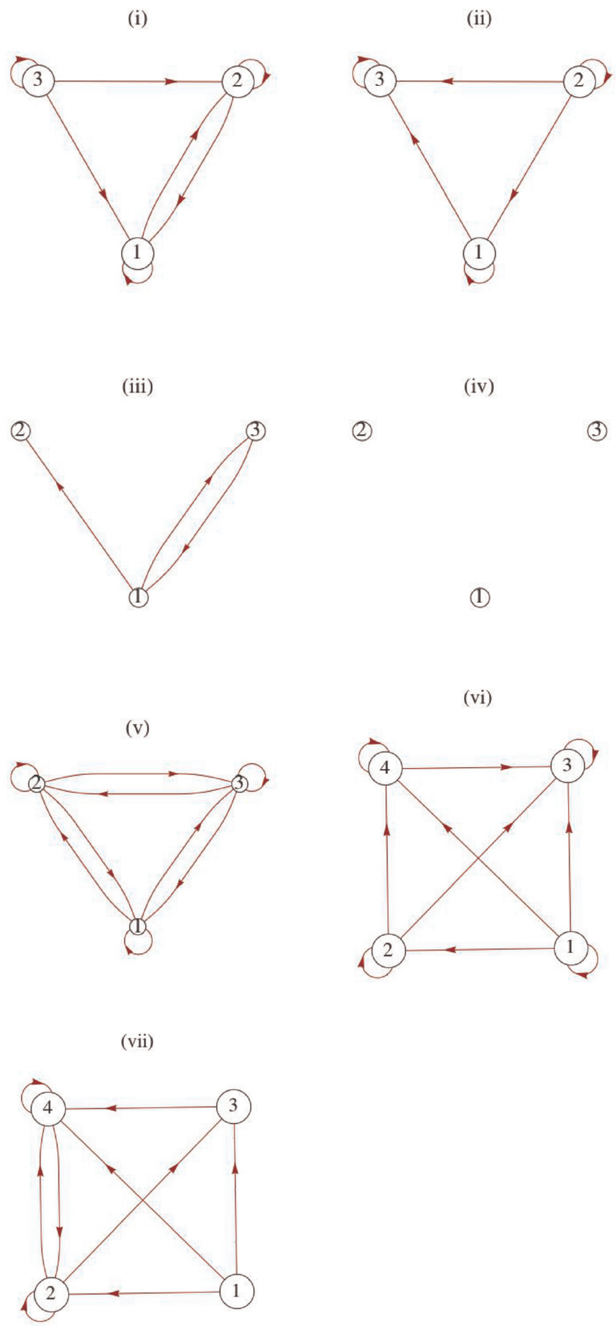



Figure 6.3.6

4. Determine which of the following are equivalence relations and/or partial ordering relations for the given sets:

   (a) $A = \{\text{lines in the plane}\}$; $x\,r\,y$ if and only if $x$ is parallel to $y$.

   (b) $A = \mathbb{R}$; $x\,r\,y$ if and only if $|x - y| \leq 7$.

5. Consider the following relation on $\{1, 2, 3, 4, 5, 6\}$. $r = \{(i, j) : |i - j| = 2\}$.

   (a) Is $r$ reflexive?

   (b) Is $r$ symmetric?

   (c) Is $r$ transitive?

(d) Draw a graph of $r$.

6. For the set of cities on a map, consider the relation $x\,r\,y$ if and only if city $x$ is connected by a road to city $y$. A city is considered to be connected to itself, and two cities are connected even though there are cities on the road between them. Is this an equivalence relation or a partial ordering? Explain.

7. Let $A = \{0, 1, 2, 3\}$ and let

$$r = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (3, 2), (2, 3), (3, 1), (1, 3)\}.$$

(a) Show that $r$ is an equivalence relation on $A$.

(b) Let $a \in A$ and define $c(a) = \{b \in A \mid a\,r\,b\}$. c(a) is called the *equivalence class of a under r*. Find $c(a)$ for each element $a \in A$.

(c) Show that $\{c(a) \mid a \in A\}$ forms a partition of A for this set A.

(d) Let $r$ be an equivalence relation on an arbitrary set $A$. Prove that the set of all equivalence classes under $r$ constitutes a partition of $A$.

8. Define $r$ on the power set of $\{1, 2, 3\}$ by $A\,r\,B \Leftrightarrow |A| = |B|$. Prove that $r$ is an equivalence relation. What are the equivalence classes under $r$?

9. Consider the following relations on $\mathbb{Z}_8 = \{0, 1, \ldots, 7\}$. Which are equivalence relations? For the equivalence relations, list the equivalence classes.

(a) $a\,r\,b$ iff the English spellings of a and b begin with the same letter.

(b) $a\,s\,b$ iff $a - b$ is a positive integer.

(c) $a\,t\,b$ iff $a - b$ is an even integer.

10. (a) Prove that conguence modulo $m$, introduced in Example 6.3.4, is a transitive.

(b) What are the equivalence classes under conguence modulo 2?

(c) What are the equivalence classes under conguence modulo 10?

## B Exercises

11. In this exercise, we prove that implication is a partial ordering. Let $A$ be any set of propositions.

(a) Verify that $q \to q$ is a tautology, thereby showing that $\Rightarrow$ is a reflexive relation on $A$.

(b) Prove that $\Rightarrow$ is antisymmetric on A. Note: we do not use = when speaking of propositions, but rather equivalence, $\Leftrightarrow$.

(c) Prove that $\Rightarrow$ is transitive on $A$.

(d) Given that $q_i$ is the proposition $n < i$ on $\mathbb{N}$, draw the Hasse diagram for the relation $\Rightarrow$ on $\{q_1, q_2, q_3, \ldots\}$.

## C Exercise

12. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ be a poset $(S, \leq)$ with the Hasse diagram shown in Figure 6.3.7. Another relation $r \subseteq S\times S$ is defined as follows: $(x, y) \in r$ if and only if there exists $z \in S$ such that $z < x$ and $z < y$ in the poset $(S, \leq)$.

(a) Prove that $r$ is reflexive.

(b) Prove that $r$ is symmetric.

(c) A compatible with respect to relation $r$ is any subset $Q$ of set $S$ such that $x \in Q$ and $y \in Q \Rightarrow (x, y) \in r$. A compatible $g$ is a maximal compatible if $Q$ is not a proper subset of another compatible. Give all maximal compatibles with respect to relation $r$ defined above.

(d) Discuss a characterization of the set of maximal compatibles for relation $r$ when $(S, \preceq)$ is a general finite poset. What conditions, if any, on a general finite poset $(S, \preceq)$ will make r an equivalence relation?

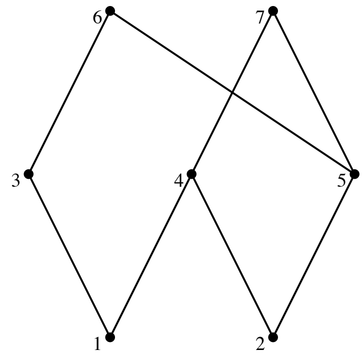



FIGURE 6.3.7

## 6.4 Matrices of Relations

We have discussed two of the many possible ways of representing a relation, namely as a digraph or as a set of ordered pairs. In this section we will discuss the representation of relations by matrices and some of its applications.

***Definition: Adjacency Matrix.*** *Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be finite sets of cardinality m and n, respectively. Let r be a relation from A into B. Then r can be represented by the $m \times n$ matrix R defined by*

$$R_{ij} = \begin{cases} 1 & \text{if } a_i \, r \, b_j \\ 0 & \text{otherwise} \end{cases}$$

*R is called the adjacency matrix (or the Boolean matrix, or the relation matrix) of r.*

**Example 6.4.1.** Let $A = \{2, 5, 6\}$ and let $r$ be the relation $\{(2, 2), (2, 5), (5, 6), (6, 6)\}$ on $A$. Since $r$ is a relation from $A$ into the same set $A$ (the $B$ of the definition), we have $a_1 = 2$, $a_2 = 5$, and $a_3 = 6$, and $b_1 = 2$, $b_2 = 5$, and $b_3 = 6$. Next, since

$2 \, r \, 2$, we have $R_{11} = 1$;

$2 \, r \, 5$, we have $R_{12} = 1$;

$5 \, r \, 6$, we have $R_{23} = 1$; and

$6 \, r \, 6$, we have $R_{33} = 1$;

All other entries of R are zero, so

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

From the definition of $r$ and of composition, we note that

$$r^2 = \{(2, 2)(2, 5)(2, 6)(5, 6)(6, 6)\},$$

The adjacency matrix of $r^2$ is

$$R^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We do not write $R^2$ only for notational purposes. In fact, $R^2$ can be obtained from the matrix product $R\,R$; however, we must use a slightly different form of arithmetic.

***Definition: Boolean Arithmetic.*** *Boolean arithmetic is the arithmetic defined on $\{0, 1\}$ using Boolean addition and Boolean multiplication, defined as:*

$$0 + 0 = 0 \qquad 0 + 1 = 1 + 0 = 1 \qquad 1 + 1 = 1$$

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 1 \cdot 0 = 0 \qquad 1 \cdot 1 = 1.$$

Notice that from Chapter 3, this is the "arithmetic of logic," where + replaces "or" and · replaces "and."

**Example 6.4.2.**

$$\text{If} \quad R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then using Boolean arithmetic, $R\,S = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $S\,R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

***Theorem 6.4.1.*** *Let $A_1$, $A_2$, and $A_3$ be finite sets where $r_1$ is a relation from $A_1$ into $A_2$ and $r_2$ is a relation from $A_2$ into $A_3$. If $R_1$ and $R_2$ are the adjacency matrices of $r_1$ and $r_2$, respectively, then the product $R_1 R_2$ using Boolean arithmetic is the adjacency matrix of the composition $r_1 r_2$.*

Remark: A convenient help in constructing the adjacency matrix of a relation from a set $A$ into a set $B$ is to write the elements from $A$ in a column preceding the first column of the adjacency matrix, and the elements of $B$ in a row above the first row. Initially, $R$ in Example 6.4.1 would be

$$\begin{array}{c} \\ 2 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccc} 2 & 5 & 6 \end{array} \\ \left(\begin{array}{ccc} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{array}\right) \end{array}$$

and $R_{ij}$ is 1 if and only if $(a_i, b_j) \in r$. So that, since the pair $(2, 5) \in r$, the entry of $R$ corresponding to the row labeled 2 and the column labeled 5 in the matrix is a 1.

**Example 6.4.3,** This final example gives an insight into how relational data base programs can systematically answer questions pertaining to large masses of information. Matrices $R$ (on the left) and $S$ (on the right) define the relations $r$ and $s$ where $a\,r\,b$ if software $a$ can be run with operating system $b$, and $b\,s\,c$ if operating system $b$ can run on computer $c$.

$$\begin{array}{c|cccc} & \text{OS1} & \text{OS2} & \text{OS3} & \text{OS4} \\ \hline \text{P1} & 1 & 0 & 1 & 0 \\ \text{P2} & 1 & 1 & 0 & 0 \\ \text{P3} & 0 & 0 & 0 & 1 \\ \text{P4} & 0 & 0 & 1 & 1 \end{array} \qquad \begin{array}{c|ccc} & \text{C1} & \text{C2} & \text{C3} \\ \hline \text{OS1} & 1 & 1 & 0 \\ \text{OS2} & 0 & 1 & 0 \\ \text{OS3} & 0 & 0 & 1 \\ \text{OS4} & 0 & 1 & 1 \end{array}$$

Although the relation between the software and computers is not implicit from the data given, we can easily compute this information. The matrix of $rs$ is $RS$, which is

$$\begin{array}{c|ccc} & \text{C1} & \text{C2} & \text{C3} \\ \hline \text{P1} & 1 & 1 & 1 \\ \text{P2} & 1 & 1 & 0 \\ \text{P3} & 1 & 1 & 1 \\ \text{P4} & 0 & 1 & 1 \end{array}$$

This matrix tells us at a glance which software will run on the computers listed. In this case, all software will run on all computers with the exception of program P2, which will not run on the computer C3, and program P4, which will not run on the computer C1.

## EXERCISES FOR SECTION 6.4

### A Exercises

1. Let $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{4, 5, 6\}$, and $A_3 = \{6, 7, 8\}$. Let $r_1$ be the relation from $A_1$ into $A_2$ defined by $r_1 = \{(x, y) \mid y - x = 2\}$, and let $r_2$ be the relation from $A_2$ into $A_3$ defined by $r_2 = \{(x, y) \mid y - x = 1\}$.

(a) Determine the adjacency matrices of $r_1$ and $r_2$ .

(b) Use the definition of composition to find $r_1 r_2$ .

(c) Verify the result in part by finding the product of the adjacency matrices of $r_1$ and $r_2$.

2. (a) Determine the adjacency matrix of each relation given via the digraphs in Exercise 3 of Section 6.3.

(b) Using the matrices found in part (a) above, find $r^2$ of each relation in Exercise 3 of Section 6.3.

(c) Find the digraph of $r^2$ directly from the given digraph and compare your results with those of part (b).

3. Suppose that the matrices in Example 6.4.2 are relations on $\{1, 2, 3, 4\}$. What relations do $R$ and $S$ describe?

4. Let D be the set of weekdays, Monday through Friday, let W be a set of employees $\{1, 2, 3\}$ of a tutoring center, and let V be a set of computer languages for which tutoring is offered, {A(PL), B(asic), C(++), J(ava), L(isp), P(ython)}. We define $s$ (schedule) from $D$ into $W$ by $d\,s\,w$ if $w$ is scheduled to work on day $d$. We also define $r$ from $W$ into $V$ by $w\,r\,l$ if $w$ can tutor students in language $l$. If $s$ and $r$ are defined by matrices

$$S = \begin{array}{c} \\ M \\ T \\ W \\ Th \\ F \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \end{array} \quad \text{and } R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & J & L & P \end{array} \\ \left(\begin{array}{cccccc} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \end{array}$$

(a) compute $S\,R$ using Boolean arithmetic and give an interpretation of the relation it defines, and

(b) compute $S\,R$ using regular arithmetic and give an interpretation of the result describes.

5. How many different reflexive, symmetric relations are there on a set with three elements? (Hint: Consider the possible matrices.)

6. Let $A = \{a, b, c, d\}$. Let $r$ be the relation on $A$ with adjacency matrix

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a) Explain why $r$ is a partial ordering on $A$.

(b) Draw its Hasse diagram.

7. Define relations $p$ and $q$ on $\{1, 2, 3, 4\}$ by $p = \{(a, b) : |a - b| = 1\}$ and $q = \{(a, b) \mid a - b \text{ is even}\}$

(a) Represent $p$ and $q$ as both graphs and matrices.

(b) Determine $p\,q$, $p^2$, and $q^2$; and represent them clearly in any way.

## B Exercises

8. (a) Prove that if $r$ is a transitive relation on a set $A$, then $r^2 \subseteq r$.

(b) Find an example of a transitive relation for which $r^2 \neq r$.

9. We define $\leq$ on the set of all $n \times n$ relation matrices by the rule that if $R$ and $S$ are any two $n \times n$ relation matrices, $R \leq S$ if and only if $R_{ij} \leq S_{ij}$ for all $1 \leq i, j \leq n$.

(a) Prove that $\leq$ is a partial ordering on all $n \times n$ relation matrices.

(b) Prove that $R \leq S \Rightarrow R^2 \leq S^2$ , but the converse is not true.

(c) If $R$ and $S$ are matrices of equivalence relations and $R \leq S$, how are the equivalence classes defined by $R$ related to the equivalence classes defined by $S$?

## 6.5 Closure Operations on Relations

In Section 6.1, we studied relations and one important operation on relations, namely composition. This operation enables us to generate new relations from previously known relations. In Section 6.3, we discussed some key properties of relations. We now wish to consider the situation of constructing a new relation $r^+$ from a previously known relation $r$ where, first, $r^+$ contains $r$ and, second, $r^+$ satisfies the transitive property.

Consider a telephone network in which the main office $a$ is connected to, and can communicate to, individuals $b$ and $c$. Both $b$ and $c$ can communicate to another person, $d$; however, the main office cannot communicate with $d$. Assume communication is only one way, as indicated. This situation can be described by the relation $r = \{(a, b), (a, c), (b, d), (c, d)\}$. We would like to change the system so that the main office a can communicate with person d and still maintain the previous system. We, of course, want the most economical system.

This can be rephrased as follows; Find the smallest relation $r^+$ which contains $r$ as a subset and which is transitive; $r^+ = \{(a, b), (a, c), (b, d), (c, d), (a, d)\}$.

***Definition: Transitive Closure.*** *Let A be a set and r be a relation on A. The transitive closure of r, denoted by $r^+$, is the smallest transitive relation that contains r as a subset.*

**Example 6.5.1.** Let $A = \{1, 2, 3, 4\}$, and let $\mathcal{S} = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on $A$. This relation is called the successor relation on $A$ since each element is related to its successor. How do we compute $\mathcal{S}^+$ ? By inspection we note that $(1, 3)$ must be in $\mathcal{S}^+$ . Let's analyze why. This is so because $(1, 2) \in \mathcal{S}$ and $(2, 3) \in \mathcal{S}$, and the transitive property forces $(1, 3)$ to be in $\mathcal{S}^+$.

In general, it follows that if $(a, b) \in \mathcal{S}$ and $(b, c) \in S$, then $(a, c) \in \mathcal{S}+$ . This condition is exactly the membership requirement for the pair (a, c) to be in the composition $\mathcal{S}\mathcal{S} = \mathcal{S}^2$. So every element in $\mathcal{S}^2$ must be an element in $\mathcal{S}^+$ . So far, $\mathcal{S}^+$ contains at least $\mathcal{S} \cup \mathcal{S}^2$ . In particular, for this example, since $\mathcal{S} = \{(1, 2), (2, 3), (3, 4)\}$ and $\mathcal{S}^2 = \{(1, 3), (2, 4)\}$, we have

$$\mathcal{S} \cup \mathcal{S}^2 = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4)\}.$$

Is the relation $\mathcal{S} \cup \mathcal{S}^2$ transitive? Again, by inspection, $(1, 4)$ is not an element of $\mathcal{S} \cup \mathcal{S}^2$ , but it must be an element of $\mathcal{S}^+$ since $(1, 3)$ and $(3, 4)$ are required to be in $\mathcal{S}^+$. From above, $(1, 3) \in \mathcal{S}^2$ and $(3, 4) \in \mathcal{S}$. Therefore, the composite $\mathcal{S}^2\mathcal{S} = \mathcal{S}^3$ produces $(1, 4)$. This shows that $\mathcal{S}^3 \subseteq \mathcal{S}^+$ . This process must be continued until the resulting relation is transitive. If A is finite, as is true in this example, the transitive closure will be obtained in a finite number of steps. For this example,

$$\mathcal{S}^+ = \mathcal{S} \cup \mathcal{S}^2 \cup \mathcal{S}^3 = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\} .$$

***Theorem 6.5.1.*** *If r is a relation on a set A and $|A| = n$, then the transitive closure of r is the union of the first n powers of r. That is,*

$$r^+ = r \cup r^2 \cup r^3 \cup \cdots \cup r^n.$$

Let's now consider the matrix analogue of the transitive closure.

**Example 6.5.2.** Consider the relation

$$r = \{(1, 4), (2, 1), (2, 2), (2, 3), (3, 2), (4, 3), (4, 5), (5, 1)\}$$

on the set $A = \{1, 2, 3, 4, 5\}$. The matrix of r is

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall that $r^2, r^3, \ldots$ can be determined through computing the matrix powers $R^2, R^3, \ldots$. Here,

$$R^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad R^3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$R^4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \qquad R^5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

How do we relate $\bigcup_{i=1}^{5} r^i$ to the powers of $R$?

***Theorem 6.5.2.*** *Let r be a relation on a finite set and let $R^+$ be the matrix of $r^+$ , the transitive closure of r. Then $R^+ = R + R^2 + \cdots + R^n$, using Boolean arithmetic.*

Using this theorem, we find $R^+$ is the $5 \times 5$ matrix consisting of all $1's$, thus, $r^+$ is all of $A \times A$.

## WARSHALL'S ALGORITHM

Let $r$ be a relation on the set $\{1, 2, \ldots, n\}$ with relation matrix $R$. The matrix of the transitive closure $R^+$, can be computed by the equation $R^+ = R + R^2 + \cdots + R^n$. By using ordinary polynomial evaluation methods, you can compute $R^+$ with $n - 1$ matrix multiplications:

$$R^+ = R(I + R(I + (\cdots R(I + R)\cdots))).$$

For example, if $n = 3$, $R = R(I + R(I + R))$.

We can make use of the fact that if $T$ is a relation matrix, $T + T = T$ due to the fact that $1 + 1 = 1$ in Boolean arithmetic. Let $S_k = R + R^2 + \cdots + R^k$. Then

$$R = S_1$$

$$S_1 (I + S_1) = R (I + R) = R + R^2 = S_2$$

$$\begin{aligned} S_2 (I + S_2) &= (R + R^2)(I + R + R^2) \\ &= \left(R + R^2\right) + \left(R^2 + R^3\right) + (R^3 + R^4) \\ &= R + R^2 + R^3 + R^4 = S_4 \end{aligned}$$

Similarly,

$$S_4(I + S_4) = S_8$$

etc..

Notice how each matrix multiplication doubles the number of terms that have been added to the sum that you currently have computed. In algorithmic form, we can compute $R^2$ as follows.

---

**Algorithm 6.5.1: Transitive Closure Algorithm 1.** Let R be a known relation matrix and let $R^+$ be its transitive closure matrix, which is to be computed.

1.0. $T := R$

2.0. Repeat

  2.1 $S := T$

  2.2 $T := S(I + S)$ // using Boolean arithmetic

 Until $T = S$

3.0. Terminate with $T = R^+$.

---

Notes:

(a) Often the higher-powered terms in $S_n$ do not contribute anything to $R^+$. When the condition $T = S$ becomes true in Step 2, this is an indication that no higher-powered terms are needed.

(b) To compute $R^+$ using this algorithm, you need to perform no more than $\lceil \log_2 n \rceil$ matrix multiplications, where $\lceil x \rceil$ is the least integer that is greater than or equal to $x$. For example, if $r$ is a relation on 25 elements, no more than $\lceil \log_2 25 \rceil = 5$ matrix multiplications are needed.

A second algorithm, Warshall's Algorithm, reduces computation time to the time that it takes to perform one matrix multiplication.

---

**Algorithm 6.5.2; Warshall's Algorithm.** Let $R$ be a known relation matrix and let $R^+$ be its transitive closure matrix, which is to be computed.

1.0 $T := R$

2.0 FOR $k := 1$ to $n$ DO

  FOR $i := 1$ to $n$ DO

   FOR $j := 1$ to $n$ DO

    $T[i, j] := T[i, j] + T[i, k] \cdot T[k, j]$

3.0 Terminate with $T = R^+$.

---

## EXERCISES FOR SECTION 6.5

### A Exercises

1. Let $A$ and $\mathcal{S}$ be as in Example 6.5.1. Compute $\mathcal{S}^+$ as in Example 6.5.2. Verify your results by checking against the relation $\mathcal{S}^+$ obtained in Example 6.5.1.

2. Let $A$ and $r$ be as in Example 6.5.2. Compute the relation $r^+$ as in Example 6.5.1. Verify your results.

3. (a) Draw digraphs of the relations $\mathcal{S}$, $\mathcal{S}^2$, $\mathcal{S}^3$ , and $\mathcal{S}^+$ of Example 6.5.1.

(b) Verify that in terms of the graph of $\mathcal{S}$, $a \mathcal{S}^+ b$ if and only if $b$ is reachable from $a$ along a path of any finite nonzero length.

4. Let $r$ be the relation represented by the digraph in Figure 6.5,1.

(a) Find $r^+$ .

(b) Determine the digraph of $r^+$ directly from the digraph of $r$.

(c) Verify your result in part (b) by computing the digraph from your result in part (a).

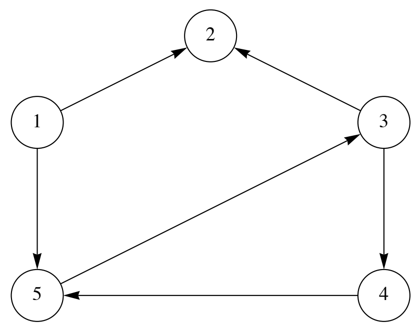



**FIGURE 6.5.1**

5. (a) Define reflexive closure and symmetric closure by imitating the definition of transitive closure.

(b) Use your definitions to compute the reflexive and symmetric closures of Examples 6.5.1 and 6.5.2.

(c) What are the transitive reflexive closures of these examples?

(d) Convince yourself that the reflexive closure of the relation $<$ on the set of positive integers $\mathbb{P}$ is $\leq$.

6. What common relations on $\mathbb{Z}$ are the transitive closures of the following relations?

(a) $a S b$ if and only if $a + 1 = b$.

(b) $a R b$ if and only if $| a - b | = 2$.

### B Exercise

7. (a) Let $A$ be any set and $r$ a relation on $A$, prove that $(r^+)^+ = r^+$.

(b) Is the transitive closure of a symmetric relation always both symmetric and reflexive? Explain.