Group Actions

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Definition: Let G be a group and X a set. G *acts on* X if there is a function $\alpha : G\times X \to X$ such that

(a) $\alpha(e, x) = x$ and

(b) $\alpha(g_1\, \alpha(g_2, x)) = \alpha(g_1\, g_2, x)$

α is called an action on X. w

$\alpha(g, x)$ is sometimes abbreviated $g\, x$.

Examples. (a) G action on itself by left translation:

$\tau_g(x) = g\, x$

Cayley's Theorem says any group G is isomorphic to a subgroup of S_G:

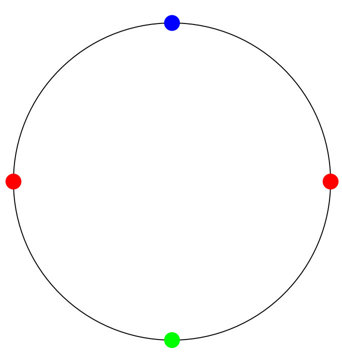
$G \cong \{\tau_g \mid g \in G\}$ with an isomorpism being $g \longmapsto \tau_g$.

(b) G acts on itself by *conjugation*. $\gamma\,(g, x) = g\, x\, g^{-1}$.

$$\gamma(e, x) = e\, x\, e^{-1} = e\, x\, e = x$$

$$\begin{aligned}\gamma(g_1\, \gamma(g_2, x)) &= \gamma\left(g_1, g_2\, x\, g_2^{-1}\right)\\ &= g_1\left(g_2\, x\, g_2^{-1}\right) g_1^{-1}\\ &= \left(g_1\, g_2\right) x \left(g_2^{-1}\, g_1^{-1}\right)\\ &= (g_1\, g_2)\, x\, (g_1\, g_2)^{-1}\\ &= \gamma(g_1\, g_2, x)\end{aligned}$$

(c) Consider the fabrication of circular bracelets with four jewels (Rubies, Emeralds, and Saphires) evenly spaced on the bracelet. For example one bracelet could look like this:



One of 81 placements of jewels

Note: Not all of the different jewels must be used. For example we could have a bracelet with four rubies.

For the time being imagine that the positions of the jewels cannot be changed. Let X be the set of possible arrangement of jewels in a bracelet. $|X| = 3^4 = 81$. The group of symmetries of a square, D_4 acts on X in a natural way.

The stablizer of x: $G_x = \{g \in G \mid \alpha(g, x) = x\} \subseteq G$

For the case of conjugation:

$$G_x = \{g \in G \mid g\,x\,g^{-1} = x\} = \{g \in G \mid g\,x = x\,g\}$$
$$= \text{the centralizer of } x.$$

For the bracelet above, the stablizer is the cyclic subgroup of D_4 generated by the vertical reflection.

Theorem. $G_x \leq G$.
Proof: First note that $e \in G_x$ since $\alpha(e, x) = x$. So G_x is never empty. If $g \in G_x$, then $x = e\,x = (g^{-1}\,g)\,x = g^{-1}(g, x) = g^{-1}\,x \Rightarrow \in G_x$. Finally, if $g, h \in G_x$,

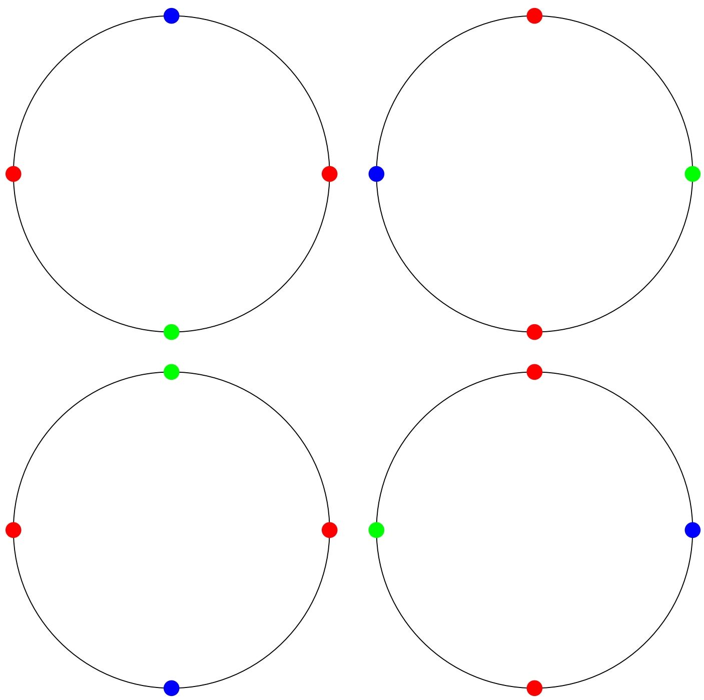
$$(g\,h)\,x = g(h\,x) = g\,x = x \Rightarrow g\,h \in G_x$$

The orbit of x: $\mathcal{O}(x) = \{g\,x \mid g \in G\} \subseteq X$

For the case of conjugation:

$$\mathcal{O}(x) = \{g\,x\,g^{-1} \mid g \in G\}$$
$$= \text{the conjugate class of } x$$

The orbit of the bracelet above is



The Center of G:

$$Z(G) = \{x \in G \mid g\,x = x\,g \text{ for all } g \in G\}$$
$$= \{x \in G \mid \text{the conjugate class of } x \text{ is } \{x\}\}$$

If $n \geq 3$, $Z(S_n) = \{i\}$

The Class Equation: If G is a finite group and $\{g_1, g_2, \ldots, g_m\}$ are selected as representatives of the conjugate classes containing more than one element, then

$$|G| = |Z(G)| + \sum_{k=1}^{m} |\mathcal{O}(g_k)|$$

Theorem. The number of elements in the conjugate class of x, $|\mathcal{O}(x)|$ is $[G : G_x]$, the number of distinct cosets of G_x in G.

Proof: Let G/G_x be the set of all cosets of G_x in G. Define $\phi : \mathcal{O}(x) \to G/G_x$ as follows: If $y \in \mathcal{O}(x)$, then $y = g\,x$ for some g. Then define $\phi(y) = g\,G_x$.

ϕ is well-defined.
ϕ is one-to-one
ϕ is onto

The Class Equation, revised: $|G| = |Z(G)| + \sum_{k=1}^{m} \left[G : G_{g_k}\right]$

Theorem If G is a finite group whose order is divisible by a prime p, then G contains an element of order p.

Lemma

(i) Let G act on X. If $x \in X$ and $\sigma \in G$, then $G_{\sigma x} = \sigma G_x \sigma^{-1}$.

(ii) If finite group G acts on finite set X and if x and y lie in the same orbit, then $\left|G_y\right| = |G_x|$

Burnside's Lemma. Let G acto on a finite set X. If N is the number of orbits, then

$$N = \frac{1}{|G|} \sum_{\tau \in G} F(\tau)$$

where $F(\tau)$ is the number of $x \in X$ fixed by τ.