## Solutions and Hints to Odd-Numbered Exercises

## CHAPTER 1

## Section 1.1

1. (a) $8,15,22,29$
(b) apple, pear, peach, plum These solutions are not unique.
(c) $1 / 2,1 / 3,1 / 4,1 / 5$
(d) $-8,-6,-4,-2$
(e) $6,10,15,21$
3.(a) $\{2 k+1: k \in \mathbb{Z}, 2 \leqslant k \leqslant 39\}$
(b) $\{x \in \mathbb{Q}:-1<x<1\}$
(c) $\{2 n: n \in \mathbb{Z}\}$
(d) $\{9 n: n \in \mathbb{Z},-2 \leqslant n\}$
5.(a) True
(b) False
(e) True
(d) True
(e) False
(f) True
(g) False
(h) True

Section 1.2

1. (a) $\{2,3\}$
(b) $\{0,2,3\}$
(c) $\{0,2,3\}$
(d) $\{0,1,2,3,5,9\}$
(e) $\{0\} \quad$ (f) $\emptyset \quad$ (g) $\{1,4,5,6,7,8,9\} \quad$ (h) $\{0,2,3,4,6,7,8\}$
(i) $\varnothing \quad$ (j) $\{0\}$
2. These are all true for any sets $A, B$, and $C$.
3. (a) $\{1,4\} \subseteq A \subseteq\{1,2,3,4\}$
(b) $\{2\} \subseteq A \subseteq\{1,2,4,5\} \quad$ (c) $A=\{2,4,5\}$
4. 


(a)

(b)

(c)

9. (a)
? Select

Select $[$ list, crit $]$ picks out all elements $e_{i}$ of list for which crit $\left[e_{i}\right]$ is True.
Select $[$ list, crit, $n]$ picks out the first $n$ elements for which crit $\left[e_{i}\right]$ is True. >>
? PrimeQ

PrimeQ[expr] yields True if expr is a prime number, and yields False otherwise. >>
(b)

## Select[Range[2000, 2099], PrimeQ[\#] \&]

$\{2003,2011,2017,2027,2029,2039,2053,2063,2069,2081,2083,2087,2089,2099\}$

## Section 1.3

1. (a) $\{(0,2),(0,3),(2,2),(2,3),(3,2),(3,3)\}$
(b) $\{(2,0),(2,2),(2,3),(3,0),(3,2),(3,3)\}$
(c) $\{(0,2,1),(0,2,4),(0,3,1),(0,3,4),(2,2,1),(2,2,4),(2,3,1),(2,3,4),(3,2,1),(3,2,4),(3,3,1),(3,3,4)\}$
(d) $\varnothing$
(e) $\{(0,1),(0,4),(2,1),(2,4),(3,1),(3,4)\}$
(f) $\{(2,2),(2,3),(3,2),(3,3)\}$
$(\mathrm{g})\{(2,2,2),(2,2,3),(2,3,2),(2,3,3),(3,2,2),(3,2,3),(3,3,2),(3,3,3)\}$
(h) $\{(2, \phi),(2,\{2\}),(2,\{3\}),(2,\{2,3\}),(3, \phi),(3,\{2\}),(3,\{3\}),(3,\{2,3\})\}$
$3 .\{a, b\},\{a, c\},\{a, d\},\{b, c\},\{b, d\},\{c, d\}$
5 . There are $n$ singleton subsets, one for each element.
2. (a) $\{+00,+01,+10,+11,-00,-01,-10,-11\}$ (b) 16 and 512
3. When $A=B$

## Section 1.4

1.(a) 11111
(b) 100000
(c) 1010
(d) 1100100
3. (a) 18
(b) 19
(c) 42
(d) 1264
5.There is a bit for each power of 2 , starting with the zeroth power. The number 1990 is between $2^{10}=1024$ and $2^{11}=2048$, so there are $10+1$ (the 0 power of 2) bits
(a) 11
(b) 12
(c) 13
(d) 8
7.It must be a multiple of four.

## Section 1.5

1.(a) 24
(b) 6
(c) $3,7,15,31$
(d) $1,4,9,16$
3. (a) $\frac{1}{1(1+1)}+\frac{1}{2(2+1)}+\frac{1}{3(3+1)}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$
(b) $\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}=\frac{3}{3+1}$
(c) $1+2^{3}+3^{3}+\cdots+n^{3}=\left(\frac{1}{4}\right) n^{2}(n+1)^{2}$

$$
\begin{array}{cc}
1+4+27=36 & \left(\frac{1}{4}\right)(3)^{2}(3+1)^{2}=36 \\
5 .(x+y)^{n}=\binom{n}{0} x^{n}+\binom{n}{1} x^{n-1} y+\binom{n}{2} x^{n-2} y^{2}+\cdots+\binom{n}{n-1} x y^{n-1}+\binom{n}{n} y^{n}
\end{array}
$$

7.(a) $\{x \in \mathbb{Q} \mid 0<x \leqslant 5\}$
(b) $\varnothing$
(c) $\{x \in \mathbb{Q} \mid-5<x<5\}=B_{5}$
(d) $\{x \in \mathbb{Q} \mid-1<x<1\}=B_{1}$
9.(a) 36
(b) 105

## Supplementary Exercises-Chapter 1

1. (a) $\{2,1\} \quad$ (b) $\varnothing$ (c) $\{\mathrm{i},-\mathrm{i}\}$
2. (a) $\{0,3,4,5,6,7,8,9\} \quad$ (b) $\{3,6,9\} \quad$ (c) $\{0,1,2\}$
3. (a) $A \cup B=\{1,2,3,4,5,6,7,9,12\} \Rightarrow|(A \cup B)|=9$
$|A|=6,|B|=5 A \cap B=\{2,5\} \Rightarrow|(A \cap B)|=2$
$|A|+|B|-|(A \cap B)|=6+5-2=9$
(b) $10,8,2$
(c) $(A \cup B \cup C)=|((A \cup B) \cup C)|$
by part (a) $\quad=|(A \cup B)|+|C|-|((A \cup B) \cap C)|$
Distributive $\quad=|(A \cup B)|+|C|-|((A \cap C) \cup(B \cap C))|$
by part (a) $\quad=|A|+|B|-|(A \cap B)|+|C|-[|(A \cap C)|$
$+|(B \cap C)|-|((A \cap C) \cap(B \cap C))|]$
Simplify $=|A|+|B|+|C|-|(A \cap B)|-|(A \cap C)|$ $-|(B \cap C)|+|(A \cap B \cap C)|$
4. (a) $\{(4,4)\}$
(b) $\{(2,4),(4,4),(6,4)\}$
(c) $\{(4,4,4)\}$
(cl) $\{(4,2),(4,4),(4,6)\}$
(e) $\{(2,4,1),(2,4,5),(4,4,1),(4,4,5),(6,4,1),(6,4,5)\}$
5. (a) $A_{0}=\{0\}, A_{1}=\{0,1,2,3\}, A_{2}=\{0,1,2, \ldots, 6\}, A 3=\{0,1,2, \ldots, 9\}$
(b) $(0,1,2,3\}$
(c) $\{0\}$
(d) $\{0,1,2, \ldots, 9\}$
(e) $\{0,1,2, \ldots, 9\}$
6. Parts $\mathrm{a}, \mathrm{b}$, and d are true with multiplication replacing addition.

## CHAPTER 2

## Section 2.1

1. If there are $m$ horses in race 1 and $n$ horses in race 2 then there are $m \cdot n$ possible daily doubles.
$3.72=4 \cdot 6 \cdot 3$
$5.720=6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
2. If we always include the blazer in the outfit we would have 6 outfits. If we consider the blazer optional then there would be 12 outfits. When we add a sweater we have the same type of choice. Considering the sweater optional produces 24 outfits.
3. (a) $2^{8}=256$ (b) $2^{4}=16$. Here we are concerned only with the first four bits, since the last four must be the same.
(c) $2^{7}=128$, you have no choice in the last bit.
4. (a) 16
(b) 30
5. (a) 3
(b) 6
6. 18
7. (a)

(b) $5^{6}$
8. $2^{n-1}-1$ and $2^{n}-2$

## Section 2.2

1. $P(1000 ; 3)$
2. With repetition: $26^{8} \approx 2.0883 \times 10^{11}$

Without repetition: $P(26 ; 8) \approx 6.2991 \times 10^{10}$
5. 15!
7. (a) $P(15 ; 5)=360360$
(b) $2 \cdot 14 \cdot 13 \cdot 12 \cdot 11=48048$
9. $2 \cdot P(3 ; 3)=12$
11. (a) $P(4 ; 2)=12$ (b) $P(n ; 2)=n(n-1)$
(c) Case 1: $m>n$. Since the coordinates must be different, this case is impossible.

Case 2: $m \leqslant n . P(n ; m)$.

## Section 2.3

1. $\{\{a\},\{b\},\{c\}\},\{\{a, b\},\{c\}\},\{\{a, c\},\{b\}\},\{\{a\},\{b, c\}\},\{\{a, b, c\}\}$
2. No. By this definition it is possible that an element of $A$; might not belong to $A$.
3. The first subset is all the even integers and the second is all the odd integers.

These two sets do not intersect and they cover the integers completely.
7. Let $J, A, N$ stand for the set of people who jog, do aerobics, and do Nautilus.

Then, using law of addition \#2,
$90=30+30+30-25-20-10+|(J \bigcap A \cap N)|$.
9. Assume $\left|\left(A_{1} \cup A_{2}\right)\right|=\left|A_{1}\right|+\left|A_{2}\right|-\left|\left(A_{1} \cap A_{2}\right)\right|$.
$\left|\left(A_{1} \cup A_{2} \cup A_{3}\right)\right|=\left|\left(\left(A_{1} \cup A_{2}\right) \cup A_{3}\right)\right|$

| by 1st law | $=\left\|\left(A_{1} \cup A_{2}\right)\right\|+\left\|A_{3}\right\|-\left\|\left(\left(A_{1} \cup A_{2}\right) \cap A_{3}\right)\right\|$ |
| :--- | :--- |
| Distributive | $=\left\|\left(A_{1} \cup A_{2}\right)\right\|+\left\|A_{3}\right\|-\left\|\left(\left(A_{1} \cap A_{3}\right) \cup\left(A_{2} \cap A_{3}\right)\right)\right\|$ |

1st law (twice) $=\left|A_{1}\right|+\left|A_{2}\right|-\left|\left(A_{1} \cap A_{2}\right)\right|+\left|A_{3}\right|$

$$
-\left[\left|\left(A_{1} \cap A_{3}\right)\right|+\left|\left(A_{2} \cap A_{3}\right)\right|-\left|\left(\left(A_{1} \cap A_{3}\right) \cap\left(A_{2} \cap A_{3}\right)\right)\right|\right]
$$

Simplify

$$
\begin{aligned}
& =\left|A_{1}\right|+\left|A_{2}\right|+\left|A_{3}\right|-\left|\left(A_{1} \cap A_{2}\right)\right|-\left|\left(A_{1} \cap A_{3}\right)\right| \\
& -\left|\left(A_{2} \cap A_{3}\right)\right|+\left|\left(A_{1} \cap A_{2} \cap A_{3}\right)\right| .
\end{aligned}
$$

(b) $\left|\left(A_{1} \cup A_{2} \cup A_{3} \cup A_{4}\right)\right|=\left|A_{1}\right|+\left|A_{2}\right|+\left|A_{3}\right|+\left|A_{4}\right|-\left|\left(A_{1} \cap A_{2}\right)\right|$
$-\left|\left(A_{1} \cap A_{3}\right)\right|-\left|\left(A_{1} \cap A_{4}\right)\right|-\left|\left(A_{2} \cap A_{3}\right)\right|-\left|\left(A_{2} \cap A_{4}\right)\right|$
$-\left|\left(A_{3} \cap A_{4}\right)\right|+\left|\left(A_{1} \cap A_{2} \cap A_{3}\right)\right|+\left|\left(A_{1} \cap A_{2} \cap A_{4}\right)\right|$
$+\left|\left(A_{1} \cap A_{3} \cap A_{4}\right)\right|+\left|\left(A_{2} \cap A_{3} \cap A_{4}\right)\right|-\left|\left(A_{1} \cap A_{2} \cap A_{3} \cap A_{4}\right)\right|$
Derivation:

| $\left\|\left(A_{1} \cup A_{2} \cup A_{3} \cup A_{4}\right)\right\|=\left\|\left(\left(A_{1} \cup A_{2} \cup A_{3}\right) \cup A_{4}\right)\right\|$ |  |
| ---: | :--- |
| 1st law $\quad=$ | $\left\|\left(A_{1} \cup A_{2} \cup A_{3}\right)\right\|+\left\|A_{4}\right\|-\left\|\left(\left(A_{1} \cup A_{2} \cup A_{3}\right) \cap A_{4}\right)\right\|$ |
| Distributive $\quad=$ | $\left\|\left(A_{1} \cup A_{2} \cup A_{3}\right)\right\|+\left\|A_{4}\right\|-\mid\left(\left(A_{1} \cap A_{4}\right) \cup\left(A_{2} \cap A_{4}\right)\right.$ |
|  | $\cup\left(A_{3} \cap A_{4}\right) \mid$ |

2nd law (twice) | $=$ | $\left\|A_{1}\right\|+\left\|A_{2}\right\|+\left\|A_{3}\right\|-\left\|\left(A_{1} \cap A_{2}\right)\right\|-\left\|\left(A_{1} \cap A_{3}\right)\right\|$ |
| ---: | :--- |
|  | $-\left\|\left(A_{2} \cap A_{3}\right)\right\|+\left\|\left(A_{1} \cap A_{2} \cap A_{3}\right)\right\|+\left\|A_{4}\right\|-\left[\left\|\left(A_{1} \cap A_{4}\right)\right\|\right.$ |
|  | $+\left\|\left(A_{2} \cap A_{4}\right)\right\|+\left\|\left(A_{3} \cap A_{4}\right)\right\|-\left\|\left(\left(A_{1} \cap A_{4}\right) \cap\left(A_{2} \cap A_{4}\right)\right)\right\|$ |
|  | $-\left\|\left(\left(A_{1} \cap A_{4}\right) \cap\left(A_{3} \cap A_{4}\right)\right)\right\|-\left\|\left(\left(A_{2} \cap A_{4}\right) \cap\left(A_{3} \cap A_{4}\right)\right)\right\|$ |
|  | $\left.+\left\|\left(\left(A_{1} \cap A_{4}\right) \cap\left(A_{2} \cap A_{4}\right) \cap\left(A_{3} \cap A_{4}\right)\right)\right\|\right]$ |

| Simplify $=\left\|A_{1}\right\|+\left\|A_{2}\right\|+\left\|A_{3}\right\|+\left\|A_{4}\right\|-\left\|\left(A_{1} \cap A_{2}\right)\right\|-\left\|\left(A_{1} \cap A_{3}\right)\right\|$ |
| ---: | :--- |
| $-\left\|\left(A_{2} \cap A_{3}\right)\right\|-\left\|\left(A_{1} \cap A_{4}\right)\right\|-\left\|\left(A_{2} \cap A_{4}\right)\right\|-\left\|\left(A_{3} \cap A_{4}\right)\right\|$ |
| $+\left\|\left(A_{1} \cap A_{2} \cap A_{3}\right)\right\|+\left\|\left(A_{1} \cap A_{2} \cap A_{4}\right)\right\|+\left\|\left(A_{1} \cap A_{3} \cap A_{4}\right)\right\|$ |
| $+\left\|\left(A_{2} \cap A_{3} \cap A_{4}\right)\right\|-\left\|\left(A_{1} \cap A_{2} A_{3} \cap A_{4}\right)\right\|$ |

11. Hint: Partition the set of fractions into blocks, where each block contains fractions that are numerically equivalent. Describe how you would determine whether two fractions belong to the same block. Redefine the rational numbers to be this partition. Each rational number is a set of fractions.

## Section 2.4

1. $C(10 ; 3) \cdot C(25 ; 4)=1,518,000$
2. $C(10 ; 7)+C(10 ; 8)+C(10 ; 9)+C(10 ; 10)$
3. $16 x^{4}-96 x^{3} y+216 x^{2} y^{2}-216 x y^{3}+81 y^{4}$
4. (a) $C(52 ; 5)=2,598,960$
(b) $C(52 ; 5) \cdot C(47 ; 5) \cdot C(42 ; 5) \cdot C(37 ; 5)$
5. $C(4 ; 2) C(48 ; 3)$
6. $C(12 ; 3) \cdot C(9 ; 4) \cdot C(5 ; 5)$
7. (a) $C(10 ; 2)=45$
(b) $C(10 ; 3)=120$
8. Assume $|A|=n$. If we let $x=y=1$ in the Binomial Theorem, we obtain
$2^{n}=C(n ; 0)+C(n ; 1)+\cdots+C(n ; n)$, and as a consequence of Example
2.4.7 we realize that the right side of this equation says the sum of all subsets
of $A$. Hence $|P(A)|=2^{|A|}$
9. 999, 400, 119, 992.

## Supplementary Exercises-Chapter 2

1. (a) $10 \cdot 9 \cdot 8=720$
(b) $10 \cdot 10 \cdot 10=1000$
2. (a)

(b) If you imagine drawing a tree diagram for this general case, from the starting point, there will be $m$ branches, one for each element of $A$. From the end of each of the " $A$ branches" there will be $n$ branches, one for each element of $B$. Therefore, there are $m \cdot n$ pairs in $A \times B$. 5. (a) If couple $A$ is seated, couple $B$ can be either to their left or right and couple $C$ sits in the other position; therefore, there are two possible arrangements.
(b) $2 \cdot 2^{3}=48$
3. (a) $5!=120 \quad$ (b) $5!-2 \cdot 4!=72$. (Here, we subtract the ways that the
two could be seated together from the total number of arrangements.)
4. (a) $P(10 ; 4)$
(b) $C(10 ; 4) \cdot C(6 ; 3)$
5. $C(10 ; 2) \cdot 8=360$
6. (a) $C(11 ; 5)=462$ (b) $C(10 ; 4)=210$
(c) $C(2 ; 1) \cdot C(9 ; 4)+C(9 ; 3)$
7. (a) $3 P(3 ; 2)=18$
(b) $2 P(3 ; 2)=12$

## CHAPTER 3

## Section 3.1

1. (a) $d \wedge c \quad$ (b) $s \vee \neg c$
(c) $\neg(d \wedge s) \quad$ (d) $\neg s \wedge \neg c$
2. (a) $2>5$ and 8 is an even integer. False.
(b) If $2 \leqslant 5$ then 8 is an even integer. True.
(c) If $2 \leqslant 5$ and 8 is an even integer then 11 is a prime number. True.
(d) If $2 \leqslant 5$ then either 8 is an even integer or 11 is not a prime number. True.
(e) If $2 \leqslant 5$ then either 8 is an odd integer or 11 is not a prime number. False
(f) If 8 is not an even integer then $2>5$. True.
3. Only the converse of $d$ is true.

## Section 3.2

1. (a) | $p \quad p \bigvee p$ |  |
| :---: | :---: |
| 1 | 0 |
| 1 | 1 |

(b) | $p$ | $\neg p$ | $p \wedge p$ |
| :---: | :---: | :---: |
| 0 | 1 | 0 |
| 1 | 0 | 0 |

(c) | $p$ | $\neg p$ | $p \wedge(\neg p)$ |
| :---: | :---: | :---: |
| 0 | 1 | 1 |
| 1 | 0 | 1 |

(d) | $p$ | $p \wedge p$ |
| :---: | :---: |
| 0 | 0 |
| 1 | 1 |

3. (a) $\neg(p \wedge q) \bigvee s$
(b) $(p \bigvee q) \wedge(r \bigvee q)$
4. $2^{4}=16$

## Section 3.3

1. $a \Leftrightarrow e, d \Leftrightarrow f, g \Leftrightarrow h$
2. No. In symbolic form the question is: Is $(p \rightarrow q) \Leftrightarrow(q \rightarrow p)$ ?

| $p$ | $q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \leftrightarrow(q \rightarrow p)$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

This table indicates that an implication is not always equivalent to its converse.
5. Let $x$ be any proposition generated by $p$ and $q$. The truth table for $x$ has 4 rows and there are 2 choices for a truth value for $x$ for each row, so there are
$2 \cdot 2 \cdot 2 \cdot 2=2^{4}$ possible propositions.
(See Table 13.6.1 for an illustration.)
$7.0 \rightarrow p$ and $p \rightarrow 1$ are tautologies.

## Section 3.4

1. Let $s=$ "I will study", $t=$ "I will learn." The argument is: $((s \rightarrow t) \wedge(\neg t)) \rightarrow(\neg s)$, call the argument $a$.

| $s$ | $t$ | $s \rightarrow t$ | $(s \rightarrow t) \wedge(\neg t)$ | $a$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 |

Since $a$ is a tautology, the argument is valid.
3. In any true statement $S$, replace; $\wedge$ with $\vee, \vee$ with $\wedge, 0$ with 1,1 with $0, \Leftarrow$ with $\Rightarrow$, and $\Rightarrow$ with $\Leftarrow$. Leave all other connectives unchanged.
5. (a) If not EOF then repeat

Read (ch);
Count := Count +1
Until EOF
Law used: involution law, Not Not EOF $\Leftrightarrow \mathrm{EOF}\}$
(b) $\quad S:=0 ; K:=1 ; N:=100$;

If $K<=N$ then do
Repeat

$$
\begin{aligned}
& S:=S+K \\
& K:=K+1 \\
& \text { Until } K>N
\end{aligned}
$$

No Law of logic is really used here, only a law of integers:
$\operatorname{Not}(K \leq N) \Leftrightarrow K>N$.

## Section 3.5

1. (a)

| $p$ | $q$ | $(p \bigvee q) \wedge \neg q$ | $((p \bigvee q) \wedge \neg q) \rightarrow p$ |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |

(b)

| $p$ | $q$ | $(p \rightarrow q) \wedge \neg q$ | $\neg p$ | $(p \rightarrow q) \wedge(\neg q)$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 |

3. (a) Direct proof:
(1) $d \rightarrow(a \vee c)$
(2) $d$
(3) $a \vee c$
(4) $a \rightarrow b$
(5) $\neg a \bigvee b$
(6) $c \rightarrow b$
(7) $\neg c \bigvee b$
(8) $(\neg a \bigvee b) \wedge(\neg c \bigvee b)$
(9) $(\neg a \wedge \neg c) \bigvee b$
(10) $\neg(a \vee c) \bigvee b$
(11)

Indirect proof:

| (1) $\neg b$ | Negated conclusion |
| :--- | :--- |
| (2) $a \rightarrow b$ | Premise |
| (3) $\neg a$ | Indirect Reasoning (1), (2) |
| (4) $c \rightarrow b$ | Premise |
| (5) $\neg c$ | Indirect Reasoning (1), (4) |
| (6) $\neg a \wedge \neg c)$ | Conjunctive (3), (5) |
| (7) $\neg(a \vee c)$ | DeMorgan's law (6) |
| (8) $d \rightarrow(a \vee c)$ | Premise |
| (9) $\neg d$ | Indirect Reasoning (7), (8) |
| (10) $d$ | Premise |
| (11) 0 | (9), (10) ■ |

(b) Direct proof:
(1) $(p \rightarrow q) \wedge(r \rightarrow s)$
(2) $p \rightarrow q$
(3) $(p \rightarrow t) \wedge(s \rightarrow u)$
(4) $q \rightarrow t$
(5) $p \rightarrow t$
(6) $r \rightarrow s$
(7) $s \rightarrow u$
(8) $r \rightarrow u$
(9) $p \rightarrow r$
(10) $p \rightarrow u$
(11) $p \rightarrow(t \wedge u) \quad$ Use $(x \rightarrow y) \wedge(x \rightarrow z) \Leftrightarrow x \rightarrow(y \wedge z)$
(12) $\neg(t \wedge u) \rightarrow \neg p$
(13) $\neg(t \wedge u)$
(14) $\neg p$ ■

Indirect proof:
(1) $p$
(2) $p \rightarrow q$
(3) $q$
(4) $q \rightarrow t$
(5) $t$
(6) $\neg(t \wedge u)$
(7) $\neg t \vee \neg u$
(8) $\neg u$
(9) $s \rightarrow u$
(10) $\neg s$
(11) $r \rightarrow s$
(12) $\neg r$
(13) $p \rightarrow r$
(14) $r$
(15) $0 ■$
(c) Direct proof:

| (1) $\neg s \vee p$ | Premise |
| :--- | :--- |
| (2) $s$ | Added premise (conditional conclusion) |
| (3) $\neg(\neg s)$ | Involution (2) |
| (4) $p$ | Disjunctive simplification (1), (3) |
| (5) $p \rightarrow(q \rightarrow r)$ | Premise |
| (6) $q \rightarrow r$ | Detachment (4), (5) |


| (7) $q$ | Premise |
| :--- | :--- |
| (8) $r$ | Detachment (6), (7) |

Indirect proof:

| (1) $\neg(s \rightarrow r)$ | Negated conclusion |
| :--- | :--- |
| (2) $\neg(\neg s \vee r)$ | Conditional equivalence (I) |
| (3) $s \wedge \neg r$ | DeMorgan (2) |
| (4) $s$ | Conjunctive simplification (3) |
| (5) $\neg s \vee p$ | Premise |
| (6) $s \rightarrow p$ | Conditional equivalence (5) |
| (7) $p$ | Detachment (4), (6) |
| (8) $p \rightarrow(q \rightarrow r)$ | Premise |
| (9) $q \rightarrow r$ | Detachment (7), (8) |
| (10) $q$ | Premise |
| (11) $r$ | Detachment (9), (10) |
| (12) $\neg r$ | Conjunctive simplification (3) |
| (13) 0 | Conjunction (11), (12) ■ |

(d) Direct proof:
(1) $p \rightarrow q$
(2) $q \rightarrow r$
(3) $p \rightarrow r$
(4) $p \bigvee r$
(5) $\neg p \bigvee r$
(6) $(p \bigvee r) \wedge(\neg p \bigvee r)$
(7) $(p \wedge \neg p) \bigvee r$
(8) $0 \bigvee r$
(9) $r$

Indirect proof:

| (1) $\neg r$ | Negated conclusion |
| :--- | :--- |
| (2) $p \vee r$ | Premise |
| (3) $p$ | (1), (2) |
| (4) $p \rightarrow q$ | Premise |
| (5) $q$ | Detachment (3), (4) |
| (6) $q \rightarrow r$ | Premise |
| (7) $r$ | Detachment (5), (6) |
| (8) 0 | (1), (7) |

5. (a) Let $W$ stand for "wages will increase," $I$ stand for "there will be inflation," and $C$ stand for "cost of living will increase." Therefore the argument is: $W \rightarrow I, \quad \neg I \rightarrow \neg C, \quad W \Rightarrow C$.. The argument is invalid. The easiest way to see this is through a truth table. Let $x$ be the conjunction of all premises.

| $W$ | $I$ | $C$ | $\neg I$ | $\neg C$ | $W \rightarrow I$ | $\neg I \rightarrow \neg C$ | $x$ | $x \rightarrow C$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |

(b) Let $r$ stand for "the races are fixed," $c$ stand for "casinos are crooked," $t$ stand for "the tourist trade will decline," and $p$ stand for "the police will be happy." Therefore, the argument is:
$(r \vee c) \rightarrow t, t \rightarrow p, \neg p \rightarrow \neg r$. The argument is valid. Proof:

| (1) $t \rightarrow p$ | Premise |
| :--- | :--- |
| (2) $\neg p$ | Premise |
| (3) $\neg t$ | Indirect Reasoning (1), (2) |
| (4) $(r \vee c) \rightarrow t$ | Premise |
| (5) $\neg(r \vee c)$ | Indirect Reasoning (3), (4) |
| (6) $(\neg r) \wedge(\neg c)$ | DeMorgan (5) |
| (7) $\neg r$ | Conjunction simplification (6) |

7. $p_{1} \rightarrow p_{k}$ and $p_{k} \rightarrow p_{k+1}$ implies $p_{1} \rightarrow p_{k+1}$. It takes two steps to get to $p_{1} \rightarrow p_{k+1}$ from $p_{1} \rightarrow p_{k}$ This means it takes $2(100-1)$ steps to get to $p_{1} \rightarrow p_{100}$ (subtract 1 because $p_{1} \rightarrow p_{2}$ is stated as a premise). A final step is needed to apply detachment to imply $p_{100}$

## Section 3.6

1. (a) $\{\{1\},\{3\},\{1,3\}, \phi\}$
(b) $\{\{3\},\{3,4\},\{3,2\},\{2,3,4\}\}$
(c) $\{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\}\}$
(d) $\{\{2\},\{3\},\{4\},\{2,3\},\{2,4\},\{3,4\}\}$
(e) $\{A \subseteq U:|A|=2\}$
2. There are $2^{3}=8$ subsets of $U$, allowing for the possibility of $2^{8}$ nonequivalent propositions over $U$.
3. $s$ is odd and $(s-1)(s-3)(s-5)(s-7)=0$
$7 . b$ and $c$

## Section 3.7

1. We wish to prove that $P(n): 1+3+5+\cdots+(2-n)=n^{2}$ is true for $n \geqslant 1$. Note: The $n$th odd positive integer is $2 \mathrm{n}-1$.

Basis: for $n=1: 1=1^{2}$
Induction: Assume that for some $n \geqslant 1, p(n)$ is true. Then:

$$
\begin{aligned}
1+3+\cdots+( & (n+1)-1)+1=[1+3+\cdots+(2 n-1)] \\
& +(2(n+1)-1) \\
& =n^{2}+(2 n+1) \text { by } p(n) \text { and basic algebra } \\
& =(n+1)^{2}
\end{aligned}
$$

3. Proof:
(a) Basis: $1=1(2)(3) / 6=1$
(b) Induction: $\sum_{1}^{n+1} k^{2}=\sum_{1}^{n} k^{2}+(n+1)^{2}$

$$
\begin{aligned}
& =\frac{n(n+1)(2 n+1)}{6}+(n+1)^{2} \\
& =\frac{(n+1)\left(2 n^{2}+7 n+6\right)}{6} \\
& =\frac{(n+1)(n+2)(2 n+3)}{6}
\end{aligned}
$$

5. Basis: For $n=1$, we observe that $\frac{1}{(1 \cdot 2)}=\frac{1}{(1+1)}$

Induction: Assume that for some $n \geqslant 1$, the formula is true.
Then: $\frac{1}{(1 \cdot 2)}+\cdots+\frac{1}{((n+1)(n+2))}=\frac{n}{(n+1)}+\frac{1}{((n+1)(n+2))}$

$$
\begin{aligned}
& =(n+2)(n)+\frac{1}{((n+1)(n+2))} \\
& =\frac{(n+1)^{2}}{((n+1)(n+2))} \\
& =\frac{(n+1)}{(n+2)}
\end{aligned}
$$

7. Let $A_{n}$ be the set of strings of zeros and ones of length $n$ (we assume that $\left|A_{n}\right|=2^{n}$ is known), $E_{n}=$ the even strings, and $E_{n}^{c}=$ the odd strings. The problem is to prove that for $n \geqslant 1,\left|E_{n}\right|=2^{n-1}$. Clearly, $\left|E^{1}\right|=1$, and, if for some $n \geqslant 1,\left|E_{n}\right|=2^{n-1}$, it follows that $\left|E_{n+1}\right|=2^{n}$ by the following reasoning:
$E_{n+1}=\left\{1 s: s\right.$ in $\left.E_{n}^{c}\right\} \bigcup\left\{0 s: s\right.$ in $\left.E_{n}\right\}$
Since $\left\{1 s: s\right.$ in $\left.E_{n}^{c}\right\}$ and $\left\{0 s: s\right.$ in $\left.E_{n}\right\}$ are disjoint, we can apply the addition law. Therefore, $\left|E_{n+1}\right|=\left|E_{n}^{c}\right|+\left|E_{n}\right|$

$$
=2^{n-1}+\left(2^{n}-2^{n-1}\right)=2^{n} .
$$

9. Assume that for $n$ persons $(n \geqslant 1), \frac{(n-1) n}{2}$ handshakes take place. If one more person enters the room, he or she will shake hands with $n$ people, $\frac{(n-1) n}{2}+n=\frac{\left(n^{2}-n+2 n\right)}{2}=\frac{n(n+1)}{2}$

$$
=\frac{((n+1)-1)(n+1)}{2}
$$

Also, for $n=1$, there are no handshakes: $\frac{(1-1)(1)}{2}=0$.
11. Let $p(n)$ be " $a_{1}+a_{2}+\cdots+a_{n}$ has the same value no matter how it is evaluated."

Basis: $a_{1}+a_{2}+a_{3}$ may be evaluated only two ways. Since + is associative, $\left(a_{1}+a_{2}\right)+a_{3}=a_{1}+\left(a_{2}+a_{3}\right)$. Hence $p$ (3) is true.
Induction: Assume that for some $n \geqslant 3 p(3), p(4), \ldots, p(n)$ are all true. Now consider the sum $a_{1}+a_{2}+\cdots+a_{n+1}$. Any of the $n$ additions in this expression can be applied last. If the $j$ th addition is applied last, we have $c_{j}=\left(a_{1}+a_{2}+\cdots+a_{j}\right)+\left(a_{j+1}+\cdots+a_{n+1}\right)$. No matter how the expression to the left and right of the $j^{\text {th }}$ addition are evaluated, the result will always be the same by the induction hypothesis, specifically $p(j)$ and $p(n+1-j)$. We now can prove that $c_{1}=c_{2}=\cdots=c_{n}$. If $i<j$,

$$
\begin{aligned}
& c_{i}=\left(a_{1}+a_{2}+\cdots+a_{i}\right)+\left(\left(a_{i+1}+\cdots+a_{j}\right)+\left(a_{j+1}+\cdots+a_{n+1}\right)\right. \\
& =\left(a_{1}+a_{2}+\cdots+a_{i}\right)+\left(\left(a_{i+1}+\cdots+a_{j}\right)+\left(a_{j+1}+\cdots+a_{n+1}\right)\right. \\
& =\left(\left(a_{1}+\cdots+a_{i}\right)+\left(a_{i+1}+\cdots+a_{j}\right)\right)+\left(a_{j+1}+\cdots+a_{n+1}\right) \quad \text { by } p(3) \\
& =c_{j} \\
& c_{i}=\left(a_{1}+a_{2}+\cdots+a_{i}\right)+\left(a_{i+1}+\cdots+a_{j}+a_{j+1}+\cdots+a_{n+1}\right) \quad \text { definition of } c_{i} \\
& =\left(a_{1}+a_{2}+\cdots+a_{i}\right)+\left(\left(a_{i+1}+\cdots+a_{j}\right)+\left(a_{j+1}+\cdots+a_{n+1}\right) \quad \text { by } p(n+1-i)\right. \\
& =\left(\left(a_{1}+\cdots+a_{i}\right)+\left(a_{i+1}+\cdots+a_{j}\right)\right)+\left(a_{j+1}+\cdots+a_{n+1}\right) \quad \text { by } p(3) \\
& =\left(a_{1}+\cdots+a_{i}+a_{i+1}+\cdots+a_{j}\right)+\left(a_{j+1}+\cdots+a_{n+1}\right) \quad \text { by } p(i) \\
& =c_{j} \quad \quad \text { definition of } c_{j}
\end{aligned}
$$

13. For $m \geqslant 1$, let $p(m)$ be $x^{n+m}=x^{n} x^{m}$ for all $n \geqslant 1$. The basis for this proof follows directly from the basis for the definition of exponentiation. Induction: Assume that for some $m \geqslant 1, p(m)$ is true. Then
$x^{n+(m+1)}=x^{(n+m)+1} \quad$ by associativity of integer addition

$$
\begin{array}{ll}
=x^{n+m} x^{1} & \text { by recursive definition } \\
=x^{n} x^{m} x^{1} & \\
=x^{n} x^{m+1} & \\
\text { inductive hypothesis } \\
\text { recursive definition }
\end{array}
$$

## Section 3.8

1. (a) $(\forall x)(F(x) \rightarrow G(x))$
(b) There are objects in the sea which are not fish.

Every fish lives in the sea.
3. (a) There is a book with a cover that is not blue.
(b) Every mathematics book that is published in the United States has a blue cover.
(c) There exists a mathematics book with a cover that is not blue.
(d) There exists a book that appears in the bibliography of every mathematics book.
(e) $(\forall x)(B(x) \rightarrow M(x))$
(f) $(\exists x)(M(x) \wedge \neg U(x))$
(g) $(\exists x)((\forall y)(\neg R(x, y))$
5. The equation $4 u^{2}-9=0$ has a solution in the integers. (False)
7. (a) Every subset of $U$ has a cardinality different from its complement. (True)
(b) There is a pair of disjoint subsets of $U$ both having cardinality 5. (False)
(c) $A-B=B^{c}-A^{c}$ is a tautology. (True)
9. $(\forall a)_{\mathbb{Q}}(\forall b)_{\mathbb{Q}}(a+b$ is a rational number. $)$
11. Let $I=\{1,2,3, \ldots, n\}$
(a) $(\exists i)_{I}\left(x \in A_{i}\right)$
(b) $(\forall i)_{I}\left(x \in A_{i}\right)$

## Section 3.9

1. The given statement can be written in if $\ldots$, then $\ldots$ format as: If $x$ and $y$ are two odd positive integers, then $x+y$ is an even integer.

Proof: Assume $x$ and $y$ are two positive odd integers. It can be shown that $x+y=2 \cdot$ (some positive integer).
$x$ odd $\Rightarrow x=2 m+1$ for some $m \in \mathbb{P}$,
$y$ odd $\Rightarrow y=2 n+1$ for some $n \in \mathbb{P}$.
Therefore, $x+y=(2 m+1)+(2 n+1)=2((m+n)+1)=2$.(some positive integer) so $x+y$ is even.
3. Proof: (Indirect) Assume to the contrary, that $\sqrt{2}$ is a rational number. Then there exists $p, q \in \mathbb{Z},(q \neq 0)$ where $\frac{p}{q}=\sqrt{2}$ and where $\frac{p}{q}$ is in lowest terms, that is, $p$ and $q$ have no common factor other than 1.
$\frac{p}{q}=\sqrt{2} \Rightarrow \frac{p^{2}}{q^{2}}=2 \Rightarrow p^{2}=2 q^{2} \Rightarrow p^{2}$ is an even integer $\Rightarrow p$ is an even integer (see Exercise 2) 4 is a factor of $p^{2} \Rightarrow q^{2} \Rightarrow$ is even $\Rightarrow q$ is even. Hence both $p$ and $q$ have a common factor, namely 2 . Contradiction.
5. Proof: (Indirect) Assume $x, y \in \mathbb{R}$ and $x+y \leqslant 1$. Assume to the contrary that ( $x \leqslant \frac{1}{2}$ or $y \leqslant \frac{1}{2}$ ) is false, which is equivalent to $x>\frac{1}{2}$ and $y>\frac{1}{2}$. Hence $x+y>\frac{1}{2}+\frac{1}{2}=1$. This contradicts the assumption that $x+y \leqslant 1$.

## Supplementary Exercises-Chapter 3

1.(a) | $p$ | $p \bigvee p$ |
| :---: | :---: |
| 0 | 0 |
| 1 | 1 |

(b) | $p$ | $\neg p$ | $p \wedge \neg p$ |
| :---: | :---: | :---: |
| 0 | 1 | 0 |
| 1 | 0 | 0 |

(c) | $p$ | $\neg p$ | $p \bigvee \neg p$ |
| :---: | :---: | :---: |
| 0 | 1 | 1 |
| 1 | 0 | 1 |

(d) | $p$ | $p \wedge p$ |
| :---: | :---: |
| 0 | 0 |
| 1 | 1 |

3. 

| $p$ | $q$ | $\neg p$ | $q \wedge \neg p$ | $p \bigvee(q \wedge \neg p)$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 |

5. Let $a=p \rightarrow \neg q, b=q \bigvee r$, and $c=(p \rightarrow \neg q) \wedge(q \bigvee r) \wedge \neg r$

| $p$ | $q$ | $r$ | $\neg q$ | $a$ | $b$ | $a \bigwedge b$ | $\neg r$ | $c$ | $\neg p$ | $c \rightarrow \neg p$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |

7. An implication is always equivalent to its contrapositive, as can be seen in the table below.

| $p$ | $q$ | $p \Rightarrow q$ | $\neg q \Rightarrow \neg p$ | $(p \Rightarrow q) \leftrightarrow(\neg q \Rightarrow \neg p)$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

9. The truth tables of $p \rightarrow(p \wedge q)$ and $x$ must be equal.

| $p$ | $q$ | $p \rightarrow(p \wedge q)$ | $x$ |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

11. (a) " 3 is not a prime number or it is odd," or " 3 is a composite number or it is odd."
(b) "4 is not a prime number and it is even," or "4 is a composite number and it is even."
(c) I can exhibit an example of a statement and I cannot prove it.
(d) $x^{2}-7 x+12=0$ and $x \neq 3$ and $x \neq 8$
12. (a)

| $p$ | $q$ | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ | $(p \rightarrow q) \leftrightarrow(\neg q \rightarrow \neg p)$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

(b)

| $p$ | $q$ | $p \leftrightarrow q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \wedge(q \rightarrow p)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

Columns 3 and 6 are the same so $(p \leftrightarrow q) \Leftrightarrow[(p \rightarrow q) \wedge(q \rightarrow p)]$ is a tautology.
15. Let $q=$ "I quit my job," $s=$ "will starve," and $w=$ "I did my work." In
symbolic form, the argument is $(q \rightarrow s) \wedge(\neg w \rightarrow q) \wedge w \Rightarrow \neg s$. The truth
table for $x=(q \rightarrow s) \wedge(\neg w \rightarrow q) \wedge w \rightarrow \neg s$ does not consist of all ones.
For example, when $q$ is false, $s$ is true and when $w$ is true, $x$ is false. Therefore,
$x$ is not a tautology and the argument is not valid.
17. $[(m \rightarrow p) \wedge(e \vee \neg p) \wedge \neg e] \Rightarrow \neg m$. Valid

Proof: (Direct)
(1) $e \bigvee \neg p$
Premise
(2) $\neg e$
Premise
(3) $\neg p$
(1), (2), disjunctive simplification
(4) $m \rightarrow p$
Premise
(5) $\neg m$
(3), (4), indirect reasoning
19. $[(\neg p \rightarrow \neg q \wedge \neg r \wedge(p \rightarrow s) \wedge(q \bigvee r) \Rightarrow s . \quad$ Valid

Proof: (Direct)
(1) $\neg p \rightarrow \neg q$

Premise
(2) $q \rightarrow p$
(1), Contrapositive
(3) $p \rightarrow s$

Premise
(4) $q \rightarrow s$
(2), (3), Chain rule
(5) $q \bigvee r$

Premise
(6) $\neg r$

Premise
(7) $q$
(5), (6), Disjunctive simplification
(8) $s$
(4), (7), Detachment
(Indirect)
(1) $\neg s$

Negated conclusion
(2) $p \rightarrow s$

Premise
(3) $\neg p$
(1), (2), Indirect Reasoning
(4) $\neg p \rightarrow \neg q$

Premise
(5) $\neg q$ (3), (4), Detachment
(6) $q \bigvee r$

Premise
(7) $r$
(5), (6), Disjunctive simplification
(8) $\neg r$

Premise
(9) 0
(7), (8)
21. $e \rightarrow i, i \rightarrow d, d \rightarrow w \Rightarrow e \rightarrow w$

Proof:
(1) $e \rightarrow i \quad$ Premise
(2) $i \rightarrow d$

Premise
(3) $e \rightarrow d$
(1), (2), Chain rule
(4) $d \rightarrow w$

Premise
(5) $e \rightarrow w$
(3), (4), Chain rule
23. Valid. Statement: $t \bigvee d, \neg e \bigvee j$, $\neg j \bigvee r \Rightarrow \neg t \rightarrow r$

Proof: (direct)
(1) $t \bigvee d$
Premise
(2) $\neg t \rightarrow d$
(1), Conditional equivalence
(3) $\neg d \bigvee j$
Premise
(4) $d \rightarrow j$
(3), Conditional equivalence
(5) $\neg j \bigvee r$
Premise
(6) $j \rightarrow r$
(5), Conditional equivalence
(7) $\neg t \rightarrow r$
(2), (4), (6), Chain rule
25. (1) First show $T_{p \wedge q} \subseteq T_{p} \cap T_{q}$

$$
\begin{aligned}
a \in T_{p \wedge q} \quad & \Rightarrow a \text { makes } p \wedge q \text { true } \\
& \Rightarrow a \text { makes } p \text { true and } a \text { makes } q \text { true } \\
& \Rightarrow a \in T_{p} \text { and } a \in T_{q} \\
& \Rightarrow a \in T_{p} \cap T_{q}
\end{aligned}
$$

(2) To prove $T_{p} \cap T_{q} \subseteq T_{p \wedge q}$ reverse the above steps.
$27.60=6 \cdot 10=2 \cdot 3 \cdot 2 \cdot 5=2^{2} \cdot 3 \cdot 5$

$$
120=2 \cdot 60=2^{3} \cdot 3 \cdot 5
$$

29 (a) $\binom{n}{k-1}+\binom{n}{k}=\frac{n!}{(n-(k-1))!(k-1)!}+\frac{n!}{(n-k)!k!}$

$$
=\frac{n!}{(n-k+1)!(k-1)}+\frac{n!}{(n-k)!k!}
$$

$$
=\frac{n!k+n!(n-k+1)}{(n-k+1)!k!}
$$

$$
=\frac{n!(n+1)}{(n-k+1)!k!}=\frac{(n+1)!}{(n+1-k)!k!}
$$

$$
=\binom{n+1}{k}
$$

(b) Basis: $\quad(n=1):(x+y)^{1}=x+y$.

$$
\sum_{k=0}^{1}\binom{1}{k} x^{1-k} y^{k}=\binom{1}{0} x+\binom{1}{1} y=x+y
$$

Induction: Assume $n \geqslant 1$ and $(x+y)^{n}=\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k}$.
We will now prove $(x+y)^{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k} x^{n+1-k} y^{k}$.

$$
\begin{aligned}
(x+y)^{n+1} & =(x+y)(x+y)^{n} & \\
& =(x+y) \sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k} & \text { By induction hypothesis } \\
& =x \sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k}+y \sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k} & \text { distribution } \\
& =\sum_{k=0}^{n}\binom{n}{k} x^{n+1-k} y^{k}+\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k}+1 &
\end{aligned}
$$

Let $k=k-1$ in the second summation; remember to increase top.

$$
\begin{aligned}
& =\sum_{k=0}^{n}\binom{n}{k} x^{n+1-k} y^{k}+\sum_{k=1}^{n+1}\binom{n}{k-1} x^{n-k+1} y^{k} \\
& =\binom{n}{0} x^{n+1}+\sum_{k=1}^{n}\binom{n}{k} x^{n+1-k} y^{k}+\sum_{k=1}^{n}\binom{n}{k-1} x^{n+1-k} y^{k}+\binom{n}{n} y^{n+1} \\
& =\binom{n}{0} x^{n+1}+\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right] x^{n+1-k} y^{k}+\binom{n}{n} y^{n+1} \\
& =\binom{n}{0} x^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k} x^{n+1-k} y^{k}+\binom{n}{n} y^{n+1}, \\
& \text { But }\binom{n}{0}=\binom{n+1}{0}=1 \text { and }\binom{n}{n}=\binom{n+1}{n+1}=1 \\
& =\sum_{k=0}^{n+1}\binom{n+1}{0} x^{n+1-k} y^{k} .
\end{aligned}
$$

31. $\neg(\exists x)((\forall y)(D(x) \wedge T(y))) \Rightarrow(\forall x)(\neg(\forall y)(D(x) \wedge T(y)))$

$$
\begin{aligned}
& \Rightarrow(\forall x)((\exists y)(\neg(D(x) \wedge T(y)))) \\
& \Rightarrow(\forall x)((\exists y)(\neg D(x) \bigvee \neg T(y))) \\
& \Rightarrow \text { All sailing is not dangerous or some fishing is } \\
& \text { not tedious. }
\end{aligned}
$$

33. (a) Let $U$ be the universe of all fish, $k(x)=" x$ is kind to children," and $s(x)=" x$ is a shark." $(\forall x)_{U}(\neg s(x) \rightarrow k(x))$
(b) Let $w(x)=" x$ is a wine drinker,"

$$
c(x)=" x \text { is very communicative," }
$$

$$
p(x)=\text { " } x \text { is a pawnbroker," }
$$

and $\quad h(x)=" x$ is honest";
then $(\forall x)((\exists y)((w(x) \rightarrow c(x)) \bigvee(p(y) \rightarrow(h(x) \bigwedge \neg w(y)))))$
(c) Let $p(x)=" x$ is a clever philosopher,"
$c(x)=" x$ is a cynic,"
and $w(x)=$ " $x$ is a woman"; then
$(\forall x)((\exists y)(((p(x) \rightarrow c(x)) \wedge(w(x) \rightarrow p(x))) \rightarrow(p(y) \rightarrow(w(y) \rightarrow c(y)))))$
35. $(\forall a)_{\mathbb{R}}+(\forall b)_{\mathbb{R}}+(\exists n)_{\mathbb{P}}(n a>b)$

## CHAPTER 4

## Section 4.1

1. (a) Assume that $x \in A$ (condition of the conditional conclusion $A \subseteq C$ ). Since $A \subseteq B, x \in B$ by the definition of $\subseteq$. $B \subseteq C$ and $x \in B$ implies that $x \in C$ Therefore, if $x \in A$, then $x \in C$.
(b) (Proof that $A-B \subseteq A \bigcap B^{c}$ ) Let $x$ be in $A-B$. Therefore, x is in $A$, but it is not in B ; that is, $x \in A$ and $\quad x \in B^{c} \Rightarrow x \in A \cap B^{c}$.
(c) ( $\Rightarrow$ ) Assume that $A \subseteq B$ and $A \subseteq C$. Let $x \in A$. By the two premises, $x \in B$ and $x \in C$. Therefore, by the definition of intersection, $x \in B \bigcap C$.
(d) $(\Rightarrow)$ (Indirect) Assume that $A \subseteq C$ and $B^{c}$ is not a subset of $A^{c}$. Therefore, there exists $x \in B^{c}$ that does not belong to $A^{c}$. $x \notin A^{c} \Rightarrow x \in A$. Therefore, $x \in A$ and $x \notin B$, a contradiction to the assumption that $A \subseteq B$.
2. (a) If $A=\mathbb{Z}$ and $B=\emptyset, A-B=\mathbb{Z}$, while $B-A=\varnothing$.
(b) If $A=\{0\}$ and $B=\{1\},(0,1) \in A \times B$, but $(0,1)$ is not in $B \times A$.
(c) Let $A=\emptyset, B=\{0\}$, and $C=\{1\}$.
3. Proof: Let $p(n)$ be

$$
\begin{aligned}
A \cap\left(B_{1} \cup B_{2} \cup\right. & \left.\cdots \cup B_{n}\right) \\
& =\left(A \bigcap B_{1}\right) \cup\left(A \cap B_{2}\right) \cup \cdots \cup\left(A \cap B_{n}\right) .
\end{aligned}
$$

Basis: We must show that $p(2): A \bigcap\left(B_{1} \cup B_{2}\right)=\left(A \bigcap B_{1}\right) \cup\left(A \cap B_{2}\right)$ is true. This was done by several methods in section 4.1 . Induction: Assume for some $n \geq 2$ that $p(n)$ is true. Then

$$
\begin{aligned}
& A \cap\left(B_{1} \cup B_{2} \cup \cdots \cup B_{n+1}\right)=A \cap\left[\left(B_{1} \cup B_{2} \cup \cdots \cup B_{n}\right) \cup B_{n+1}\right] \\
&=\left(A \bigcap\left(B_{1} \cup B_{2} \cup \cdots \cup B_{n}\right)\right) \cup\left(A \bigcap B_{n+1}\right) \\
& \quad\left(\left(A \cap B_{1}\right) \cup \cdots \cup\left(A \cap B_{n}\right)\right) \cup\left(A \cap B_{n+1}\right) \quad \text { by } p(2) \\
&=\left(A \cap B_{1}\right) \cup \cdots \cup\left(A \cap B_{n}\right) \cup\left(A \bigcap B_{n+1}\right) \quad \text { by the induction hypothesis }
\end{aligned}
$$

## Section 4.2

1. (a)


A H


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$\mathrm{H}-\mathrm{F}$

(b) $\begin{array}{ccccccc}A & B & A^{c} & B^{c} & A \cup B & (A \cup B)^{c} \quad A^{c} \cap B^{c}\end{array}$

| 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 |

The last two columns are the same so the two sets must be equal.
(c) (i) $x \in A \cup A \Rightarrow x \in A$ or $x \in A$ by the definition of $\cap$

$$
\Rightarrow x \in A \text { by the idempotent law of logic }
$$

Therefore, $A \cup A \subseteq A$.
(ii) $x \in A \Rightarrow x \in A$ and $x \in A$ by conjunctive addition

$$
\Rightarrow x \in A \bigcup A
$$

Therefore, $A \subseteq A \cup A$ and so we have $A \cup A=A$.
3. For all parts of this exercise, a reason should be supplied for each step. We have supplied reasons for part a only and left them out of the other parts to give you further practice.
(a) $A \cup(B-A)=A \bigcup\left(B \cap A^{c}\right)$

$$
\begin{aligned}
& =(A \cup B) \cap\left(A \cup A^{c}\right) \\
& =(A \bigcup B) \cap U \\
& =(A \bigcup B) \quad \text { by }
\end{aligned}
$$

by Exercise 2b of Section 4.1
by the distributive law
by the null law
by the identity law
(b) $A-B=A \bigcap B^{c}$

$$
\begin{aligned}
& =B^{c} \bigcap A \\
& =B^{c} \cap\left(A^{c}\right)^{c} \\
& =B^{c}-A^{c}
\end{aligned}
$$

(c) Select any element, $x$, in $A \cap C$. One such element exists since $A \cap C$ is not empty.

$$
\begin{aligned}
& x \in A \cap C \Rightarrow x \in A \text { and } x \in C \\
& \Rightarrow x \in B \text { and } x \in C
\end{aligned}
$$

$\Rightarrow x \in B \cap C$
$\Rightarrow B \cap C \neq \emptyset$
(d) $A \cap(B-C)=A \cap\left(B \cap C^{c}\right)$

$$
=\left(A \cap B \cap A^{c}\right) \cup\left(A \cap B \cap C^{c}\right)
$$

$$
=(A \cap B) \cap\left(A^{c} \cup C^{c}\right)
$$

$$
=(A \cap B) \cap(A \cup C)^{c}
$$

$$
=(A-B) \cap(A-C)
$$

(e) $A-(B \cup C)=A \cap(B \cup C)^{c}$

$$
\begin{aligned}
& =A \cap\left(B^{c} \cap C^{c}\right) \\
& =\left(A \cap B^{c}\right) \cap\left(A \cap C^{c}\right) \\
& =(A-B) \cap(A-C)
\end{aligned}
$$

$3 \quad 12$
5. (a) $A \cup B^{c} \cap C$
(b) $A \cap B \cup C \cap B$
(c) $A \cup B \cup C^{c}$

## Section 4.3

1. (a) $\{1\},\{2,3,4,5\},\{6\},\{7,8\},\{9,10\}$
(b) $2^{5}$, as compared with $2^{10} .\{1,2\}$ is one of the 992 sets that can't be generated.
2. $B_{1}=\{00,01,10,11\}$ and $B_{2}=\{0,00,01\}$ generate minsets $\{00,01\},\{0\},\{10,11\}$, and $\{\lambda, 1\}$. Note: $\lambda$ is the null string, which has length zero.
3. (a) $B_{1} \cap B_{2}=\varnothing$

$$
\begin{aligned}
& B_{1} \cap B^{c}{ }_{2}=\{0,2,4\} \\
& B^{c}{ }_{1} \cap B_{2}=\{1,5\} \\
& B_{1}^{c} \cap B^{c}{ }_{2}=\{3\}
\end{aligned}
$$

(b) $2^{3}$, since there are 3 nonempty minsets.
7. Let $a \in A$. For each $i, a \in B_{i}$, or $a \in B_{i}^{c}$, since $B_{i} \cup B_{i}{ }^{c}=A$ by the complement law. Let $D_{i}=B_{i}$ if $a \in B_{i}$, and $D=B_{i}{ }^{c}$ otherwise. Since $a$ is in each $D_{i}$, it must be in the minset $D_{1} \cap D_{2} \cdots \cap D_{n}$. Now consider two different minsets $M_{1}=D_{1} \cap D_{2} \cdots \cap D_{n}$, and $M_{2}=G_{1} \cap G_{2} \cdots \cap G_{n}$, where each $D_{i}$ and $G_{i}$ is either $B_{i}$ or $B_{i}{ }^{c}$. Since these minsets are not equal, $D_{i} \neq G_{i}$, for some $i$. Therefore, $M_{1} \cap M_{2}=D_{1} \cap D_{2} \cdots \cap D_{n} \cap G_{1} \cap G_{2} \cdots \cap G_{n}=\phi$, since two of the sets in the intersection are disjoint. Since every element of A is in a minset and the minsets are disjoint, the nonempty minsets must form a partition of A.

## Section 4.4

1. (a) $A \cap(B \cup A)=A$
(b) $A \cap\left(\left(B^{c} \cap A\right) \cup B\right)^{c}=\varnothing$
(c) $\left(A \cap B^{c}\right)^{c} \cup B=A^{c} \cup B$
2. (a) $(p \wedge \neg(\neg q \wedge p) \bigvee g)) \Leftrightarrow 0$
(b) $(\neg(p \vee(\neg q)) \wedge q) \Leftrightarrow((\neg p) \wedge q)$
3. The maxsets are:

$$
B_{1} \cup B_{2}=\{1,2,3,5\}
$$

$$
\begin{aligned}
& B_{1} \cup B_{2}^{c}=\{1,3,4,5,6\} \\
& B_{1}^{c} \cup B_{2}=\{1,2,3,4,6\} \\
& B_{1}^{c} \cup B_{2}^{c}=\{2,4,5,6\}
\end{aligned}
$$

They do not form a partition of A since it is not true that the intersection of any two of them is empty. A set is said to be in maxset normal form when it is expressed as the intersection of distinct nonempty maxsets or it is the universal set $U$.

## Supplementary Exercises-Chapter 4

1. (a) Proof: $(\Rightarrow)$ (Indirect) Assume $A \subseteq B$ and $A \cup(U-B) \neq \emptyset$. To prove that this cannot occur, let $x \in A \cap(U-B)$.

$$
\begin{array}{ll}
x \in A \cap(U-B) & \\
\text { Definition of complement } & \Rightarrow x \in A \cap B^{c} \\
\text { Definition of } \bigcap & \Rightarrow x \in A \text { and } x \in B^{c} \\
\text { Definition of complement } & \Rightarrow x \in A \text { and } x \notin B \\
\text { Definition of subset } & \Rightarrow A \text { is not a subset of } B
\end{array}
$$

This contradicts the premise that $A \subseteq B$. Hence this part of the statement is proven.
$(\Leftarrow)$ (Indirect) Assume $A \bigcap(U-B)=\emptyset$, and $A$ is not a subset of $B$. To prove that this cannot occur, let $x \in A$ such that $x \notin B$.

$$
\begin{array}{lc}
x \in A \text { and } x \notin B & \\
\text { Definition of complement } & \Rightarrow x \in A \text { and } x \in B^{c} \\
\text { Definition of } \cap & \Rightarrow x \in A \cap B^{c} \\
\text { Definition of complement } & \Rightarrow x \in A \cap(U-B) \\
\text { Definition of disjoint } \Rightarrow A \cap(U-B) \neq \emptyset
\end{array}
$$

But this cannot happen because it contradicts the assumption that $A \bigcap(U-B)=\phi$. Hence this part of the statement is proven and the proof is complete.
(b) Proof: (Indirect) Assume $U=A \cup B, A \cap B=\emptyset$, and $A \neq U-B$. One way in which $A$ and $U-B$ can be not
equal is that $A$ is not a subset of $U-B$. Let $x \in A$ and $x \notin U-B$.

$$
\begin{array}{ll}
x \in A \text { and } x \notin U-B & \\
\text { Definition of complement } & \Rightarrow x \in A \text { and } x \in B \\
\text { Definition of } \cap & \Rightarrow x \in A \cap B \\
\text { Definition of disjoint } & \Rightarrow A \cap B \neq \emptyset
\end{array}
$$

But this cannot happen because it contradicts the assumption that $A \cap B=\emptyset$. The other way $A$ and $U-B$ can diffe is if $U-B$ is not a subset of $A$, Let $x \notin A$ and $x \in U-B$. We could infer from this assumption thatx $x \notin A \cup B$. Therefore, any way that we assume that $A \neq U-B$ leads to a contradiction.
(c) Proof: $(\Rightarrow)$ (Direct) Let $x \in A$.
$A$ and $B$ are disjoint
Definition of disjoint $\quad \Rightarrow x \notin B$
Definition of complement $\quad \Rightarrow x \in B^{c}$
Therefore, $A \subseteq B^{c}$
$(\Leftarrow)$ (Indirect) Assume that $A \subseteq B^{c}$ and $x \in A \bigcap B$.
$x \in A \bigcap B$
Definition of intersection $\quad \Rightarrow x \in A$ and $x \in B$
Definition of complement $\quad \Rightarrow x \in A$ and $x \notin B^{c}$
Definition of subset $\quad \Rightarrow A$ is not a subset of $B^{c}$
3. (a) Proof: (Direct) Let $A, B$, and $C$ be sets.

Let $(x, y) \in(A \cup B) \times C$.

$$
\begin{array}{ll}
\text { Definition of Cartesian product } & \Rightarrow x \in(A \cup B) \text { and } y \in C \\
\text { Definition of } \cup & \Rightarrow(x \in A \text { or } x \in B) \text { and } y \in C \\
\text { Distributive law of logic } & \Rightarrow(x \in A \text { and } y \in C) \text { or }(x \in B \text { and } y \in C) \\
\text { Definition of Cartesian product } & \Rightarrow((x, y) \in A \times C) \text { or } \\
& ((x, y) \in B \times C) \\
\text { Definition of } \cup & \Rightarrow(x, y) \in(A \times C) \cup(B \times C) \square
\end{array}
$$

(b) We proved $(A \cup B) \times C \subseteq(A \times C) \cup(B \times C)$ in part a; we now must show $(A \times C) \cup(B \times C) \subseteq(A \bigcup B) \times C$ and we will be finished.
5. Proof: (Indirect) Assume $A, B$, and $C$ are subsets of $U, A \subseteq B, B \subseteq C$ and $C^{c}$ is not a subset of $A^{c}$. To prove that this cannot occur, let $x \in C^{c}$ and $x \notin A^{c}$ by definition of subset.

| $x \in C^{c}$ and $x \notin A^{c}$ |  |
| :--- | :--- |
| Definition of complement | $\Rightarrow x \notin C$ and $x \in A$ |
| Premise | $\Rightarrow A \subseteq B$ |
| Definition of subset | $\Rightarrow x \in B$ |
| Premise | $\Rightarrow B \subseteq C$ |
| Definition of subset | $\Rightarrow x \in C$ (Contradiction) |

7. (a) Proof: (Indirect) Let $A, B$, and $C$ be sets. Assume $A \cup C \neq B \cup C$ and $A=B$.

$$
\begin{array}{ll}
A=B \Rightarrow A \subseteq B & \\
x \in A \cup C & \\
\text { Definition of union } & \Rightarrow x \in A \text { or } x \in C \\
\text { Definition of subset } & \Rightarrow x \in B \text { or } x \in C \\
\text { Definition of union } & \Rightarrow x \in B \cup C
\end{array}
$$

Therefore, $A \cup C \subseteq B \bigcup C$. By a similar line of reasoning we can infer $B \bigcup C \subseteq A \cup C$, which proves that $A \cup C=B \cup C$, a contradiction.
(b) Proof: (Direct) Assume $A \neq B$ and show $A^{c} \neq B^{c}$. Since $A \neq B$ we can assume that $A$ is not a subset of $B$. The alternative is that $B$ is not a subset of $A$ and the remaining logic would be identical.

| $A$ not a subset of $B$ |  |
| :--- | :--- |
| Definition of subset | $\Rightarrow x \in A$ and $x \notin B$ |
| Definition of complement | $\Rightarrow x \notin A^{c}$ and $x \in B^{c}$ |
| Definition of subset | $\Rightarrow B^{c}$ is not a subset of $A^{c}$ |
| Definition of inequality | $\Rightarrow A^{c} \neq B^{c} \quad \square$ |

9. (a) The minsets are $B_{1} \cap B_{2}=\{3\}, B_{1}{ }^{c} \cap B_{2}=\{2,5\}, B_{1} \cap B_{2}{ }^{c}=\{1\}$, and $B_{1}{ }^{c} \cap B_{2}{ }^{c}=\{4,6\}$
(b) The minsets are disjoint and
$\left(B_{1} \cap B_{2}\right) \cup\left(B_{1}{ }^{c} \cap B_{2}\right) \cup\left(B_{1} \cap B_{2}{ }^{c}\right) \cup\left(B_{1}{ }^{c} \cap B_{2}{ }^{c}\right)=U$,
so the minsets form a partition of U .

## CHAPTER 5

## Sections 5.1-5.3

1. For parts $\mathrm{c}, \mathrm{d}$ and i of this exercise, only a verification is needed. Here, we supply the result that will appear on both sides of the equality.
(a) $\quad A B=\left(\begin{array}{cc}-3 & 6 \\ 9 & -13\end{array}\right) \quad B A=\left(\begin{array}{cc}2 & 3 \\ -7 & -18\end{array}\right)$
(b) $\left(\begin{array}{cc}1 & 0 \\ 5 & -2\end{array}\right)$
(c) $\left(\begin{array}{cc}3 & 0 \\ 15 & -6\end{array}\right)$
(d) $\left(\begin{array}{ccc}18 & -15 & 15 \\ -39 & 35 & -35\end{array}\right)$
(e) $\left(\begin{array}{ccc}-12 & 5 & -5 \\ 5 & -25 & 25\end{array}\right)$
(f) $B+0=B$
(g) $\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$
(h) $\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$
(i) $\left(\begin{array}{cc}5 & -5 \\ 10 & 15\end{array}\right)$
2. $\left(\begin{array}{cc}1 / 2 & 0 \\ 0 & 1 / 3\end{array}\right)$
3. $A^{3}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27\end{array}\right) \quad A^{15}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 32768 & 0 \\ 0 & 0 & 14348907\end{array}\right)$
4. (a) $A x=\binom{2 x_{1}+1 x_{2}}{1 x_{1}-1 x_{2}}$ equals $\binom{3}{1}$ if and only if both of the equalities

$$
2 x_{1}+x_{2}=3 \text { and } x_{1}-x_{2}=1 \text { are true. }
$$

(b) (i) $\quad A=\left(\begin{array}{cc}2 & -1 \\ 1 & 1\end{array}\right) \quad x=\binom{x_{1}}{x_{2}} \quad B=\binom{4}{0}$
(ii) $\quad A=\left(\begin{array}{ccc}1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1\end{array}\right) x=\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right) \quad B=\left(\begin{array}{c}1 \\ -1 \\ 5\end{array}\right)$
(iii) $\quad A=\left(\begin{array}{lll}1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3\end{array}\right) \quad x=\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right) \quad B=\left(\begin{array}{l}3 \\ 5 \\ 6\end{array}\right)$

## Section 5.4

1. (a) $\left(\begin{array}{cc}-1 / 5 & 3 / 5 \\ 2 / 5 & -1 / 5\end{array}\right)$ (b) $\left(\begin{array}{ll}1 & 3 \\ 0 & 1\end{array}\right)$ (c) No inverse exists.
(d) $A^{-1}=A \quad$ (e) $\left(\begin{array}{ccc}1 / 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 / 5\end{array}\right)$
2. Let A and B be $n$ by $n$ invertible matrices. Prove $(A B)^{-1}=B^{-1} A^{-1}$.

$$
\text { Proof: } \begin{aligned}
\left(B^{-1} A^{-1}\right)(A B) & =\left(B^{-1}\right)\left(A^{-1}(A B)\right) \\
& \left.=\left(B^{-1}\right)\left(\left(A^{-1} A\right) B\right)\right) \\
& =\left(B^{-1}\right)(I B) \\
& =\left(B^{-1}\right)(B) \\
& =I
\end{aligned}
$$

Similarly, $(A B)\left(B^{-1} A^{-1}\right)=I$.
By Theorem 5.4.1, $B^{-1} A^{-1}$ is the only inverse of $A B$, If we tried to invert $A B$ with $A^{-1} B^{-1}$, we would be unsuccessful since we cannot rearrange the order of the matrices.
5. (b) $1=\operatorname{det} I=\operatorname{det}\left(A A^{-1}\right)=\operatorname{det} A \operatorname{det} A^{-1}$. Now solve for $\operatorname{det} A^{-1}$.
7. Basis: $(n=1): \operatorname{det} A^{1}=\operatorname{det} A=(\operatorname{det} A)^{1}$.

Induction: Assume $\operatorname{det} A^{n}=(\operatorname{det} A)^{n}$ for some $n \geq 1$.

$$
\begin{aligned}
\operatorname{det} A^{n+1}= & \operatorname{det}\left(A^{n} A\right) & \text { by the definition of exponents } \\
& =\operatorname{det}\left(A^{n}\right) \operatorname{det}(A) & \text { by exercise } 5 \\
& =(\operatorname{det} A)^{n}(\operatorname{det} A) & \text { by the induction hypothesis } \\
& =(\operatorname{det} A)^{n+1} &
\end{aligned}
$$

9. (a) Assume $A=B D B^{-1}$

Basis: $(m=1)$ : $A^{\wedge} 1=B D^{1} B^{-1}$ is given.
Induction: Assume that for some positive integer $m, A^{m}=B D^{m} B^{-1}$

$$
\begin{aligned}
A^{m+1} & =A^{m} A \\
& =\left(\mathrm{BD}^{m} B^{-1}\right)\left(\mathrm{BDB}^{-1}\right) \text { by the induction hypothesis } \\
& =B D^{m} D B^{-1} \quad \text { by associativity, definition of inverse } \\
& =B D^{m+1} B^{-1} \quad
\end{aligned}
$$

(b) $A^{10}=B D^{10} B^{-1}=\left(\begin{array}{ll}-9206 & 15345 \\ -6138 & 10231\end{array}\right)$

## Section 5.5

1. (1) Let $A$ and $B$ be $m$ by $n$ matrices. Then $A+B=B+A$,
(2) Let $A, B$, and $C$ be $m$ by $n$ matrices. Then $A+(B+C)=(A+B)+C$.
(3) Let $A$ and $B$ be $m$ by $n$ matrices, and let $c \in \mathbb{R}$. Then $c(A+B)=c A+c B$,
(4) Let $A$ be an $m$ by $n$ matrix, and let $c_{1}, c_{2} \in \mathbb{R}$. Then ( $c_{1}+c_{2}$ ) $A=c_{1} A+c_{2} A$.
(5) Let $A$ be an $m$ by $n$ matrix, and let $c_{1}, c_{2} \in \mathbb{R}$. Then $c_{1}\left(c_{2} A\right)=\left(c_{1} c_{2}\right) A$
(6) Let $\mathbf{0}$ be the zero matrix, of size $m$ by $n$, and let $A$ be a matrix of size $n$ by $r$. Then $\mathbf{0} A=\mathbf{0}=$ the $m$ by $r$ zero matrix.
(7) Let $A$ be an $m$ by $n$ matrix, and $0=$ the number zero. Then $0 A=0=$ the $m$ by $n$ zero matrix.
(8) Let $A$ be an $m$ by $n$ matrix, and let $\mathbf{0}$ be the $m$ by $n$ zero matrix. Then $A+\mathbf{0}=A$.
(9) Let $A$ be an $m$ by $n$ matrix. Then $A+(-1) A=\mathbf{0}$, where $\mathbf{0}$ is the $m$ by $n$ zero matrix.
(10) Let $A, B$, and $C$ be $m$ by $n, n$ by $r$, and $n$ by $r$ matrices respectively. Then $A(B+C)=A B+A C$.
(11) Let $A, B$, and $C$ be $m$ by $n, r$ by $m$, and $r$ by $m$ matrices respectively. Then $(B+C) A=\mathrm{BA}+\mathrm{CA}$.
(12) Let $A, B$, and $C$ be $m$ by $n, n$ by $r$, and $r$ by $p$ matrices respectively. Then $A(B C)=(A B) C$.
(13) Let $A$ be an $m$ by $n$ matrix, $I_{m}$ the $m$ by $m$ identity matrix, and $I_{n}$ the $n$ by $n$ identity matrix. Then $I_{m} A=A I_{n}=A$
(14) Let $A$ be an $n$ by $n$ matrix. Then if $A^{-1}$ exists, $\left(A^{-1}\right)^{-1}=A$.
(15) Let $A$ and $B$ be $n$ by $n$ matrices. Then if $A^{-1}$ and $B^{-1}$ exist, $(A B)^{-1}=B^{-1} A^{-1}$.
2. (a) $\mathrm{AB}+\mathrm{AC}=\left(\begin{array}{ccc}21 & 5 & 22 \\ -9 & 0 & -6\end{array}\right)$
(b) $A(B+C)=A B+A C$
(c) $A^{-1}=\left(\begin{array}{cc}1 & 2 \\ 0 & -1\end{array}\right)=A$
(d) $\left(A^{2}\right)^{-1}=(A A)^{-1}=\left(A A^{-1}\right)^{-1}=I^{-1}=I$ by part c

## Section 5.6

1. In elementary algebra (the algebra of real numbers), each of the given oddities does not exist.
(i) $A B$ may be different from $B A$. Not so in elementary algebra, since $a b=b a$ by the commutative law of multiplication.
(ii) There exist matrices $A$ and $B$ such that $A B=\mathbf{0}$, yet $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$. In elementary algebra, the only way $a b=0$ is if either $a$ or $b$ is zero. There are no exceptions.
(iii) There exist matrices $A, A \neq \mathbf{0}$, yet $A^{2}=\mathbf{0}$. In elementary algebra, $a^{2}=0 \Leftrightarrow a=0$.
(iv) There exist matrices $A^{2}=A$. where $A \neq \mathbf{0}$ and $A \neq I$. In elementary algebra, $a^{2}=a \Leftrightarrow a=0$ or 1 .
(v) There exist matrices $A$ where $A^{2}=I$ but $A \neq I$ and $A \neq-I$. In elementary algebra, $a^{2}=1 \Leftrightarrow a=1$ or -1 .
2. (a) $\operatorname{det} A \neq 0 \Rightarrow A^{-1}$ exists, and if you multiply the equation $A^{2}=A$ on both sides by $A^{-1}$, you obtain $A=I$.
(b) Counterexample: $A=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$
3. (a) $A^{-1}=\left(\begin{array}{cc}1 / 3 & 1 / 3 \\ 1 / 3 & -2 / 3\end{array}\right) \quad x_{1}=4 / 3$, and $x_{2}=1 / 3$
(b) $A^{-1}=\left(\begin{array}{ll}1 & -1 \\ 1 & -2\end{array}\right) \quad x_{1}=4$, and $x_{2}=4$
(c) $A^{-1}=\left(\begin{array}{cc}1 / 3 & 1 / 3 \\ 1 / 3 & -2 / 3\end{array}\right) \quad x_{1}=2 / 3$, and $x_{2}=-1 / 3$
(d) $A^{-1}=\left(\begin{array}{cc}1 / 3 & 1 / 3 \\ 1 / 3 & -2 / 3\end{array}\right) \quad x_{1}=0$, and $x_{2}=1$
(e) The matrix of coefficients for this system has a zero determinant; therefore, it has no inverse. The system cannot be solved by this method. In fact, the system has no solution.

## Supplementary Exercises-Chapter 5

1. $\left(\begin{array}{cc}x+y & 5 \\ -2 & x-y\end{array}\right)=\left(\begin{array}{cc}3 & 5 \\ -2 & 4\end{array}\right) \Rightarrow\left\{\begin{array}{l}x+y=3 \\ x-y=4\end{array} \Rightarrow\left\{\begin{array}{l}y=-1 / 2 \\ x=7 / 2\end{array}\right.\right.$
2. For $n \geq 1$ let $p(n)$ be $A B^{n}=B^{n} A$

Basis: $(n=1): \mathrm{AB}^{1}=B^{1} A$ is true as given in the statement of the problem. Therefore, $p(1)$ is true.
Induction: Assume $n \geq 1$ and $p(n)$ is true.

$$
\begin{aligned}
A B^{n+1} & =\left(A B^{n}\right) B & & \\
& =\left(B^{n} A\right) B & & \text { By the induction hypothesis } \\
& =\left(B^{n} B\right) A & & \text { By } p(l) \\
& =B^{n+1} A & & \square
\end{aligned}
$$

5. $A^{-1} A^{3}=A^{2}=\left(\begin{array}{ll}7 & 18 \\ 6 & 19\end{array}\right)$
6. $D$ has no inverse if $\operatorname{det} D=0$.

$$
\operatorname{det} D=0 \Leftrightarrow 3 c-f(15)=3 c-60=0 \Leftrightarrow c=20
$$

9. (a) $(A+B)^{2}=A^{2}+A B+B A+B^{2}$
(b) $(A+B)^{2}=A^{2}+2 A B+B^{2}$ only if $A B=B A$.
10. The implication is false. Both $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ and $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ are self-inverting, but their product is not.
11. Yes, matrices of the form $A=\left(\begin{array}{cc}a & b \\ (1-a)^{2} / b & -a\end{array}\right)$ also solve $A^{2}=I$

## CHAPTER 6

## Section 6.1

1. (a) $(2,4),(2,8)$
(b) $(2,3),(2,4),(5,8)$
(c) $(1,1),(2,4)$
2. (a) $r=\{(1,2),(2,3),(3,4),(4,5)\}$
(b) $r^{2}=\{(1,3),(2,4),(3,5)\}=\{(x, y): y=x+2, x, y \in A\}$
(c) $r^{3}=\{(1,4),(2,5)\}=\{(x, y): y=x+3, x, y \in A\}$
3. (a) When $n=3$, there are 27 pairs in the relation.
(b) Imagine building a pair of disjoint subsets of $S$. For each element of $S$ there are three places that it can go: into the first set of the ordered pair, into the second set, or into neither set. Therefore the number of pairs in the relation is $3^{n}$, by the product rule.

## Section 6.2

1. 


3. See Figure 13.1.1 of Section 13.1.
5. A Hasse diagram cannot be used because not every set is related to itself. Also, $\{a\}$ and $\{b\}$ are related in both directions.

## Section 6.3



(c) The graphs are the same if we disregard the names of the vertices.
3. (a) (i)
reflexive (ii)
not symmetric
not antisymmetric
transitive
(iv) not reflexive (v) symmetric antisymmetric transitive
reflexive (iii)
not symmetric
antisymmetric
transitive not reflexive symmetric not antisymmetric transitive transitive
(iv) symmetric not symmetric not antisymmetric antisymmetric transitive transitive
(vii) not reflexive
not symmetric not antisymmetric not transitive
(b) Graphs ii and vi show partial ordering relations. Graph $v$ is of an equivalence relation.
5. (a) No, since for example $|1-1|=0 \neq 2$
(b) Yes, since $|i-j|=|j-i|$
(c) No, since $|2-4|=2$ and $|4-6|=2$, but $|2-6|=4 \neq 2$.
(d)

7. (b) $c(0)=\{0\}, c(1)=\{1,2,3\}=c(2)=c(3)$
(c) $c(0) \cup c(1)=A$ and $c(0) \cap c(1)=\varnothing$
(d) Let $A$ be any set and let $r$ be an equivalence relation on $A$. Let $a$ be any element of $A$. $a \in c(a)$ since $r$ is reflexive, so each element of $A$ is in some equivalence class. Therefore, the union of all equivalence classes equals $A$. Next we show that any two equivalence classes are either identical or disjoint and we are done. Let $c(a)$ and $c(b)$ be two equivalence classes, and assume that $c(a) \cap c(b) \neq \emptyset$. We want to show that $c(a)=c(b)$. To show that $c(a) \subseteq c(b)$, let $x \in c(a) . x \in c(a) \Rightarrow a r x$. Also, there exists an element, $y$, of $A$ that is in the intersection of $c(a)$ and $c(b)$ by our assumption. Therefore,

$$
\begin{aligned}
\text { ary and } b r y & \Rightarrow a r y \text { and } y r b(r \text { is symmetric }) \\
& \Rightarrow a r b \text { (transitivity of } r)
\end{aligned}
$$

Next,

$$
a r x \text { and } a r b \Rightarrow x r a \text { and } a r b
$$

$$
\begin{aligned}
& \Rightarrow x r b \\
& \Rightarrow b r x \\
& \Rightarrow x \in c(b)
\end{aligned}
$$

Similarly, $c(b) \subseteq c(a)$.
9. (a) Equivalence Relation
$c(0)=\{0\}, c(1)=\{1\}, c(2)=\{2,3\}=c(3), c(4)=\{4,5\}=c(5)$,
$c(6)=\{6,7\}=c(7)$
(b) Not an Equivalence Relation
(c) Equivalence Relation
$c(0)=\{0,2,4,6\}=c(2)=c(4)=c(6)$
$c(1)=\{1,3,5,7\}=c(3)=c(5)=c(7)$
11. (b) The proof follows from the biconditional equivalence in Table 3.4.2.
(c) Apply the chain rule.
(d)


## Section 6.4

1. (a)

| $\square$ | 4 |
| :--- | :---: |
| 1 |  |
| 2 |  |
| 3 |  |
| 4 |  |\(\left(\begin{array}{lll}0 \& 0 \& 0 <br>

1 \& 0 \& 0 <br>
0 \& 1 \& 0 <br>
0 \& 0 \& 1\end{array}\right)\)

and | $\square$ | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: |
| 4 |  |  |  |
| 5 |  |  |  |
| 6 |  |  |  |\(\left(\begin{array}{ccc}0 \& 0 \& 0 <br>

1 \& 0 \& 0 <br>
0 \& 1 \& 0\end{array}\right)\)
(b)

| $\square$ | 6 | 7 | 8 |
| :--- | :--- | :--- | :--- |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |\(\left(\begin{array}{lll}0 \& 0 \& 0 <br>

0 \& 0 \& 0 <br>
1 \& 0 \& 0 <br>
0 \& 1 \& 0\end{array}\right)\)
3. R: xry if and only if $|x-y|=1$.
$\mathrm{S}: x s y$ if and only if $x$ is less than $y$.
5. The diagonal entries of the matrix for such a relation must be 1 . When the three entries above the diagonal are determined, the entries below are also determined. Therefore, the answer is $2^{3}$.
7. (a)
$\square$
1
2
3
4 \(\left(\begin{array}{cccc}1 \& 2 \& 3 \& 4 <br>
0 \& 1 \& 0 \& 0 <br>
1 \& 0 \& 1 \& 0 <br>
0 \& 1 \& 0 \& 1 <br>

0 \& 0 \& 1 \& 0\end{array}\right) \quad\) and $\quad$| $\square$ |
| :--- |
| 1 |
| 2 |
| 3 |
| 4 |\(\left(\begin{array}{cccc}1 \& 2 \& 3 \& 4 <br>

1 \& 0 \& 1 \& 0 <br>
0 \& 1 \& 0 \& 1 <br>
1 \& 0 \& 1 \& 0 <br>
0 \& 1 \& 0 \& 1\end{array}\right)\)
(b)

$$
\begin{array}{r}
\mathrm{PQ}=\begin{array}{c}
\square \\
1 \\
2 \\
3 \\
4
\end{array}\left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right) \\
\\
\square \\
1 \\
P^{2}=\begin{array}{llll}
0 \\
2 \\
3 \\
4
\end{array}\left(\begin{array}{llll}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right)=Q^{2}
\end{array}
$$

9. (a) Reflexive: $R_{i j}=R_{i j}$ for all $i, j$, therefore $R_{i j} \leq R_{i j}$

Antisymmetric: Assume $R_{i j} \leq S_{i j}$ and $S_{i j} \leq R_{i j}$ for all $1 \leq i, j \leq n \Rightarrow R_{i j} \leq S_{i j}$
Transitive: Assume $R, S$, and $T$ are matrices where $R_{i j} \leq S_{i j}$ and $S_{i j} \leq T_{i j}$, for all $1 \leq i, j \leq n$. Then $R_{i j} \leq T_{i j}$ for all $1 \leq i$, $j \leq n$, and so $R \leq T$.
(b) $\quad\left(R^{2}\right)_{i j}=R_{i 1} R_{1 j}+R_{i 2} R_{2 j}+\cdots+R_{i n} R_{n j}$

$$
\leq S_{i l} S_{1 j}+S_{i 2} S_{2 j}+\cdots+S_{i n} S_{n j}=\left(S^{2}\right)_{i j} \Rightarrow R^{2} \leq S^{2}
$$

To verify that the converse is not true we need only one example. For $n=2$, let $R_{12}=1$ and all other entries equal 0 , and let $S$ be the zero matrix. Since $R^{2}$ and $S^{2}$ are both the zero matrix, $R^{2} \leq S^{2}$, but since $R_{12}>S_{12}, R \leq S$ is false.
(c) The matrices are defined on the same set $A=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$. Let $c\left(a_{i}\right), i=1,2, \ldots, n$ be the equivalence classes defined by $R$ and let $d\left(a_{i}\right)$ be those defined by $S$. Claim: $c\left(a_{i}\right) \subseteq d\left(a_{i}\right)$. Let $a_{j} \in c\left(a_{i}\right) \Rightarrow a_{i} r a_{j} \Rightarrow R_{i j}=1 \Rightarrow S_{i j}=1 \Rightarrow a_{i} s a_{j} \Rightarrow a_{j} \in d\left(a_{i}\right)$.

## Section 6.5

3. (a)

(b) Example, 1 s 4 and using $S$ one can go from 1 to 4 using a path of length 3 .
4. (a) Definition: Reflexive Closure. Let $r$ be a relation on $A$. A reflexive closure of $r$ is the smallest reflexive relation that contains $r$. Theorem: The reflexive closure of $r$ is the union of $r$ with $\{(x, x): x \in A\}$
5. (a) By the definition of transitive closure, $r^{+}$is the smallest relation which contains $r$; therefore, it is transitive. The transitive closure of $r^{+}$, $\left(r^{+}\right)^{+}$, is the smallest transitive relation that contains $r^{+}$. Since $r^{+}$is transitive, $\left(r^{+}\right)^{+}=r^{+}$.
(b) The transitive closure of a symmetric relation is symmetric, but it may not be reflexive. If one element is not related to any elements, then the transitive closure will not relate that element to others.

## Supplementary Exercises-Chapter 6

1. If Andy is the parent of Barbara and Barbara is the parent of Charles, then Andy is the grandparent of Charles.
2. (a) $r=\{(-1,0),(0,1),(1,2)\}$
(b) $\quad s=\{(-1,-1),(-1,1),(0,0),(1,-1),(1,1),(2,2)\}$
(c) $t=\{(-1,0),(-1,1),(-1,2),(0,-1),(0,1),(0,2)$, $(1,-1),(1,0),(1,2),(2,-1),(2,0),(2,1)\}$
3. His main office should be at node 2 . The least desirable location is at node 1 . The arrows in both directions between nodes 1 and 2 represent a two-way street.
4. (a) No.
(b) Person $a$ is friendly toward the most people so he/she would be chair person.
(c) If "great personality" has any effect then person $b$ becomes chairperson.
(d) A seating arrangement does not exist, since persons $c$ and $d$ are only friendly toward one person each and they have to be seated between two people they are friendly toward.
5. In order for the relation "living in the same house" to be an equivalence relation we must assume that a person lives in only one house.
11.(a) $r$ is an equivalence relation.
(b) $s$ is neither since $s$ is not reflexive.
(c) In order for $s$ to be a partial ordering we rephrase it slightly; xsy iff $x$ taller than $y$ or $x$ equals $y$. Why would $x$ sy iff $x$ is the same height as or taller than $y$ be wrong?
6. There are 16 places in the adjacency matrix for a relation on four elements, but for a symmetric relation those entries below the diagonal will be the same as above. Hence we are only concerned with $16-6=10$ places. Each of the remaining entries may take on a value of either 0 or 1 , so by the rule of products we have $2^{10}$ possible symmetric relations on a four element set.
7. 
8. (a)

(b) (i) $\{(a, a),(a, b),(b, a),(b, b),(c, c)\}$
(ii) $\{(a, a),(a, c),(c, a),(c, b),(c, c)\}$
(c)
(i) $R^{2}=R=\left(\begin{array}{lll}1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$
(ii) $R^{2}=\left(\begin{array}{lll}1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1\end{array}\right)$
9. 

(a)

(b)

(b) $r$ is not reflexive, not symmetric, not antisymmetric, and not transitive.
(c) $R^{+}=\left(\begin{array}{llll}1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1\end{array}\right)$
19. (a) $A_{5}$ is friendly to no one.
(b) The U.S. Ambassador $\left(A_{1}\right)$ should be the chairman of this committee, since he is friendly toward the most people.
(c) The U.S. Ambassador can communicate to everyone on the committee.

## CHAPTER 7

## Section 7.1

1. 

(a) Yes
(b) Yes
(c) No
(d) No
(e) Yes
3. (a) Range of $f=f(A)=\{a, b, c, d\}=B$
(b) Range of $g=g(A)=\{a, b, d\}$
(c) Range of $L=L(A)=\{1\}$
5. For each of the $|A|$ elements of $A$, there are $|B|$ possible images, so there are $|B| \cdot|B| \cdot \ldots \cdot|B|=|B|^{|A|}$ functions from $A$ into $B$.

## Section 7.2

1. The only one-to-one function and the only onto function is $f$.
2. (a) onto but not one-to-one $\left(f_{1}(0)=f_{1}(1)\right)$
(b) one-to-one and onto
(c) one-to-one but not onto
(d) onto but not one-to-one
(e) one-to-one but not onto
(f) one-to-one but not onto
3. Let $X=\{$ socks selected $\}$ and $Y=\{$ pairs of socks $\}$ and define $f: X \rightarrow Y$ where $f(x)=$ the pair of socks that $x$ belongs to. By the Pigeonhole principle, there exist two socks that were selected from the same pair.
4. (a) $f(n)=n$, for example
(b) $f(n)=1$, for example
(c) None exist.
(d) None exist.
5. (a) Use $s: \mathbb{N} \rightarrow \mathbb{P}$ defined by $s(x)=x+1$.
(b) Use the function $f: \mathbb{N} \rightarrow \mathbb{Z}$ defined by $f(\mathrm{x} 0=x / 2$ if $x$ is even and $f(x)=-(x+1) / 2$ if $x$ is odd.
(c)The proof is due to Georg Cantor (1845-1918), and involves listing the rationals through a definite procedure so that none are omitted and duplications are avoided. In the first row list all nonnegative rationals with denominator 1 , in the second all nonnegative rationals with denominator 2 , etc. In this listing, of course, there are duplications, for example, $0 / 1=0 / 2=0,1 / 1=3 / 3=1,6 / 4=9 / 6=3 / 2$, etc. To obtain a list without duplications follow the arrows in the given array listing only the circled numbers.


We obtain: $0,1,1 / 2,2,3,1 / 3,1 / 4,2 / 3,3 / 2,4 / 1, \ldots$ Each nonnegative rational appears in this list exactly once. We now must insert in this list the negative rationals, and follow the same scheme to obtain: $0,1,-1,1 / 2,-1 / 2,2,-2,3,-3,1 / 3,-1 / 3, \ldots$, which can be paired off with the elements of $\mathbb{N}$.
11. Let $f$ be any function from $A$ into $B$. By the Pigeonhole principle with $n=1$, there exists an element of $B$ that is the image of at least two elements of $A$. Therefore, $f$ is not an injection.
13. The proof is indirect and follows a technique called the Cantor diagonal process. Assume to the contrary that the set is countable, then the elements can be listed:

$$
n_{1}, n_{2}, n_{3}, \ldots \text { where each } n_{i} \text { is an infinite sequence of } 0 \mathrm{~s} \text { and } 1 \mathrm{~s} \text {. Consider the array: }
$$

$$
\begin{gathered}
n_{1}=n_{11} n_{12} \\
n_{13} \cdots \\
n_{2}=n_{21} \\
n_{22}
\end{gathered} n_{23} \cdots,
$$

We assume that this array contains all infinite sequences of 0 s and 1 s . Consider the sequence $s$ defined by

$$
s_{i}= \begin{cases}0 & \text { if } n_{\mathrm{ii}}=1 \\ 1 & \text { if } n_{\mathrm{ii}}=0\end{cases}
$$

$s$ differs from each $n_{i}$ in the $i$ th position and so cannot be in the list. This is a contradiction, which completes our proof.

## Section 7.3

1. (a) $g \circ f: A \rightarrow C$ is defined by $(g \circ f)(k)= \begin{cases}+ & \text { if } k=1 \text { or } k=5 \\ - & \text { if } k=2,3,4\end{cases}$
(b) No, since the domain of $f$ is not equal to the codomain of $g$.
(c) No, since $f$ is not surjective.
(d) No, since $g$ is not injective.
2. (a) The permutations of $A$ are $i, r_{1}, r_{2}, f_{1}, f_{2}$, and $f_{3}$, defined in section 15.3
(b, c)

| Permutation | Inverse of <br> the permutation | Square of <br> permutation |
| :--- | :---: | :---: |
| $i$ | $i$ | $i$ |
| $r_{1}$ | $r_{2}$ | $r_{2}$ |
| $r_{2}$ | $r_{1}$ | $r_{1}$ |
| $f_{1}$ | $f_{1}$ | $i$ |
| $f_{2}$ | $f_{2}$ | $i$ |
| $f_{3}$ | $f_{3}$ | $i$ |

(d) Apply both Theorems 7.3.3 and 7.3.4: If $f$ and $g$ are permutations of $A$, then they are both injections and their composition, $f \circ g$, is a injection, by Theorem 7.3.3. By 7.3.4, $f \circ g$ is also a surjection; therefore, $f \circ g$ is a bijection on $A$, a permutation.
(e) Proof by induction:

Basis: $(n=1)$ The number of permutations of $A$ is one, the identity function, and $1!=1$.
Induction: Assume that the number of permutations on a set with $n$ elements, $n \geq 1$, is $n$ !. Furthermore, assume that $|A|=n+1$ and that $A$ contains an element called $x$. Let $A^{\prime}=A-\{x\}$. We can reduce the definition of a permutation, $f$, on $A$ to two steps. First, we select any one of the $n$ ! permutations on $A^{\prime}$. (Note the use of the induction hypothesis.) Call it $g$. This permutation almost completely defines a permutation on $A$ by $f(a)=g(a)$ for all $a$ in $A^{\prime}$, Next, we select the image of $x$, which can be done $n+1$ different ways. To keep our function bijective, we must adjust $f$ as follows: If we select $f(x)=y$, then we must find the element, $z$, of $A$ such that $g(z)=y$, and redefine the image of $z$ to $f(z)=x$. If we had selected $f(x)=x$, then there is really no adjustment needed. By the rule of products, the number of ways that we can define $f$ is $n!(n+1)=(n+1)$ !
7. (a) $f \circ g(n)=n+3$
(b) $f^{3}(n)=n+15$
(c) $f \circ h(n)=n^{2}+5$
9. Theorem: If $f: A \rightarrow B$ and $f$ has an inverse, then that inverse is unique.

Proof: Suppose that $g$ and $h$ are both inverses of $f$.

$$
\begin{aligned}
g & =g \circ i_{A} g \\
& =g \circ(f \circ h) \\
& =(g \circ f) \circ h \\
& =i_{A} \circ h \\
& =h
\end{aligned}
$$

11. Proof of Theorem 7.3.2: Let $x, x^{\prime}$ be elements of $A$ such that $g \circ f(x)=g \circ f\left(x^{\prime}\right)$; that is, $g(f(x))=g\left(f\left(x^{\prime}\right)\right)$. Since $g$ is injective, $f(x)=f\left(x^{\prime}\right)$ and since $f$ is injective, $x=x^{\prime}$.
Proof of Theorem 7.3.3: Let $x$ be an element of $C$. We must show that there exists an element of $A$ whose image under $g \circ f$ is $x$. Since $g$ is surjective, there exists an element of $B, y$, such that $g(y)=x$. Also, since $f$ is a surjection, there exists an element of $A, z$, such that $f(z)=y$, $g \circ f(z)=g(f(z))=g(y)=x$.
12. Basis: $(n=2):\left(f_{1} \circ f_{2}\right)^{-1}=f_{2}^{-1} \circ f_{1}^{-2}$ by exercise 10 .

Induction: Assume $n \geq 2$ and $\left(f_{1} \circ f_{2} \circ \cdots \circ f_{n}\right)^{-1}=$
$f_{n}{ }^{-1} \circ \cdots \circ f_{2}^{-1} \circ f_{1}{ }^{-1}$
Consider $\left(f_{1} \circ f_{2} \circ \cdots \circ f_{n+1}\right)^{-1}$.

$$
\begin{array}{ll}
\left(f_{1} \circ f_{2} \circ \cdots \circ f_{n+1}\right)^{-1} & =\left(\left(f_{1} \circ f_{2} \circ \cdots \circ f_{n}\right) \circ f_{n+1}\right)^{-1} \\
\text { by the Basis } & =f_{n+1}^{-1} \circ\left(f_{1} \circ f_{2} \circ \cdots \circ f_{n}\right)^{-1} \\
\text { by Induction hypothesis } & =f_{n+1}{ }^{-1} \circ\left(f_{n}^{-1} \circ \cdots \circ f_{2}^{-1} \circ f_{1}^{-1}\right)
\end{array}
$$

$$
=f_{n+1}{ }^{-1} \circ \cdots \circ f_{2}^{-1} \circ f_{1}^{-1} .
$$

15. Assume all functions are functions on $A$.

$$
\begin{aligned}
& (f \circ g) \circ h=f \circ(g \circ h) \\
& f \circ i_{A}=i_{A} \circ f=f \\
& \text { If } f^{-1} \text { and } g^{-1} \text { exist, }(g \circ f)^{-1}=f^{-1} \circ g^{-1} \text { and } \\
& \text { If } f^{-1} \text { exists, }\left(f^{-1}\right)^{-1}=f .
\end{aligned}
$$

## Supplementary Exercises-Chapter 7

1. (a) $\mathbb{Z}$
(b) $\mathbb{Z}$
(c) $f(-5)=2|5|+1=11$
(d) $\{1,3,5,7,9, \ldots\}=$ the set of odd positive integers
(e) No, $a=5$ or $a=-5$.
2. No. Relations (iii) and (iv) are not functions because the domain is not all of the set $A$. The others are not functions since in each case at least one element of $A$ is mapped to 2 different elements. Example for relation (i), $a$ is mapped to both $a$ and $b$.
3. (a) The matrix of $f$ can only have one 1 in each row. So if the domain of $f$ has $n$ elements the matrix of $f$ will have $n 1 \mathrm{~s}$.
(b) If $f$ is a bijection, besides having only one 1 in each row, there can only be one 1 in each column.
4. (a) Let $f(n)=n^{2}, \forall \mathrm{n} \in \mathbb{N}$. Since $f$ is a bijection from $\mathbb{N}$ into $A=\left\{n^{2} \mid n \in \mathbb{N}\right\}, \mathbb{N}$ and $A$ have the same cardinality; so $A$ is countable.
(b) Let $B=\{1 / n \mid n \in \mathbb{P}\} . g: \mathbb{N} \rightarrow B$ defined by $g(n)=1 /(n+1)$ is the required bijection.
(c) That $C=C_{1} \cup C_{2}=\{3,9,27,81, \ldots\} \cup\{2,4,8,16, \ldots\}$ is countable follows from the proof of Exercise 8 . Without using this proof, we can still prove that $C$ is countable by using the list $2^{1}, 3^{1}, 2^{2}, 3^{2}, 2^{3}, 3^{3}$ to define $h: \mathbb{N} \rightarrow C$ where $h(a)=$ the number in position $a+1$ in the list, 9. $f: A \times B \rightarrow B \times A$ defined by $f(a, b)=(b, a)$ is a bijection, which is all that we need to prove that
$|A \times B|=|B \times A|$
5. This "code" can be viewed as a function, $a$, on the set of all finite sequences of letters. For example, $a($ hat $)=q m h$. This encoding function will not work very well because it is not a bijection. For example, no sequence with $a$ or $t$ in it is in the range. Although $a$ is not one-to-one, it is difficult to find two English words with the same image.
6. (a) $10(a+10)$
(b) $a+20$
(c) $10 a \operatorname{div} 10=a$
(d) $(a+10) \operatorname{div} 10=a \operatorname{div} 10+1$
7. (a) $f(b)=b$ and $f(c)=c$
(b) $f(b)=a$ and $f(c)=d$
8. Since $\operatorname{det}\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)=a d-b c$ there are four permutations of $\{a, b, c, d\}$ that leave the determinant invariant. These permutations are the identity function, $\alpha_{1}=\{(a, d),(b, b),(c, c),(d, a)\}, \alpha_{2}=\{(a, d),(b, c),(c, b),(d, a)\}$, and $\alpha_{3}=\{(a, a),(b, c),(c, b),(d, d)\}$.
9. $($ a) Domain $=$ positive real numbers, Codomain $=$ Real numbers.

## CHAPTER 8

## Section 8.1

```
1. \(C(5,2)=C(4,2)+C(4,1)\)
    \(=C(3,2)+C(3,1)+C(3,1)+C(3,0)\)
    \(=C(3,2)+2 C(3,1)+1\)
    \(=C(2,2)+C(2,1)+2(C(2,1)+C(2,0))+1\)
    \(=3 C(2,1)+4\)
    \(=6+4=10\)
```

3. (a) $p(x)$ in telescoping form: $((((x+3) x-15) x+0) x+1) x-10$
(b) $p(3)=((((3+3) 3-15) 3-0) 3+1) 3-10=74$
4. The basis is not reached in a finite number of steps if you try to compute $f(x)$ for a nonzero value of $x$.

## Section 8.2

1. Basis: $B(0)=3 \cdot 0+2=2$, as defined

Induction: Assume: $B(k)=3 k+2$ for some $k \geq 0$.

$$
\begin{aligned}
B(k+1)= & B(k)+3 \\
& =(3 k+2)+3 \quad \text { by the induction hypothesis } \\
& =(3 k+3)+2 \\
& =3(k+1)+2, \quad \text { as desired. }
\end{aligned}
$$

3. Imagine drawing line $k$ in one of the infinite regions that it passes through. That infinite region is divided into two infinite regions by line $k$. As line $k$ is drawn through every one of the $k-1$ previous lines, you enter another region that line $k$ divides. Therefore, the number of regions is increased by $k$.
4. For $n$ greater than zero, $M(n)=M(n-1)+1$, and $M(0)=0$.

## Section 8.3

1. $S(k)=2+9^{k}$
2. $S(k)=6(1 / 4)^{k}$
3. $S(k)=k^{2}-10 k+25$
4. $S(k)=(3+k) 5^{k}$
5. $S(k)=(12+3 k)+\left(k^{2}+7 k-22\right) 2^{k-1}$
6. $P(k)=4(-3)^{k}+2^{k}-5^{k+1}$
7. (a) The characteristic equation is a $a^{2}-a-1=0$, which has solutions $\alpha=(1+\sqrt{5}) / 2$ and $\beta=(1-\sqrt{5}) / 2$, It is useful to point out that $\alpha+\beta=1$ and $\alpha-\beta=\sqrt{5}$. The general solution is

$$
F(k)=b_{1} \alpha^{k}+b_{2} \beta^{k} .
$$

Using the initial conditions, we obtain the system: $b_{1}+b_{2}=1$ and $b_{1} \alpha+b_{2} \beta=1$. The solution to this system is

$$
\begin{array}{ll} 
& b_{1}=\alpha /(\alpha-\beta)=(5+\sqrt{5}) / 2 \sqrt{5} \\
\text { and } & b_{2}=\beta /(\alpha-\beta)=(5-\sqrt{5}) / 2 \sqrt{5}
\end{array}
$$

Therefore the final solution is

$$
F(n)=(1 / \sqrt{5})\left[((1+\sqrt{5}) / 2)^{n+1}-((1-\sqrt{5}) / 2)^{n+1}\right]
$$

(b) $C_{r}=F(r+1)$
15. (a) $D(n)=2 D(n-1)+1$ for $n \geq 2$ and $D(1)=0$.
(b) $D(n)=2^{n-1}-1$
17. Solutions to the recurrence relation and its approximation are $B(k)=(1+c)^{k}+(1-c)^{k}$ and $B_{a}(k)=1$. Note how as $k$ increases, $B(k)$ grows in size, while $B_{a}(k)$ stays constant.

## Section 8.4

1. (a) $S(n)=1 / n$ !
(c) $U(k)=1 / k$, an improvement.
(b) $T(k)=(-3)^{k} k!$, no improvement.
2. (a) $T(n)=3\left(\left\lfloor\log _{2} n\right\rfloor+1\right)$ (c) $V(n)=\left\lfloor\log _{8} n\right\rfloor+1$
(b) $T(n)=2$
3. The indicated substitution yields $S(n)=S(n+1)$. Since $S(0)=T(1) / T(0)=6, S(n)=6$ for all $n$. Therefore $T(n+1)=6 T(n) \Rightarrow T(n)=6^{n}$.
4. (a) A good approximation to the solution of this recurrence relation is based on the following observation: $n$ is a power of a power of two; that is, $n$ is $2^{m}$, where $m=2^{k}$, then $Q(n)=1+Q\left(2^{m / 2}\right)$. By applying this recurrence relation $k$ times we obtain $Q(n)=k$. Going back to the original form of $n, \log _{2} n=2^{k}$ or $\log _{2}\left(\log _{2} n\right)=k$. We would expect that in general, $Q(n)$ is $\left\lfloor\log _{2}\left(\log _{2} n\right)\right\rfloor$. We do not see any elementary
method for arriving at an exact solution.
(b) Suppose that $n$ is a positive integer with $2^{k-1} \leq n<2^{k}$. Then $n$ can be written in binary form, $\left(a_{k-1} a_{k-2} \cdots a_{2} a_{1} a_{0}\right)_{\text {two }}$ with $a_{k-1}=1$ and $R(n)$ is equal to the sum

$$
\sum_{i=0}^{k-1}\left(a_{k-1} a_{k-2} \cdots a_{i}\right)_{\mathrm{two}}
$$

If $2^{k-1} \leq n<2^{k}$, then we can estimate this sum to be between $2 n-1$ and $2 n+1$. Therefore, $R(n) \approx 2 n$.

## Section 8.5

1. (a) $1,0,0,0,0, \ldots$
(b) $5(1 / 2)^{k}$
(c) $1,1,0,0,0, \ldots$
(d) $3(-2)^{k}+3 \cdot 3^{k}$
2. (a) $1 /(1-9 z)$
(b) $(2-10 z) /\left(1-6 z+5 z^{2}\right)$
(c) $1 /\left(1-z-z^{2}\right)$
3. (a) $3 /(1-2 z)+2 /(1+2 z), 3 \cdot 2^{k}+2(-2)^{k}$
(b) $10 /(1-z)+12 /(2-z), 10+6(1 / 2)^{k}$
(c) $-1 /(1-5 z)+7 /(1-6 z), 7 \cdot 6^{k}-5^{k}$
4. (a) $11 k$
(b) $(5 / 3) k(k+1)(2 k+1)+5 k(k+1)$
(c) $\sum_{j=0}^{k}(j)(10(k-j))=10 k \sum_{j=0}^{k} j-10 \sum_{j=0}^{k} j^{2}$

$$
\begin{aligned}
& =5 k^{2}(k+1)-(5 k(k+1)(2 k+1) / 6) \\
& =(5 / 3) k(k+1)(2 k+1)
\end{aligned}
$$

(d) $k(k+1)(2 k+7) / 12$
9. Coefficients of $z^{0}$ through $z^{5}$ in $(1+5 z)(2+4 z)(3+3 z)(4+2 z)(5+z)$

| $k$ | Number of ways of getting $a$ score of $k$ |
| :---: | :---: |
| 0 | 120 |
| 1 | 1044 |
| 2 | 2724 |
| 3 | 2724 |
| 4 | 1044 |
| 5 | 120 |

## Supplementary Exercises-Chapter 8

1. Let $v(n)$ be the quantity in question. Since any positive digit can appear in a one-digit positive integer, $v(1)=9$. Given an $n$ digit number, $n \geq 2$, it can be thought of as an $n-1$ digit number times ten plus a digit. This digit cannot be the same as the units digit of the $n-1$ digit number. Therefore, by the product rule $v(n)=9 v(n-1)$ for $n \geq 2$.
2. (a) To execute Split with $L$ in $=(1,2,3,4)$, we must split the list into $L 1=(1,3)$ and $L 2=(2,4)$. If you carefully examine the algorithm for a list of length 2 , you will see that the output equals the input; therefore $L 1$ out $=(1,3)$ and $L 2$ out $=(2,4)$ and $L$ out $=(1,3,2,4)$.
(b) Examine the results for $r=1,2,3$ with numbers in binary form. Notice the symmetry with respect to the vertical line.

| $r=1$ | $L$ in | $L \text { out }$ | $r=3$ | $L$ in 000 | $\begin{gathered} L \text { out } \\ 000 \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 0 |  | 001 | 100 |
|  | 1 | 1 |  | 010 | 010 |
| $r=2$ |  |  |  | 011 | 110 |
|  | $L$ in | Lout |  | 100 | 001 |
|  | 00 | 00 |  | 101 | 101 |
|  | 01 | 10 |  | 110 | 011 |
|  | 10 | 01 |  | 111 | 111 |
|  | 11 | 11 |  |  |  |

The integers in $L$ out are sorted so that $\left(b_{r-1} b_{r-2} \cdots b_{0}\right)_{\mathrm{two}}$ appears in position $\left(b_{0} b_{1} \cdots b_{r-1}\right)_{\mathrm{two}}$.
5. This is not a closed form expression because the number of operations that are needed to compute the expression grows with $n, B$ ( $n$ ) in this form requires $n$ additions and $n-1$ multiplications.
7. Kathryn's balance on her first birthday is $\$ 1=B(1)$. If $B(n)$ is her balance on her $n$th birthday, $n \geq 2$, then $B(n)=1.1 B(n-1)+n$.

$$
\begin{aligned}
& B(n)=B^{(n)}(n)+B^{(p)}(n)=b_{1}(1.1)^{n}-(10 n+110) \\
& B(1)=1 \Rightarrow(1.1) b_{1}=121 \Rightarrow b_{1}=1.1
\end{aligned}
$$

Therefore $B(n)=121(1.1)^{n-1}-(10 n+110)$. On her 21st birthday, Kathryn will have $B(21)=121(1.1)^{20}-(210+110)=\$ 494.03$.
9. (a) If it takes $X(n)$ moves to move $n$ disks to peg 2 , then we can transfer the $n+1$ disk to peg 3 in one move and then transfer the $n$ disks from peg 2 to peg 3 in $X(n)$ moves, so $X(n+1)=X(n)+1+X(n)=2 X(n)+1$, or equivalently $X(n)=2 X(n-1)+1$.
(b) $X(n)=b_{1} \cdot 2^{n}-1$. Since it takes 1 move to transfer 1 disk from one peg to another, $X(1)=1$; so $b_{1}=1$ and $X(n)=2^{n}-1$. We verify that $X(3)=7:$

11. The solution for $n=4^{k}$ is $Q\left(4^{k}\right)=\frac{1}{3}\left(4^{k+1}-1\right)$, This can be obtained in one of two ways. Either use the substitution $S(k)=Q\left(4^{k}\right)$, which yields $S(k)=4^{k}+S(k-1)$, or note that $Q\left(4^{k}\right)=4^{k}+Q\left(4^{k-1}\right)=4^{k}+4^{k-1}+Q\left(4^{k-2}\right)=4^{k}+4^{k-1}+\cdots+4+1$. This finite geometric series has the closed form expression above. By similar means, $Q\left(2 \cdot 4^{k}\right)=2 Q\left(4^{k}\right)=\frac{2}{3}\left(4^{k+1}-1\right)$
13. $G(S ; z)=1+z+2 z^{2}+4 z^{3}+8 z^{4} \ldots$
15. $G(T ; z)=G(S ; c z)=\sum_{k=0}^{\infty} S(k)(c z)^{k}=$

$$
=\sum_{k=0}^{\infty}\left(S(k) c^{k}\right) z^{k}
$$

Therefore, $T(k)=S(k) c^{k}$.

