chapter 6

RELATIONS AND GRAPHS

GOALS

One understands a set of objects completely only if the structure of that set is made clear by the interrelationships between its elements. For example, the individuals in a crowd can be compared by height, by age, or through any number of other criteria. In mathematics, such comparisons are called relations. The goal of this chapter is to develop the language, tools, and concepts of relations.

6.1 Basic Definitions

In Chapter 1 we introduced the concept of the Cartesian product of sets. Let's assume that a person owns three shirts and two pairs of slacks. More precisely, let

\[ A = \{ \text{blue shirt, tan shirt, mint green shirt} \} \]

\[ B = \{ \text{grey slacks, tan slacks} \} \]

Then certainly \( A \times B \) is the set of all possible combinations (six) of shirts and slacks that the individual can wear. However, the individual may wish to restrict himself or herself to combinations which are color coordinated, or "related." This may not be all possible pairs in \( A \times B \) but will certainly be a subset of \( A \times B \). For example, one such subset may be \( \{ \text{blue shirt, grey slacks}, \text{blue shirt, tan slacks}, \text{mint green shirt, tan slacks} \} \).

**Definition: Relation.** Let \( A \) and \( B \) be sets. A relation from \( A \) into \( B \) is any subset of \( A \times B \).

**Example 6.1.1.** Let \( A = \{1, 2, 3\} \) and \( B = \{4, 5\} \). Then \( \{ (1, 4), (2, 4), (3, 5) \} \) is a relation from \( A \) into \( B \). Of course, there are many others we could describe; 64, to be exact.

**Example 6.1.2.** Let \( A = \{2, 3, 5, 6\} \) and define a relation \( r \) from \( A \) into \( A \) by \( (a, b) \in r \) if and only if \( a \) divides evenly into \( b \). So \( r = \{ (2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6) \} \).

**Definition: Relation on a Set.** A relation from a set \( A \) into itself is called a relation on \( A \).

The relation "divides" in Example 6.1.2 is will appear throughout the book. Here is a general definition on the whole set of integers.

**Definition: Divides.** Let \( a, b \in \mathbb{Z} \),

\[ a \mid b \text{ if and only if there exists an integer } k \text{ such that } a k = b. \]

Based on the equation \( a k = b \), we can say that \( a \mid b \) is equivalent to \( k = \frac{b}{a} \), or \( a \) divides evenly into \( b \). In fact the "divides" is short for "divides evenly into." You might find the equation \( k = \frac{b}{a} \) initially easier to understand, but in the long run we will find the equation \( a k = b \) more useful.

Sometimes it is helpful to illustrate a relation. Consider Example 6.1.1. A picture of \( r \) can be drawn as in Figure 6.1.1. The arrows indicate that 1 is related to 4 under \( r \). Also, 2 is related to 4 under \( r \), and 3 is related to 5, while the upper arrow denotes that \( r \) is a relation from the whole set \( A \) into the set \( B \).

A typical element in a relation \( r \) is an ordered pair \((x, y)\). In some cases, \( r \) can be described by actually listing the pairs which are in \( r \), as in the previous examples. This may not be convenient if \( r \) is relatively large. Other notations are used depending on personal preference or past practice. Consider the following relations on the real numbers:
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\[ r = \{(x, y) \mid y \text{ is the square of } x\}, \text{ and } s = \{(x, y) \mid x \leq y\}. \]

The notation \((4, 16) \in r \text{ or } (3, 7.2) \in s\) makes sense in both cases. However, \(r\) would be more naturally expressed as \(r(x) = x^2\) or \(r(x) = y\), where \(y = x^2\). But this notation when used for \(s\) is at best awkward. The notation \(x \leq y\) is clear and self-explanatory; it is a better notation to use than \((x, y) \in s\).

Many of the relations we will work with "resemble" the relation \(\leq\), so \(x \leq y\) is a common way to express the fact that \(x\) is related to \(y\) through the relation \(s\).

**Relation Notion.** Let \(s\) be a relation from a set \(A\) into a set \(B\). Then the fact that \((x, y) \in s\) is frequently written \(x \leq y\).

Let \(A = \{2, 3, 5, 8\}, B = \{4, 6, 16\}, \) and \(C = \{1, 4, 5, 7\}; \) let \(r\) be the relation "divides," denoted by \(|\), from \(A\) into \(B\); and let \(s\) be the relation \(\leq\) from \(B\) into \(C\). So \(r = \{(2, 4), (2, 6), (2, 16), (3, 6), (8, 16)\}\) and \(s = \{(4, 4), (4, 5), (4, 7), (6, 7)\}\).

Notice from Figure 6.1.2 that we can, for certain elements of \(A\), go through elements in \(B\) to results in \(C\). That is:

\[
\begin{align*}
2 & \mid 4 \text{ and } 4 \leq 4 \\
2 & \mid 4 \text{ and } 4 \leq 5 \\
2 & \mid 4 \text{ and } 4 \leq 7 \\
2 & \mid 6 \text{ and } 6 \leq 7 \\
3 & \mid 6 \text{ and } 6 \leq 7
\end{align*}
\]

Based on this observation, we can define a new relation, call it \(rs\), from \(A\) into \(C\). In order for \((a, c)\) to be in \(rs\), it must be possible to travel along a path in Figure 6.1.2 from \(a\) to \(c\). In other words, \((a, c) \in rs\) if and only if \((\exists b)_{B}(a \in r \text{ and } b \in s)\). The name \(rs\) was chosen solely because it reminds us that this new relation was formed by the two previous relations \(r\) and \(s\). The complete listing of all elements in \(rs\) is \([(2, 4), (2, 5), (2, 7), (3, 7)]\). We summarize in a definition.

**Definition: Composition of Relations.** Let \(r\) be a relation from a set \(A\) into a set \(B\), and let \(s\) be a relation from \(B\) into a set \(C\). The composition of \(r\) and \(s\), written \(rs\), is the set of pairs of the form \((a, c) \in A \times C\), where \((a, c) \in rs\) if and only if there exists \(b \in B\) such that \((a, b) \in r\) and \((b, c) \in s\).

Remark: A word of warning to those readers familiar with composition of functions. (For those who are not, disregard this remark. It will be repeated at an appropriate place in Chapter 7.) As indicated above, the traditional way of describing a composition of two relations is \(rs\) where \(r\) is the first relation and \(s\) the second. However, function composition is traditionally expressed "backwards"; that is, as \(sr\) (or \(s \circ r\), where \(r\) is the first function and \(s\) is the second.

**EXERCISES FOR SECTION 6.1**

**A Exercises**

1. For each of the following relations \(r\) defined on \(P\), determine which of the given ordered pairs belong to \(r\).

   (a) \(x \leq y \iff x \mid y; \quad (2, 3), (2, 4), (2, 8), (2, 17)\)

   (b) \(x \leq y \iff y \leq x; \quad (2, 3), (3, 2), (2, 4), (5, 8)\)

   (c) \(x \leq y \iff y = x^2; \quad (1, 1), (2, 3), (2, 4), (2, 6)\)
2. The following relations are on \{1, 3, 5\}. Let \( r \) be the relation \( x \rightarrow y \) iff \( y = x + 2 \) and \( s \) the relation \( x \leftarrow y \) iff \( x \leq y \).
   (a) Find \( rs \).
   (b) Find \( sr \).
   (c) Illustrate \( rs \) and \( sr \) via a diagram.
   (d) Is the relation (set) \( rs \) equal to the relation \( sr \)? Why?
3. Let \( A = \{1, 2, 3, 4, 5\} \) and define \( r \) on \( A \) by \( x \rightarrow y \) iff \( x + 1 = y \). We define \( r^2 = rr \) and \( r^3 = r^2 r \). Find:
   (a) \( r \)
   (b) \( r^2 \)
   (c) \( r^3 \)
4. Given \( s \) and \( t \), relations on \( \mathbb{Z} \), \( s = \{(1, n) : n \in \mathbb{Z}\} \) and \( t = \{(n, 1) : n \in \mathbb{Z}\} \), what are \( st \) and \( ts \)?

**B Exercises**

5. Let \( \rho \) be the relation on the power set, \( \mathcal{P}(S) \), of a finite set \( S \) of cardinality \( n \). Define \( \rho \) by \((A, B) \in \rho \) iff \( A \cap B = \emptyset \).
   (a) Consider the specific case \( n = 3 \), and determine the cardinality of the set \( \rho \).
   (b) What is the cardinality of \( \rho \) for an arbitrary \( n \)? Express your answer in terms of \( n \). (Hint: There are three places that each element of \( S \) can go in building an element of \( \rho \).)
6. Let \( r_1, r_2, \) and \( r_3 \) be relations on any set \( A \). Prove that if \( r_1 \subseteq r_2 \) then \( r_1 r_3 \subseteq r_2 r_3 \).
6.2 Graphs of Relations

In this section we will give a brief explanation of procedures for graphing a relation. A graph is nothing more than an illustration that gives us, at a glance, a clearer idea of the situation under consideration. A road map indicates where we have been and how to proceed to reach our destination. A flow chart helps us to zero in on the procedures to be followed to code a problem and/or organize the flow of information. The graph of the function \( y = 2x + 3 \) in algebra helps us to understand how the function behaves. Indeed, it tells us that the graph of this function is a straight line. The pictures of relations in the previous section gave us an added insight into what a relation is. They indicated that there are several different ways of graphing relations. We will investigate two additional methods.

**Example 6.2.1.** Let \( A = \{0, 1, 2, 3\} \) and let

\[
r = \{(0, 0), (0, 3), (1, 2), (2, 1), (3, 2), (2, 0)\}.
\]

The elements of \( A \) are called the vertices of the graph. They are represented by labeled points or occasionally by small circles. Connect vertex \( a \) to vertex \( b \) with an arrow, called an edge of the graph, going from vertex \( a \) to vertex \( b \) if and only if \( a \, r \, b \). This type of graph of a relation \( r \) is called a directed graph or digraph. The result is Figure 6.2.1. Notice that since \( 1 \, r \, 2 \) and \( 2 \, r \, 1 \), we draw a single edge between 1 and 2 with arrows in both directions.

![Figure 6.2.1](image)

The actual location of the vertices is immaterial. The main idea is to place the vertices in such a way that the graph is easy to read. Obviously, after a rough-draft graph of a relation, we may decide to relocate and/or order the vertices so that the final result will be neater. Figure 6.2.1 could be presented as in Figure 6.2.2.

![Figure 6.2.2](image)

A vertex of a graph is also called a node, point, or a junction. An edge of a graph is also referred to as an arc, a line, or a branch. Do not be concerned if two graphs of a given relation look different. It is a nontrivial problem to determine if two graphs are graphs of the same relation.

**Example 6.2.2.** Consider the relation \( s \) whose digraph is Figure 6.2.3. What information does this give us? Certainly we know that \( s \) is a relation of a set \( A \), where \( A = \{1, 2, 3\} \) and

\[
s = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 3)\},
\]

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Example 6.2.3. Let \( B = \{a, b\} \), and let \( A = \mathcal{P}(B) = \{0, \{a\}, \{b\}, \{a, b\}\} \). Then \( \subset \) is a relation on \( A \) whose digraph is Figure 6.2.4.

This graph is helpful insofar as it reminds us that each set is a subset of itself (How?) and shows us at a glance the relationship between the various subsets in \( \mathcal{P}(B) \). Some relations, such as this one, can also be conveniently depicted by what is called a Hasse, or ordering, diagram. To read a Hasse diagram for a relation on a set \( A \), remember:

1. Each vertex of \( A \) must be related to itself, so the arrows from a vertex to itself are not necessary.
2. If vertex \( b \) appears above vertex \( a \) and if vertex \( a \) is connected to vertex \( b \) by an edge, then \( a \subset b \), so direction arrows are not necessary.
3. If vertex \( c \) is above vertex \( a \) and if \( c \) is connected to \( a \) by a sequence of edges, then arc.
4. The vertices (or nodes) are denoted by points rather than by "circles."

The Hasse diagram of the directed graph depicted in Figure 6.2.4 is Figure 6.2.5.
Example 6.2.4. Consider the relation $s$ whose Hasse diagram is Figure 6.2.6.

How do we read this diagram? What is $A$? What is $s$? What does the digraph of $s$ look like? Certainly $A = \{1, 2, 3, 4, 5\}$ and $1 \leq 2$, $3 \leq 4$, $1 \leq 3$, $1 \leq 4$, $1 \leq 5$, $2 \leq 4$, $2 \leq 5$, etc., so

$s = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (1, 4), (1, 5), (1, 2), (3, 4), (3, 5), (4, 5), (2, 5)\}$

A digraph for $s$ is Figure 6.2.7. It is certainly more complicated to read than the Hasse diagram.
EXERCISES FOR SECTION 6.2

A Exercises

1. Let \( A = \{1, 2, 3, 4\} \), and let \( r \) be the relation \( \leq \) on \( A \). Draw a digraph and a Hasse diagram for \( r \).

2. Let \( B = \{1, 2, 3, 4, 6, 8, 12, 24\} \), and let \( s \) be the relation "divides," on \( B \). Draw a Hasse diagram for \( s \).

3. Draw a Hasse diagram of the relation \( \subseteq \) on \( \mathcal{P}(A) \), where \( A = \{a, b, c\} \).

4. (a) Let \( A \) be the set of strings of 0's and 1's of length 3 or less. Define the relation of \( d \) on \( A \) by \( x \ d \ y \) if \( x \) is contained within \( y \). For example, \( 01 \ d \ 101 \). Draw a Hasse diagram for this relation.

   (b) Do the same for the relation \( p \) defined by \( x \ p \ y \) if \( x \) is a prefix of \( y \). For example, \( 10 \ p \ 101 \), but \( 01 \ p \ 101 \) is false.

5. Draw the digraph for the relation \( \rho \) in Exercise 5 of Section 6.1, where \( S = \{a, b\} \). Explain why a Hasse diagram could not be used to depict \( \rho \).

6. Let \( C = \{1, 2, 3, 4, 6, 8, 12, 24\} \) and define \( t \) on \( C \) by

   \[ a \ t b \ \text{if and only if} \ a \text{ and } b \text{ share a common divisor greater than 1}. \]

   Draw a digraph for \( t \).
6.3 Properties of Relations

Consider the set \( B = \{1, 2, 3, 4, 6, 12, 36, 48\} \) and the relations "divides" and \( \leq \) on \( B \). We notice that these two relations on \( B \) have several properties in common. In fact:

1. Every element in \( B \) divides itself and is less than or equal to itself. This is called the reflexive property.
2. If we search for two elements from \( B \) where the first divides the second and the second divides the first, then we are forced to choose the same first and second number. The reader can verify that a similar result is true for the relation \( \leq \) on \( B \). This is called the antisymmetric property.
3. Next if we choose three numbers from \( B \) such that the first divides (or is \( \leq \)) the second and the second divides (or is \( \leq \)) the third, then this forces the first number to divide (or be \( \leq \)) the third. This is called the transitive property.

Sets on which relations are defined which satisfy the above properties are of special interest to us. More detailed definitions follow.

**Definitions:** Reflexive, Antisymmetric, and Transitive Relations. Let \( A \) be a set and let \( r \) be a relation on \( A \), then:

1. \( r \) is reflexive if and only if \( a r a \) for all \( a \in A \).
2. \( r \) is antisymmetric if and only if whenever \( a r b \) and \( a \neq b \) then \( b r a \) is false; or equivalently whenever \( a r b \) and \( b r a \) then \( a = b \). (The reader is encouraged to think about both conditions since they are frequently used.)
3. \( r \) is transitive if and only if whenever \( a r b \) and \( b r c \) then \( a r c \).

A word of warning about asymmetry: Students frequently find it difficult to understand this definition. Keep in mind that this term is defined through an "If . . . then . . ." statement. The question that you must ask is: Is it true that whenever there are elements \( a \) and \( b \) from \( A \) where \( a r b \) and \( a \neq b \), it follows that \( b \) is not related to \( a \)? If so, then the relation is antisymmetric. Another way to determine whether a relation is antisymmetric is to examine its graph. The relation is not antisymmetric if there exists a pair of vertices that are connected by edges in both directions. Note that the negation of antisymmetric is not symmetric. We will define the symmetric property later.

**Definition:** Partial Ordering. A relation on a set \( A \) that is reflexive, antisymmetric, and transitive is called a partial ordering on \( A \). A set on which there is a partial ordering relation defined is called a partially ordered set or poset.

**Example 6.3.1.** Let \( A \) be a set. Then \( \mathcal{P}(A) \) together with the relation \( \subseteq \) is a poset. To prove this we observe that the three properties hold:

1. Let \( B \in \mathcal{P}(A) \). We must show that \( B \subseteq B \). This is true by definition of subset. Hence, the relation is reflexive.
2. Let \( B_1, B_2 \in \mathcal{P}(A) \) and assume that \( B_1 \subseteq B_2 \) and \( B_2 \neq B_1 \). Could it be that \( B_2 \subseteq B_1 \)? No. Why? Hence, the relation is anti symmetric.
3. Let \( B_1, B_2, B_3 \in \mathcal{P}(A) \) and assume that \( B_1 \subseteq B_2 \) and \( B_2 \subseteq B_3 \). Does it follow that \( B_1 \subseteq B_3 \)? Yes. Hence, the relation is transitive.

**Example 6.3.2.** Consider the relation \( s \) defined by the Hasse diagram in Figure 6.2.6. A relation defined by a Hasse diagram is always a partial ordering. Let's convince ourselves of this.

1. First, \( s \) is reflexive? Yes, a Hasse diagram always implies that each element is related to itself.
2. Next, \( s \) is antisymmetric. From the diagram, can we find two different elements, say \( c_1 \) and \( c_2 \), such that \( c_1 s c_2 \) and \( c_2 s c_1 \)? No. If \( c_1 s c_2 \), then \( c_1 \) and \( c_2 \) are connected by a series of edges in the Hasse diagram and \( c_1 \) is below \( c_2 \). In order for \( c_2 s c_1 \) to be true, \( c_2 \) would need to be below \( c_1 \), which is impossible.
3. Finally, \( s \) is transitive. Again, this follows from the way Hasse diagrams are always interpreted. If \( c_1 s c_2 \), then \( c_1 \) and \( c_2 \) are connected by a series of edges in the Hasse diagram and \( c_1 \) is below \( c_2 \). Similarly, if \( c_2 s c_3 \) then \( c_2 \) and \( c_3 \) are connected by a series of edges in the Hasse diagram and \( c_2 \) is below \( c_3 \). Thus, \( c_1 s c_3 \) and we can patch together the two series of edges through \( c_2 \) to connect \( c_1 \) to \( c_3 \).

Another property that is frequently referred to is that of symmetry.

**Definition:** Symmetry. Let \( r \) be a relation on a set \( A \). \( r \) is symmetric if and only if whenever \( a r b \), it follows that \( b r a \).

Consider the relation of equality \( (\equiv) \) defined on any set \( A \). Certainly \( a \equiv b \) implies that \( b \equiv a \) so equality is a symmetric relation on \( A \).

Surprisingly, equality is also an antisymmetric relation on \( A \). This is due to the fact that the condition that defines the antisymmetry property, \( a = b \) and \( a \neq b \), is a contradiction. Remember, a conditional proposition is always true when the condition is false. So a relation can be both symmetric and antisymmetric on a set! Again recall that these terms are not negatives of one other.

**Definition:** Equivalence Relation. A relation \( r \) on a set \( A \) is called an equivalence relation if and only if it is reflexive, symmetric, and transitive.

The classic example of an equivalence relation is equality on a set \( A \). In fact, the term equivalence relation is used because those relations which satisfy the definition behave quite like the equality relation.

**Example 6.3.3.** Let \( \mathbb{Z}^* \) be the set of nonzero integers. One of the most basic equivalence relations in mathematics is the relation \( q \) on \( \mathbb{Z} \times \mathbb{Z} \) defined by \( (a, b) \equiv (c, d) \) if and only if \( a d = b c \). We will leave it to the reader to verify that \( q \) is indeed an equivalence relation. Be aware that since the elements of \( \mathbb{Z} \times \mathbb{Z} \) are ordered pairs, proving symmetry involves four numbers and transitivity involves six numbers. Two ordered pairs, \( (a, b) \) and \( (c, d) \), are related if the fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) are numerically equal.

**Example 6.3.4.** Consider the relation \( s \) described by the digraph in Figure 6.3.1. This relation is reflexive (Why?), not symmetric (Why?), and not transitive (Why?). Is \( s \) an equivalence relation? A partial ordering? It is neither, and among the valid reasons of stating this is that \( s \) isn't transitive.
A classic example of a partial ordering relation is ≤ on the real numbers, \( \mathbb{R} \). Indeed, when graphing partial ordering relations, it is natural to "plot" the elements from the given poset starting with the "least" element to the "greatest" and to use terms like "least," "greatest," etc. Because of this the reader should be forewarned that many texts use the \( \leq \) notation when describing an arbitrary partial ordering. This can be quite confusing for the novice, so we continue to use the general notation \( r, s, \) etc., when speaking of relations.

**EXERCISES FOR SECTION 6.3**

**A Exercises**

1. (a) Let \( B = \{a, b\} \) and \( U = \mathcal{P}(B) \). Draw a Hasse diagram for \( \subseteq \) on \( U \).
   
   (b) Let \( A = \{1, 2, 3, 6\} \). Show that divides, |, is a partial ordering on \( A \).
   
   (c) Draw a Hasse diagram for divides on \( A \).
   
   (d) Compare the graphs of parts a and c.

2. Repeat Exercise 1 with \( B = \{a, b, c\} \) and \( A = \{1, 2, 3, 5, 6, 10, 15, 30\} \).

3. (a) Consider the relations defined by the digraphs in Figure 6.3.2. Determine whether the given relations are reflexive, symmetric, antisymmetric, or transitive. Try to develop procedures for determining the validity of these properties from the graphs.

   (b) Which of the graphs in Figure 6.3.2 are of equivalence relations or of partial orderings?
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4. Determine which of the following are equivalence relations and/or partial ordering relations for the given sets:

(a) \( A = \{\text{lines in the plane}\}; x \sim y \text{ if and only if } x \text{ is parallel to } y. \)

(b) \( A = \mathbb{R}; x \sim y \text{ if and only if } |x - y| \leq 7. \)

5. Consider the following relation on \( \{1, 2, 3, 4, 5, 6\}. \) \( r = \{(i, j): |i - j| = 2\}. \)

(a) Is \( r \) reflexive?

(b) Is \( r \) symmetric?

(c) Is \( r \) transitive?
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(d) Draw a graph of $r$.

6. For the set of cities on a map, consider the relation $x r y$ if and only if city $x$ is connected by a road to city $y$. A city is considered to be connected to itself, and two cities are connected even though there are cities on the road between them. Is this an equivalence relation or a partial ordering? Explain.

7. Let $A = \{0, 1, 2, 3\}$ and let

$$r = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (3, 2), (2, 3), (3, 1), (1, 3)\}.$$

(a) Show that $r$ is an equivalence relation on $A$.

(b) Let $a \in A$ and define $c(a) = \{b \in A \mid a \, r \, b\}$. $c(a)$ is called the equivalence class of $a$ under $r$. Find $c(a)$ for each element $a \in A$.

(c) Show that $\{c(a) \mid a \in A\}$ forms a partition of $A$ for this set $A$.

(d) Let $r$ be an equivalence relation on an arbitrary set $A$. Prove that the set of all equivalence classes under $r$ constitutes a partition of $A$.

8. Define $r$ on the power set of $\{1, 2, 3\}$ by $A \, r \, B \iff |A| = |B|$. Prove that $r$ is an equivalence relation. What are the equivalence classes under $r$?

9. Consider the following relations on $\mathbb{Z}_8 = \{0, 1, \ldots, 7\}$. Which are equivalence relations? For the equivalence relations, list the equivalence classes.

(a) $a \, r \, b$ iff the English spellings of $a$ and $b$ begin with the same letter.

(b) $a \, s \, b$ iff $a - b$ is a positive integer.

(c) $a \, t \, b$ iff $a - b$ is an even integer.

10. Define $r$ on $A = \{1, 2, \ldots, 9\}$ by $x \, t \, y$ iff $x + y = 10$. Is $r$ an equivalence relation on $A$? If yes, list its equivalence classes. If no, why not?

**B Exercises**

11. In this exercise, we prove that implication is a partial ordering. Let $A$ be any set of propositions.

(a) Verify that $q \implies q$ is a tautology, thereby showing that $\implies$ is a reflexive relation on $A$.

(b) Prove that $\implies$ is antisymmetric on $A$. Note: we do not use $=$ when speaking of propositions, but rather equivalence, $\iff$.

(c) Prove that $\implies$ is transitive on $A$.

(d) Given that $q_i$ is the proposition $n < i$ on $\mathbb{N}$, draw the Hasse diagram for the relation $\implies$ on $\{q_1, q_2, q_3, \ldots\}$.

**C Exercise**

12. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ be a poset $(S, \leq)$ with the Hasse diagram shown in Figure 6.3.3. Another relation $r \subseteq S \times S$ is defined as follows: $(x, y) \in r$ if and only if there exists $z \in S$ such that $z \leq x$ and $z \leq y$ in the poset $(S, \leq)$.

(a) Prove that $r$ is reflexive.

(b) Prove that $r$ is symmetric.

(c) A compatible with respect to relation $r$ is any subset $Q$ of set $S$ such that $x \in Q$ and $y \in Q \implies (x, y) \in r$. A compatible $g$ is a maximal compatible if $Q$ is not a proper subset of another compatible. Give all maximal compatibles with respect to relation $r$ defined above.

(d) Discuss a characterization of the set of maximal compatibles for relation $r$ when $(S, \leq)$ is a general finite poset. What conditions, if any, on a general finite poset $(S, \leq)$ will make $r$ an equivalence relation?

![FIGURE 6.3.3](image-url)
6.4 Matrices of Relations

We have discussed two of the many possible ways of representing a relation, namely as a digraph or as a set of ordered pairs. In this section we will discuss the representation of relations by matrices and some of its applications.

**Definition: Adjacency Matrix.** Let \( A = \{a_1, a_2, \ldots, a_m\} \) and \( B = \{b_1, b_2, \ldots, b_n\} \) be finite sets of cardinality \( m \) and \( n \), respectively. Let \( r \) be a relation from \( A \) into \( B \). Then \( r \) can be represented by the \( m \times n \) matrix \( R \) defined by

\[
R_{ij} = \begin{cases} 
1 & \text{if } a_i r b_j \\
0 & \text{otherwise}
\end{cases}
\]

\( R \) is called the adjacency matrix (or the Boolean matrix, or the relation matrix) of \( r \).

**Example 6.4.1.** Let \( A = \{2, 5, 6\} \) and let \( r \) be the relation \( \{(2, 2), (2, 5), (5, 6), (6, 6)\} \) on \( A \). Since \( r \) is a relation from \( A \) into the same set \( A \) (the \( B \) of the definition), we have \( a_1 = 2, a_2 = 5, \) and \( a_3 = 6, \) and \( b_1 = 2, b_2 = 5, \) and \( b_3 = 6. \) Next, since \( 2 \rightarrow 2, \) we have \( R_{11} = 1; \)
\( 2 \rightarrow 5, \) we have \( R_{12} = 1; \)
\( 5 \rightarrow 6, \) we have \( R_{23} = 1; \) and \( 6 \rightarrow 6, \) we have \( R_{33} = 1; \)
All other entries of \( R \) are 0, so
\[
R = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

From the definition of \( r \) and of composition, we note that
\( r^2 = \{(2, 2)(2, 5)(2, 6)(5, 6)(6, 6)\}. \)

The adjacency matrix of \( r^2 \) is
\[
R^2 = \begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

We do not write \( R^2 \) only for notational purposes. In fact, \( R^2 \) can be obtained from the matrix product \( RR; \) however, we must use a slightly different form of arithmetic.

**Definition: Boolean Arithmetic.** Boolean arithmetic is the arithmetic defined on \( \{0, 1\} \) using Boolean addition and Boolean multiplication, defined as:

\[
\begin{align*}
0 + 0 &= 0 \\
0 + 1 &= 1 \\
1 + 0 &= 1 \\
1 + 1 &= 1 \\
0 \cdot 0 &= 0 \\
0 \cdot 1 &= 0 \\
1 \cdot 0 &= 1 \\
1 \cdot 1 &= 1
\end{align*}
\]

Notice that from Chapter 3, this is the "arithmetic of logic," where + replaces "or" and \( \cdot \) replaces "and."

**Example 6.4.2.**

If \( R = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \) and \( S = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} \)

Then using Boolean arithmetic, \( RS = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix} \) and \( SR = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \)

**Theorem 6.4.1.** Let \( A_1, A_2, \) and \( A_3 \) be finite sets where \( r_1 \) is a relation from \( A_1 \) into \( A_2 \) and \( r_2 \) is a relation from \( A_2 \) into \( A_3 \). If \( R_1 \) and \( R_2 \) are the adjacency matrices of \( r_1 \) and \( r_2 \), respectively, then the product \( R_1 R_2 \) using Boolean arithmetic is the adjacency matrix of the composition \( r_1 r_2 \).

Remark: A convenient help in constructing the adjacency matrix of a relation from a set \( A \) into a set \( B \) is to write the elements from \( A \) in a column preceding the first column of the adjacency matrix, and the elements of \( B \) in a row above the first row. Initially, \( R \) in Example 6.4.1 would be
\[
\begin{pmatrix}
2 & 5 & 6 \\
2 \\
5 \\
6
\end{pmatrix}
\]
and $R_{ij}$ is 1 if and only if $(a_i, b_j) \in r$. So that, since the pair $(2, 5) \in r$, the entry of $R$ corresponding to the row labeled 2 and the column labeled 5 in the matrix is a 1.

**Example 6.4.3.** This final example gives an insight into how relational data base programs can systematically answer questions pertaining to large masses of information. Matrices $R$ (on the left) and $S$ (on the right) define the relations $r$ and $s$ where $a r b$ if software $a$ can be run with operating system $b$, and $b s c$ if operating system $b$ can run on computer $c$.

<table>
<thead>
<tr>
<th>OS1</th>
<th>OS2</th>
<th>OS3</th>
<th>OS4</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>P2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>P3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>P4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Although the relation between the software and computers is not implicit from the data given, we can easily compute this information. The matrix of $rs$ is $RS$, which is

<table>
<thead>
<tr>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>P2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>P3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>P4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

This matrix tells us at a glance which software will run on the computers listed. In this case, all software will run on all computers with the exception of program $P_2$, which will not run on the computer $C_3$, and program $P_4$, which will not run on the computer $C_1$.

**EXERCISES FOR SECTION 6.4**

**A Exercises**

1. Let $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{4, 5, 6\}$, and $A_3 = \{6, 7, 8\}$. Let $r_1$ be the relation from $A_1$ into $A_2$ defined by $r_1 = \{(x, y) \mid y - x = 2\}$, and let $r_2$ be the relation from $A_2$ into $A_3$ defined by $r_2 = \{(x, y) \mid y - x = 1\}$.
   (a) Determine the adjacency matrices of $r_1$ and $r_2$.
   (b) Use the definition of composition to find $r_1 r_2$.
   (c) Verify the result in part (b) by finding the product of the adjacency matrices of $r_1$ and $r_2$.

2. (a) Determine the adjacency matrix of each relation given via the digraphs in Exercise 3 of Section 6.3.
   (b) Using the matrices found in part (a) above, find $r^2$ of each relation in Exercise 3 of Section 6.3.
   (c) Find the digraph of $r^2$ directly from the given digraph and compare your results with those of part (b).

3. Suppose that the matrices in Example 6.4.2 are relations on $\{1, 2, 3, 4\}$. What relations do $R$ and $S$ describe?

4. Let $D$ be the set of weekdays, Monday through Friday, let $W$ be a set of employees $\{1, 2, 3\}$ of a tutoring center, and let $V$ be a set of computer languages for which tutoring is offered, $\{A(PL), B(asic), C(++)\}$.
   We define $s$ (schedule) from $D$ into $W$ by $d s w$ if $w$ is scheduled to work on day $d$.
   We also define $r$ from $W$ into $V$ by $w r l$ if $w$ can tutor students in language $l$. If $s$ and $r$ are defined by matrices

   $M = \begin{pmatrix}
   1 & 0 & 1 \\
   0 & 1 & 1 \\
   1 & 0 & 0
   \end{pmatrix}$

   $A B C J L P = \begin{pmatrix}
   1 & 1 & 0 & 1 & 0 & 1
   \end{pmatrix}$

   (a) compute $s r$ using Boolean arithmetic and give an interpretation of the relation it defines, and
   (b) compute $s r$ using regular arithmetic and give an interpretation of the result describes.

5. How many different reflexive, symmetric relations are there on a set with three elements? (Hint: Consider the possible matrices.)

6. Let $A = \{a, b, c, d\}$. Let $r$ be the relation on $A$ with adjacency matrix

   $A B C J L P = \begin{pmatrix}
   a & b & c & d \\
   1 & 0 & 0 & 0 \\
   b & 0 & 1 & 0 \\
   c & 1 & 1 & 0 \\
   c & 0 & 1 & 0
   \end{pmatrix}$

   (a) compute $s r$ using Boolean arithmetic and give an interpretation of the relation it defines, and
   (b) compute $s r$ using regular arithmetic and give an interpretation of the result describes.
(a) Explain why \( r \) is a partial ordering on \( A \).

(b) Draw its Hasse diagram.

7. Define relations \( p \) and \( q \) on \( \{1, 2, 3, 4\} \) by 
\[
 p = \{(a, b) \mid |a - b| = 1\}
\]
and 
\[
 q = \{(a, b) \mid a - b \text{ is even}\}
\]
(a) Represent \( p \) and \( q \) as both graphs and matrices.

(b) Determine \( pq \), \( p^2 \), and \( q^2 \); and represent them clearly in any way.

**B Exercises**

8. (a) Prove that if \( r \) is a transitive relation on a set \( A \), then \( r^2 \subseteq r \).

   (b) Find an example of a transitive relation for which \( r^2 \neq r \).

9. We define \( \leq \) on the set of all \( n \times n \) relation matrices by the rule that if \( R \) and \( S \) are any two \( n \times n \) relation matrices, 
\[
 R \leq S \text{ if and only if } R_{ij} \leq S_{ij} \text{ for all } 1 \leq i, j \leq n.
\]
(a) Prove that \( \leq \) is a partial ordering on all \( n \times n \) relation matrices.

(b) Prove that \( R \leq S \Rightarrow R^2 \leq S^2 \), but the converse is not true.

(c) If \( R \) and \( S \) are matrices of equivalence relations and \( R \leq S \), how are the equivalence classes defined by \( R \) related to the equivalence classes defined by \( S \)?
6.5 Closure Operations on Relations

In Section 6.1, we studied relations and one important operation on relations, namely composition. This operation enables us to generate new relations from previously known relations. In Section 6.3, we discussed some key properties of relations. We now wish to consider the situation of constructing a new relation \( r^+ \) from a previously known relation \( r \) where, first, \( r^+ \) contains \( r \) and, second, \( r^+ \) satisfies the transitive property.

Consider a telephone network in which the main office \( a \) is connected to, and can communicate to, individuals \( b \) and \( c \). Both \( b \) and \( c \) can communicate to another person, \( d \), however, the main office cannot communicate with \( d \). Assume communication is only one way, as indicated. This situation can be described by the relation \( r = \{(a, b), (a, c), (b, d), (c, d)\} \). We would like to change the system so that the main office can communicate with person \( d \) and still maintain the previous system. We, of course, want the most economical system.

This can be rephrased as follows; Find the smallest relation \( r^+ \) which contains \( r \) as a subset and which is transitive; \( r^+ = \{(a, b), (a, c), (b, d), (c, d), (a, d)\} \).

**Definition:** Transitive Closure. Let \( A \) be a set and \( r \) be a relation on \( A \). The transitive closure of \( r \), denoted by \( r^+ \), is the smallest transitive relation that contains \( r \) as a subset.

**Example 6.5.1.** Let \( A = \{1, 2, 3, 4\} \), and let \( S = \{(1, 2), (2, 3), (3, 4)\} \) be a relation on \( A \). This relation is called the successor relation on \( A \) since each element is related to its successor. How do we compute \( S^+ \)? By inspection we note that \((1, 3)\) must be in \( S^+ \). Let’s analyze why. This is so because \((1, 2) \in S \) and \((2, 3) \in S \), and the transitive property forces \((1, 3) \) to be in \( S^+ \).

In general, it follows that if \((a, b) \in S \) and \((b, c) \in S \), then \((a, c) \in S^+ \). This condition is exactly the membership requirement for the pair \((a, c)\) to be in the composition \( SS = S^2 \). Every element in \( S^2 \) must be an element in \( S^+ \). So far, \( S^+ \) contains at least \( 1 \leq S \leq S^2 \). In particular, for this example, since \( S = \{(1, 2), (2, 3), (3, 4)\} \) and \( S^2 = \{(1, 3), (2, 4)\} \), we have \( S \cup S^2 = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4)\} \).

Is the relation \( S \cup S^2 \) transitive? Again, by inspection, \((1, 4)\) is not an element of \( S \cup S^2 \), but it must be an element of \( S^+ \) since \((1, 3) \) and \((3, 4) \) are required to be in \( S^+ \). From above, \((1, 3) \in S^2 \) and \((3, 4) \in S \), and the composite \( S^3 = S^2 \) produces \((1, 4) \). This shows that \( S^3 \subset S^+ \). This process must be continued until the resulting relation is transitive. If \( A \) is finite, as is true in this example, the transitive closure will be obtained in a finite number of steps. For this example, the transitive closure will be obtained in three steps.

\[ S^+ = S \cup S^2 \cup S^3 = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\} \]

**Theorem 6.5.1.** If \( r \) is a relation on a set \( A \) and \( |A| = n \), then the transitive closure of \( r \) is the union of the first \( n \) powers of \( r \). That is,

\[ r^+ = r \cup r^2 \cup r^3 \cup \cdots \cup r^n \]

Let’s now consider the matrix analogue of the transitive closure.

**Example 6.5.2.** Consider the relation \( r = \{(1, 4), (2, 1), (2, 2), (2, 3), (3, 2), (4, 3), (4, 5), (5, 1)\} \) on the set \( A = \{1, 2, 3, 4, 5\} \). The matrix of \( r \) is

\[
R = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Recall that \( r^2, r^3, \ldots \) can be determined through computing the matrix powers \( R^2, R^3, \ldots \). Here,

\[
R^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}, \quad R^3 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix},
\]

\[
R^4 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}, \text{ and } \quad R^5 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

How do we relate \( \bigcup_{i=1}^{5} r^i \) to the powers of \( R \)?

**Theorem 6.5.2.** Let \( r \) be a relation on a finite set and let \( R^+ \) be the matrix of \( r^+ \), the transitive closure of \( r \). Then \( R^+ = R + R^2 + \cdots + R^n \), using Boolean arithmetic.

Using this theorem, we find \( R^+ \) is the \( 5 \times 5 \) matrix consisting of all 1’s, thus, \( r^+ \) is all of \( A \times A \).
WARSHALL’S ALGORITHM
Let \( r \) be a relation on the set \( \{1, 2, \ldots, n\} \) with relation matrix \( R \). The matrix of the transitive closure \( R^+ \), can be computed by the equation
\[
R^+ = R + R^2 + \cdots + R^n.
\]
By using ordinary polynomial evaluation methods, you can compute \( R^+ \) with \( n-1 \) matrix multiplications:
\[
R^+ = R(I + R(I + \cdots R(I + R) \cdots))).
\]
For example, if \( n = 3 \), \( R = R(I + R) \).

We can make use of the fact that if \( T \) is a relation matrix,\( T + T = T \) due to the fact that \( 1 + 1 = 1 \) in Boolean arithmetic. Let \( S_k = R + R^2 + \cdots + R^k \). Then
\[
R = S_1
\]
\[
S_1(I + S_1) = R(I + R) = R + R^2 = S_2
\]
\[
S_2(I + S_2) = (R + R^2)(I + R + R^2)
\]
\[
= (R + R^2) + (R^2 + R^3) + (R^3 + R^4)
\]
\[
= R + R^2 + R^3 + R^4 = S_3
\]
Similarly,
\[
S_4(I + S_4) = S_8
\]
etc.

Notice how each matrix multiplication doubles the number of terms that have been added to the sum that you currently have computed. In algorithmic form, we can compute \( R^2 \) as follows.

**Algorithm 6.5.1: Transitive Closure Algorithm 1.** Let \( R \) be a known relation matrix and let \( R^+ \) be its transitive closure matrix, which is to be computed.
1.0. \( T := R \)
2.0. Repeat
2.1 \( S := T \)
2.2 \( T := S(I + S) \) // using Boolean arithmetic
Until \( T = S \)
3.0. Terminate with \( T = R^+ \).

Notes:
(a) Often the higher-powered terms in \( S_n \) do not contribute anything to \( R^+ \). When the condition \( T = S \) becomes true in Step 2, this is an indication that no higher-powered terms are needed.
(b) To compute \( R^+ \) using this algorithm, you need to perform no more than \( \lceil \log_2 n \rceil \) matrix multiplications, where \( \lceil x \rceil \) is the least integer that is greater than or equal to \( x \). For example, if \( r \) is a relation on 25 elements, no more than \( \lceil \log_2 25 \rceil = 5 \) matrix multiplications are needed.

A second algorithm, Warshall’s Algorithm, reduces computation time to the time that it takes to perform one matrix multiplication.

**Algorithm 6.5.2; Warshall’s Algorithm.** Let \( R \) be a known relation matrix and let \( R^+ \) be its transitive closure matrix, which is to be computed.
1.0 \( T := R \)
2.0 FOR \( k := 1 \) to \( n \) DO
2.1 FOR \( i := 1 \) to \( n \) DO
2.2 FOR \( j := 1 \) to \( n \) DO
\[
T[i, j] := T[i, j] + T[i, k] \cdot T[k, j]
\]
3.0 Terminate with \( T = R^+ \).

**EXERCISES FOR SECTION 6.5**

**A Exercises**
1. Let \( A \) and \( \Sigma \) be as in Example 6.5.1. Compute \( \Sigma^+ \) as in Example 6.5.2. Verify your results by checking against the relation \( \Sigma^+ \) obtained in Example 6.5.1.
Chapter 6 - Relations

2. Let $A$ and $r$ be as in Example 6.5.2. Compute the relation $r^+$ as in Example 6.5.1. Verify your results.

3. (a) Draw digraphs of the relations $S$, $S^2$, $S^3$, and $S^+$ of Example 6.5.1.
   (b) Verify that in terms of the graph of $S$, $a S^+ b$ if and only if $b$ is reachable from $a$ along a path of any finite nonzero length.

4. Let $r$ be the relation represented by the digraph in Figure 6.5.1.
   (a) Find $r^+$. 
   (b) Determine the digraph of $r^+$ directly from the digraph of $r$.
   (c) Verify your result in part (b) by computing the digraph from your result in part (a).

5. (a) Define reflexive closure and symmetric closure by imitating the definition of transitive closure.
   (b) Use your definitions to compute the reflexive and symmetric closures of Examples 6.5.1 and 6.5.2.
   (c) What are the transitive reflexive closures of these examples?
   (d) Convince yourself that the reflexive closure of the relation $<$ on the set of positive integers $\mathbb{P}$ is $\leq$.

6. What common relations on $\mathbb{Z}$ are the transitive closures of the following relations?
   (a) $a S b$ if and only if $a + 1 = b$.
   (b) $a R b$ if and only if $|a - b| = 2$.

B Exercise
7. (a) Let $A$ be any set and $r$ a relation on $A$, prove that $(r^+)^+ = r^+$.
   (b) Is the transitive closure of a symmetric relation always both symmetric and reflexive? Explain.
SUPPLEMENTARY EXERCISES FOR CHAPTER 6

Section 6.1
1. Give an example to illustrate how the relation "is a grandparent of" is a composition of the relation "is a parent of" on people.

2. Three students, Melissa, John, and Ted, would like to set up a tutorial program in the languages Pascal, FORTRAN, and COBOL. Melissa is proficient in all three languages, John in Pascal and FORTRAN, and Ted in just FORTRAN.
   (a) Let $S = \{\text{three students}\}$, $L = \{\text{three Languages}\}$, and let $p$ be the relation "is proficient in the language of". Describe this relation as a set of ordered pairs and illustrate the relation by a diagram similar to that of Figure 6.1.1.
   (b) Two P.C.s are available for tutoring purposes; one has software for Pascal and FORTRAN, and the second only for Pascal. Describe by a composite relation which student can tutor on each machine. Illustrate this composite relationship.

Section 6.2
3. Let $A = \{-1, 0, 1, 2\}$. List the ordered pairs and draw the digraphs of each of the following relations on $A$.
   (a) $r = \{(x, y) \mid y = x + 1\}$
   (b) $s = \{(x, y) \mid x^2 = y^2\}$
   (c) $t = \{(x, y) \mid x \neq y\}$

4. List the ordered pairs and draw the digraph of the relation $s^2$ for the relation $s$ of Exercise 2, Section 6.2.

5. In Figure 6.2.1, assume the nodes stand for four separate cities where a manufacturer has warehouses, while the arrows represent one-way streets. Where should the manufacturer place his main office? Where is the least desirable location? How can we interpret the arrows in both directions between nodes 1 and 2?

6. The problem of computer compatibility is an important one. In Figure 6.2.2 interpret the four nodes as representing computers, and an arrow from one node to another as "is compatible with". Note that some software does not go both ways.
   (a) Is there any one computer that is not compatible with any other?
   (b) Is it possible to create a network where any computer could be linked with any other using at most two links? If not, what software should be created to enhance compatibility?
   (c) If an arrow from a node to itself is interpreted as "high flexibility" of the system, does this affect your answer in part b?

Section 6.3
7. In Figure 6.3.2 (vii), interpret the four nodes as representing people, and an arrow from one node to another as "being friendly toward". Note that some friendships are not mutual.
   (a) Is there any individual in this group unfriendly to everyone else?
   (b) If this group were a committee, who is most likely to be the chairperson; that is, who is friendly toward the most people?
   (c) If an arrow from one vertex to itself is interpreted as "great person ality," does your answer to part b still hold?
   (d) The four people are to be seated at a round table. A person is to be seated between two people only if he is friendly toward both of them. Does a seating arrangement exist? Is there more than one?

8. Let $A = \{a, b, c, d, e\}$ and let $r, s,$ and $t$ be the following relations on $A$:
   
   $r = \{(a, a), (a, b), (b, b), (b, c), (c, c), (c, d), (d, d), (d, e), (e, e), (e, a)\}$
   $s = \{(a, a), (a, b), (a, d), (b, b), (b, d), (c, c), (d, a), (d, b), (d, d), (e, e)\}$
   $t = \{(a, a), (a, b), (b, b), (c, c), (d, d), (e, e)\}$

   (a) Which relation is a partial ordering? Draw its Hasse diagram.
   (b) Which relation is an equivalence relation? List its equivalence classes.

9. Demonstrate that the relation "living in the same house" on the set of people in a given city is an equivalence relation. State the necessary assumption for this to be the case.

10. Let $A = \{00, 01, 10, 11\}$, the set of strings of 0s and 1s with length two. Given $r$ and $s$ defined by

   $x y \Rightarrow x$ and $y$ differ in exactly one position (for example 01 r 11, but not 10 r 01), and
   $x y \Rightarrow x$ and $y$ have the same number of 0s.

   (a) Draw a directed graph of $r$.
   (b) Which of the adjectives, reflexive, symmetric, antisymmetric, and transitive, describe $r$? Explain your answers.
   (c) Which of the adjectives, reflexive, symmetric, antisymmetric, and transitive, describe $s$?
   (d) Describe with a directed graph the relation $r s$.

11. Determine whether the following relations are partial orderings and/or equivalence relations on the given set:

   (a) $C = \{\text{students in this class}\}$: $x y$ iff $x$ and $y$ have the same grade point average.
   (b) $C = \{\text{students in this class}\}$: $x y$ iff $x$ is taller than $y$.
   (c) Rephrase (slightly) the relation in part b so it is a partial ordering relation.
12. Let \( A = \{a, b, c, d\} \). Draw the graph of a relation where the relation is:
   (a) reflexive, symmetric, but not transitive.
   (b) transitive, but not symmetric and not reflexive.
   (c) both an equivalence relation and a partial ordering.

Section 6.4

13. How many symmetric relations can there be on a four-element set? Hint: Think of the possible relation matrices.

14. Let \( A = \{1, 2, 3, 4, 5, 6\} \) and let \( p = \{(i, j) \mid i \text{ divides } j\} \) be a relation on \( A \).
   (a) List the elements in \( p \).
   (b) Determine the relation matrix of \( p \).
   (c) Construct the digraph and the Hasse diagram of \( p \).

15. Let \( A = \{a, b, c\} \). The following matrices describe relations on \( A \):
   \[
   \begin{bmatrix}
   1 & 1 & 0 \\
   1 & 1 & 0 \\
   0 & 0 & 1
   \end{bmatrix}
   \]
   (i)
   \[
   \begin{bmatrix}
   1 & 0 & 1 \\
   0 & 0 & 0 \\
   1 & 1 & 1
   \end{bmatrix}
   \]
   (ii)
   (a) Draw the graph of the relation.
   (b) Describe each relation as a set of ordered pairs.
   (c) Compute \( r^2 \) for each relation \( r \).

Section 6.5

16. Let the relation \( s \) on the set \( \{a, b, c, d, e\} \) be given by the matrix
   \[
   \begin{bmatrix}
   0 & 0 & 0 & 0 \\
   1 & 0 & 1 & 0 \\
   0 & 1 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   1 & 1 & 0 & 0
   \end{bmatrix}
   \]
   (a) Draw the digraph of \( s \).
   (b) Find the transitive closure of \( s \). Give the adjacency matrix or the digraph or the set of ordered pairs.

17. Consider the relation \( r \) on \( \{1, 2, 3, 4\} \) whose Boolean matrix is
   \[
   \begin{bmatrix}
   1 & 2 & 3 & 4 \\
   1 & 0 & 0 & 1 \\
   2 & 0 & 1 & 0 \\
   3 & 0 & 0 & 1 \\
   4 & 1 & 0 & 0
   \end{bmatrix}
   \]
   (a) Draw the graph of \( r \).
   (b) Determine whether \( r \) is reflexive, symmetric, antisymmetric, and/or transitive. Explain fully.
   (c) Find the transitive closure of \( r \) and draw the graph of \( r^+ \).

18. In a small town a bank (\( b \)), school (\( s \)), town hall (\( t \)), and shopping mall (\( m \)) are connected by a series of narrow one-way streets; a street from the town hall to the bank, one from the bank to the school, one from the school to the shopping mall, and one from the shopping mall to the town hall.
   (a) Draw a digraph of this system of roads.
   (b) Find the matrix representation of the digraph in part a.
   (c) Assuming that the given streets cannot be widened, assist the mayor in planning the construction of new roads to increase traffic flow. Assume that if there is a one-way street from point \( a \) to \( b \) and one from point \( b \) to \( c \), there should be one from point \( a \) to \( c \).
   (d) If you have not done so yet, draw the matrix representation and the graph of your answer to part c and interpret the results for the mayor.

19. The ambassadors of four countries are to meet with the ambassador of the United States \( A_1 \) to discuss world problems. Some countries are friendly to each other, some are not, and in certain situations the friendship is one-way. The U.S. ambassador's daughter (who obviously took a discrete structures course) assists her father in diagnosing this complex situation and, using the relation "is friendly toward," has come up with the following digraph.
(a) Is there any person friendly to no one?

(b) Who should be the chairman of this committee; that is, who is friendly to most people?

(c) The U.S. ambassador would like to graph all friendships that can be developed through intermediaries on the committee. That is, if person $a$ is friendly toward person $b$ and person $b$ is friendly toward person $c$, then $a$ can communicate to $c$ through $b$. Draw this digraph. Can the U.S. ambassador communicate to every person on the committee through some person(s)? If not, what friendships should he develop to do so?