

The Cubic and Quartic Equations

■ The Cubic Equation

```
Clear[b, c, d, e, x]
```

```
dcubic = Expand[(x^3 + b x^2 + c x + d) /. {x -> y - b/3}]
```

$$\frac{2b^3}{27} - \frac{yb^2}{3} - \frac{cb}{3} + y^3 + d + cy$$

```
CoefficientList[dcubic, y]
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$$\left\{ \frac{2b^3}{27} - \frac{cb}{3} + d, c - \frac{b^2}{3}, 0, 1 \right\}$$

```
Clear[γ, δ, η, y]
```

```
Solve[y^3 + γ y + δ == 0, y]
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$$\left\{ y \rightarrow \frac{\sqrt[3]{\sqrt{108\gamma^3 + 729\delta^2} - 27\delta}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}\gamma}{\sqrt[3]{\sqrt{108\gamma^3 + 729\delta^2} - 27\delta}} \right\},$$

$$\left\{ y \rightarrow \frac{(1+i\sqrt{3})\gamma}{2^{2/3}\sqrt[3]{\sqrt{108\gamma^3 + 729\delta^2} - 27\delta}} - \frac{(1-i\sqrt{3})\sqrt[3]{\sqrt{108\gamma^3 + 729\delta^2} - 27\delta}}{6\sqrt[3]{2}} \right\},$$

$$\left\{ y \rightarrow \frac{(1-i\sqrt{3})\gamma}{2^{2/3}\sqrt[3]{\sqrt{108\gamma^3 + 729\delta^2} - 27\delta}} - \frac{(1+i\sqrt{3})\sqrt[3]{\sqrt{108\gamma^3 + 729\delta^2} - 27\delta}}{6\sqrt[3]{2}} \right\}$$

Solve[$x^3 + b x^2 + c x + d == 0$, x]

$$\left\{ \left\{ x \rightarrow -\frac{b}{3} + \frac{\sqrt[3]{-2b^3 + 9cb - 27d + \sqrt{4(3c - b^2)^3 + (-2b^3 + 9cb - 27d)^2}}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}(3c - b^2)}{3\sqrt[3]{-2b^3 + 9cb - 27d + \sqrt{4(3c - b^2)^3 + (-2b^3 + 9cb - 27d)^2}}} \right\}, \right.$$

$$\left\{ x \rightarrow -\frac{b}{3} - \frac{(1 - i\sqrt{3})\sqrt[3]{-2b^3 + 9cb - 27d + \sqrt{4(3c - b^2)^3 + (-2b^3 + 9cb - 27d)^2}}}{6\sqrt[3]{2}} + \frac{(1 + i\sqrt{3})(3c - b^2)}{3 \cdot 2^{2/3} \sqrt[3]{-2b^3 + 9cb - 27d + \sqrt{4(3c - b^2)^3 + (-2b^3 + 9cb - 27d)^2}}} \right\},$$

$$\left\{ x \rightarrow -\frac{b}{3} - \frac{(1 + i\sqrt{3})\sqrt[3]{-2b^3 + 9cb - 27d + \sqrt{4(3c - b^2)^3 + (-2b^3 + 9cb - 27d)^2}}}{6\sqrt[3]{2}} + \frac{(1 - i\sqrt{3})(3c - b^2)}{3 \cdot 2^{2/3} \sqrt[3]{-2b^3 + 9cb - 27d + \sqrt{4(3c - b^2)^3 + (-2b^3 + 9cb - 27d)^2}}} \right\} \left. \right\}$$

■ Quartic

dquartic = Expand[($x^4 + b x^3 + c x^2 + d x + e$) /. { $x \rightarrow y - \frac{b}{4}$ }]

$$-\frac{3b^4}{256} + \frac{yb^3}{8} - \frac{3y^2b^2}{8} + \frac{cb^2}{16} - \frac{db}{4} - \frac{c y b}{2} + y^4 + c y^2 + e + d y$$

CoefficientList[dquartic, y]

$$\left\{ -\frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e, \frac{b^3}{8} - \frac{cb}{2} + d, c - \frac{3b^2}{8}, 0, 1 \right\}$$

Solve[$y^4 + \gamma y^2 + \delta y + \eta = 0, y$]

$$\left\{ \left\{ y \rightarrow \frac{1}{2} \sqrt{\left(-\frac{2\gamma}{3} + \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right. \right.$$

$$\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} \right) - \right.$$

$$\left. \frac{1}{2} \sqrt{\left(-\frac{4\gamma}{3} - \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right.$$

$$\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} \right) - \right.$$

$$\left. (2\delta) / \left(\left(-\frac{2\gamma}{3} + \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right. \right. \right.$$

$$\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} \right) \right) \right\},$$

$$\left\{ y \rightarrow \frac{1}{2} \sqrt{\left(-\frac{2\gamma}{3} + \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right.$$

$$\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} \right) + \right.$$

$$\left. \frac{1}{2} \sqrt{\left(-\frac{4\gamma}{3} - \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right.$$

$$\begin{aligned}
& \frac{\sqrt[3]{2} (\gamma^2 + 12 \eta)}{3 \sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}} - \\
(2 \delta) / & \left(\sqrt[3]{\left(-\frac{2 \gamma}{3} + \frac{\sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}}{3 \sqrt[3]{2}} \right)} + \right. \\
& \left. \frac{\sqrt[3]{2} (\gamma^2 + 12 \eta)}{3 \sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}} \right) \Bigg) \Bigg) \Bigg) \\
\{y \rightarrow -\frac{1}{2} & \sqrt[3]{\left(-\frac{2 \gamma}{3} + \frac{\sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}}{3 \sqrt[3]{2}} \right)} + \right. \\
& \left. \frac{\sqrt[3]{2} (\gamma^2 + 12 \eta)}{3 \sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}} \right) \Bigg) - \\
\frac{1}{2} & \sqrt[3]{\left(-\frac{4 \gamma}{3} - \frac{\sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}}{3 \sqrt[3]{2}} \right)} - \\
& \frac{\sqrt[3]{2} (\gamma^2 + 12 \eta)}{3 \sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}} + \\
(2 \delta) / & \left(\sqrt[3]{\left(-\frac{2 \gamma}{3} + \frac{\sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}}{3 \sqrt[3]{2}} \right)} + \right. \\
& \left. \frac{\sqrt[3]{2} (\gamma^2 + 12 \eta)}{3 \sqrt[3]{2 \gamma^3 - 72 \eta \gamma + 27 \delta^2 + \sqrt{(2 \gamma^3 - 72 \eta \gamma + 27 \delta^2)^2 - 4 (\gamma^2 + 12 \eta)^3}}} \right) \Bigg) \Bigg) \Bigg)
\end{aligned}$$

$$\left\{ y \rightarrow \frac{1}{2} \sqrt{\left(-\frac{4\gamma}{3} - \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right. \\
\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} + \right. \right. \\
(2\delta) / \left(\sqrt{\left(-\frac{2\gamma}{3} + \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right. \\
\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} \right) \right) \left. \right. \\
\left. \left. \frac{1}{2} \sqrt{\left(-\frac{2\gamma}{3} + \frac{\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}}{3\sqrt[3]{2}} \right.} \right. \right. \\
\left. \left. \frac{\sqrt[3]{2}(\gamma^2 + 12\eta)}{3\sqrt[3]{2\gamma^3 - 72\eta\gamma + 27\delta^2 + \sqrt{(2\gamma^3 - 72\eta\gamma + 27\delta^2)^2 - 4(\gamma^2 + 12\eta)^3}}} \right) \right) \left. \right\}$$

Solve[$x^4 + bx^3 + cx^2 + dx + e = 0$, x]

$$\left\{ x \rightarrow -\frac{b}{4} - \frac{1}{2} \sqrt{\left(\frac{b^2}{4} - \frac{2c}{3} + \frac{1}{3\sqrt[3]{2}} \right.} \right. \\
\left. \left. \left((2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3}) \right)^{1/3} \right. \right. \\
\left. \left. + \frac{\sqrt[3]{2}(c^2 - 3bd + 12e)}{3(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3})^{1/3}} \right) \right. \\
\left. \left. \frac{1}{2} \sqrt{\left(\frac{b^2}{2} - \frac{4c}{3} - \frac{1}{3\sqrt[3]{2}} \left((2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3}) \right)^{1/3} \right.} \right. \right. \\
\left. \left. \left(\sqrt[3]{2}(c^2 - 3bd + 12e) \right) / \left(3(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3}) \right)^{1/3} \right) \right. \right. \\
\left. \left. (-b^3 + 4cb - 8d) / \left(4 \sqrt{\left(\frac{b^2}{4} - \frac{2c}{3} + \frac{1}{3\sqrt[3]{2}} \left((2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3}) \right)^{1/3} \right.} \right. \right. \right.$$

$$\begin{aligned}
& \left. \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} \right] \right] \right] \right], \\
\{x \rightarrow & -\frac{b}{4} + \frac{1}{2} \sqrt{\left(\frac{b^2}{4} - \frac{2c}{3} + \frac{1}{3\sqrt[3]{2}} \left((2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \right.\right. \right. \\
& \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} + \right.\right. \\
& \left. \left. \left. (\sqrt[3]{2} (c^2 - 3bd + 12e)) / \left(3(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \right.\right. \right. \right. \\
& \left. \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} \right) \right) \right] + \\
\frac{1}{2} \sqrt{\left(\frac{b^2}{2} - \frac{4c}{3} - \frac{1}{3\sqrt[3]{2}} \left((2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \right.\right. \right. \\
& \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} \right) - \right. \\
& \left. \left. \left. (\sqrt[3]{2} (c^2 - 3bd + 12e)) / \left(3(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \right.\right. \right. \right. \\
& \left. \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} \right) \right) + \right. \\
& \left. \left. (-b^3 + 4cb - 8d) / \left(4 \sqrt{\left(\frac{b^2}{4} - \frac{2c}{3} + \frac{1}{3\sqrt[3]{2}} \left((2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \right.\right. \right. \right. \right. \right. \\
& \left. \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} \right) \right) \right) + \right. \\
& \left. \left. \left. (\sqrt[3]{2} (c^2 - 3bd + 12e)) / \left(3(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e + \right.\right. \right. \right. \\
& \left. \left. \left. \left. \sqrt{(2c^3 - 9bdc - 72ec + 27d^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12e)^3} \right)^{(1/3)} \right) \right) \right) \right] \right] \right] \right]
\end{aligned}$$

■ Quintic

`Solve[x5 + b x4 + c x3 + d x2 + e x + f == 0, x]`

`{x -> Root[#15 + b #14 + c #13 + d #12 + e #1 + f &, 1]}`,

`{x -> Root[#15 + b #14 + c #13 + d #12 + e #1 + f &, 2]}`, `{x -> Root[#15 + b #14 + c #13 + d #12 + e #1 + f &, 3]}`,

`{x -> Root[#15 + b #14 + c #13 + d #12 + e #1 + f &, 4]}`, `{x -> Root[#15 + b #14 + c #13 + d #12 + e #1 + f &, 5]}`