Perfect Delaunay Polytopes and Perfect Quadratic Functions on Lattices

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Abstract. A polytope D, whose vertices belong to a lattice of rank d, is *Delaunay* if it can be circumscribed by an ellipsoid $\mathcal{E}$ with interior free of lattice points, and so that the vertices of D are the only lattice points on the quadratic surface $\mathcal{E}$. If in addition $\mathcal{E}$ is uniquely determined by D, we call D a *perfect Delaunay polytope*. Thus, in the perfect case, the lattice points on $\mathcal{E}$, which are the vertices of D, uniquely determine the quadratic surface $\mathcal{E}$. We have been able to construct infinite sequences of perfect Delaunay polytopes, one perfect polytope in each successive dimension starting at some initial dimension; we have been able to construct an infinite number of such sequences. Perfect Delaunay polytopes play an important role in the theory of Delaunay polytopes and in Voronoi's theory of lattice types.

1. Introduction

Consider the lattice $\mathbb{Z}^d$ and a $\mathbb{Z}^d$-polytope D. The polytope D is called a *Delaunay polytope* if it can be circumscribed by an *ellipsoid* $\mathcal{E}$ so that: (1) $\mathcal{E}$ has no interior $\mathbb{Z}^d$-elements and (2) the only $\mathbb{Z}^d$-elements on the surface $\mathcal{E}$ are the vertices of D. We will say that a circumscribing ellipsoid $\mathcal{E}$ that has these two properties, is an *empty ellipsoid* for D. The Delaunay polytope D with a circumscribing empty ellipsoid $\mathcal{E}$ is called *perfect* when the surface $\mathcal{E}$ is uniquely determined by the vertex set of D among all quadratic surfaces.

If D is a Delaunay polytope on $\mathbb{Z}^d$ with circumscribing empty ellipsoid $\mathcal{E}$, the vertices of D are given by $\mathcal{E}\bigcap D = \mathcal{E}\bigcap \mathbb{Z}^d$. Typically, the quadratic surface $\mathcal{E}$ through this vertex set can be deformed continuously, but when D is a *perfect* Delaunay polytope this surface is uniquely determined by $\mathcal{E}\bigcap D$. In this case we refer to $\mathcal{E}$ as a *perfect ellipsoid* around D.

We have studied perfect Delaunay polytopes by constructing infinite sequences of them, one perfect Delaunay polytope in each successive dimension, starting at some initial dimension. We have been able to construct an infinite number of such infinite sequences of perfect Delaunay polytopes. Typically, the perfect Delaunay polytopes we construct are on $\mathbb{Z}^d$, but this does not compromise generality.

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The perfect Delaunay polytopes in each sequence have similar combinatorial and geometric properties. One of our constructions is the sequence of *G-topes,* $G^d,\ d \geq 6$. The initial term, G^6, is the well-known 6-dimensional Gossett polytope with 27 vertices; G^6 is Coxeter's semiregular 2_{21}. Each G-tope is asymmetric with respect to central inversion, and G^d has $\frac{d(d+1)}{2} + d$ vertices. Each term in the sequence is a combinatorial analogue of the initial Gossett polytope G^6. Another of our constructions is the sequence of *C-topes,* $C^d,\ d \geq 7$, with the initial term, C^7 the 7-dimensional Gossett polytope with 56 vertices (3_{31} in Coxeter's notation). Each C-tope is *centrally symmetric,* and C^d has $2\binom{d+1}{2}$ vertices. As with the sequence of G-topes, each term is a combinatorial analogue of the initial Gossett polytope C^7. Just as the 6-dimensional Gossett polytope can be represented as a section of the 7-dimensional one, each G^d of the asymmetric sequence can be represented as a section of C^{d+1} of the symmetric sequence.

In addition to being fascinating geometrical objects there are a number of other reasons why perfect Delaunay polytopes are interesting objects to study.

Maximality: It follows directly from the definition that the perfect Delaunay polytopes on $\mathbb{Z}^d$ are maximal in the class of all Delaunay polytopes on $\mathbb{Z}^d$. That is, if D is perfect it can't be inscribed vertex-to-vertex in any larger Delaunay polytope on $\mathbb{Z}^d$ (see Subsection 2.1 for details).

Delaunay tilings: Each perfect Delaunay polytope D uniquely determines a positive definite quadratic form on $\mathbb{Z}^d$ which, in turn, determines a Delaunay tiling for the lattice $\mathbb{Z}^d$ with D as one of its Delaunay tiles (details on how these tilings are constructed are given in the following section). Therefore, each of our sequences of perfect Delaunay polytopes determines a corresponding sequence of Delaunay tilings. Just as the perfect Delaunay polytopes in the sequence are combinatorial analogues, the corresponding Delaunay tilings have similar combinatorial properties. For example, each perfect Delaunay polytope in the sequence of G-topes uniquely determines a Delaunay tiling that is an analogue of that for the root lattice E_6. Similarly, each term in the sequence of C-topes uniquely determines a Delaunay tiling that is an analogue of that for the root lattice E_7.

Perfection property: A positive definite quadratic form φ in d variables is called *perfect* if it is uniquely determined by its minimal vectors through the system of linear equations $\chi(\mathbf{m}) = m$, where m is the minimal value achieved by φ on the non-zero elements of $\mathbb{Z}^d$, and where $\mathbf{m} \in \mathbb{Z}^d$ belongs to the set of minimal vectors for φ. Since a quadratic form is determined by $\frac{n(n+1)}{2}$ coefficients, the set of minimal vectors must be sufficiently numerous so that the coefficients of φ are uniquely determined.

There is an inhomogeneous analogue to perfect form that plays a role in the study of Delaunay polytopes. Let $\mathcal{E}$ be an empty ellipsoid around a d-dimensional Delaunay polytope D on $\mathbb{Z}^d$. Then $\mathcal{E}$ determines an inhomogeneous quadratic function f so that $f(\mathbf{x}) = 0$ is an equation for $\mathcal{E}$; f is determined by $\mathcal{E}$ up to a scale factor. If we assume that this scale factor is chosen so that f has negative values on the interior of $\mathcal{E}$, then, since $\mathcal{E}$ is empty, f has non-negative values on $\mathbb{Z}^d$. In this case f assumes its minimum value zero on $\mathcal{E} \bigcap \mathbb{Z}^d$, which is the vertex set for D. When D is perfect the empty ellipsoid $\mathcal{E}$ is uniquely determined by D, and accordingly, the coefficients of f are determined, up to a scale factor, by the linear system of equations $\{f(\mathbf{z}) = 0 \,|\, \mathbf{z} \in \mathcal{E} \bigcap \mathbb{Z}^d\}$. The rank of this homogeneous linear system must be $\frac{d(d+1)}{2} + d$, which requires a perfect Delaunay polytope to have at least this number of vertices.

The conditions that a perfect quadratic function must satisfy are more severe than those for a perfect form, which requires at least $\frac{n(n+1)}{2}$ pairs of minimal vectors. This leads to the speculation that the growth of the number of arithmetic types of perfect quadratic functions with dimension will be much slower than the growth of the number of arithmetic types of perfect forms. This has been borne out by the data now available on perfect Delaunay polytopes (see `http://www.liga.ens.fr/~dutour/`). In fact, the growth of the number of arithmetic types of perfect functions seems slower that that for most classes of fundamental geometric objects studied in the geometry of numbers.

When the Delaunay polytope D is perfect it's vertices are used to determine the coefficients of f in much the same way that the minimal vectors are used to determine the coefficients of a perfect form. For this reason that we have referred to D as perfect, and will refer to the associated circumscribing ellipsoid $\mathcal{E}$, and associated inhomogeneous quadratic function f, as *perfect*.

Perfect forms have been introduced by Korkine and Zolotareff (1873), who found all perfect forms through the dimension five (the term `perfect` was coined by Voronoi who confirmed that Korkine and Zolotareff's list was complete). In addition, they found several perfect forms in higher dimensions, including the forms corresponding to the root lattices A_n, D_n (both for $n \in \mathbb{N}$), E_6, E_7, and E_8. Korkine and Zolotareff used different roman letters to name these root lattices, which gives a partial explanation why these lattices are rarely associated with them. More recently, the study of perfect forms has been taken up by Martinet who published a comprehensive monograph on the topic in 2003. As of the end of 2006 perfect forms have been completely classified through eight dimensions (see `http://fma2.math.uni-magdeburg.de/~latgeo/`).

Inhomogeneous domains: Perfect Delaunay polytopes and infinite cylinders with perfect Delaunay polytopes as bases provide labels for *edge functions* for the *inhomogeneous domains* of Delaunay polytopes. Associated with each lattice Delaunay polytope D is a *polyhedral cone of inhomogeneous quadratic functions,* which are positive (≥ 0) on $\mathbb{Z}^d$ and vanish on $\mathsf{vert}\, D$. We refer to this cone as the inhomogeneous domain of D and to the functions lying on the extreme rays of this cone as edge functions for D. These domains play an important role in the structure theory of Delaunay polytopes (Erdahl, 1992) and are discussed in details in Section 2.

Homogeneous domains: A significant role for perfect Delaunay polytopes is to provide geometric labels for a class of *edge forms* for the *homogeneous domains* of Delaunay tilings of $\mathbb{Z}^d \otimes \mathbb{R}$, which are defined in Section 2. These domains, also referred to as *domains of lattice types* (*L-type* domains), are polyhedral cones of positive forms, and the edge forms are those that lie on the extreme rays. L-type domains have been first introduced and studied by Voronoi, and they play a central role in his classification theory for lattices, his theory of lattice types. Delaunay associated these polyhedral cones with Delaunay tilings. Prior to the discovery of the sequence $\{G^d\}$ by Erdahl and Rybnikov in 2001 only finitely many arithmetic types of edge forms of *full rank* were known. The significance of *extreme* L-types is much due to their relation to the structure of Delaunay and Voronoi tilings for lattices. The Voronoi-Dirichlet polytope V_φ for a form φ contained in an L-type domain Φ_φ is a weighted Minkowski sum of linear transforms of Voronoi-Dirichlet polytopes for each of the edge forms of Φ_φ (H.-F. Loesch, 1990). There is a corresponding dual statement on the structure of Delaunay polytopes: the Delaunay tiling on $\mathbb{Z}^n$ for a form φ lying in an L-type domain Φ_φ is the intersection of the Delaunay tilings for the edge forms of Φ_φ (Erdahl, 2000).

Method of variation of parameters: Delaunay tilings of lattices and Voronoi's theory of L-types are intrinsically connected to the theory of moduli spaces of abelian varieties. On a philosophical level this connection stems from the idea of parametric approach to the study of geometric objects. In geometry of numbers and number theory this approach was pioneered by Hermite (1850). Korkine, Zolotareff, and Voronoi referred to this approach as Hermite's method of continuous variation of parameters. It is worth noting that Voronoi's interest in geometry of numbers developed as a result of his work on irrationalities of the third degrees and, in particular, elliptic curves. The curious reader may consult Delone [Delaunay] and Faddeev (1964). For recently discovered connections between moduli spaces of abelian varieties and Voronoi's theories of perfect and L-types see Alexeev (2002) and Shepard-Barron (2006). We are not aware for any results interpreting perfect Delaunay polytopes and L-types defined by them, but we certainly anticipate them.

1.1. Historical notes and recent developments. Perfect Delaunay polytopes were first considered by Erdahl (1975) in connection with lattice polytopes arising from the quantum mechanics of many electrons. He observed (1975) that Delaunay tilings of 0- and 1-dimensional lattices consist entirely of perfect Delaunay polytopes and showed that the Gosset polytope $G^6 = 2_{21}$ with 27 vertices was a perfect Delaunay polytope in the root lattice E_6. He also showed that there were no perfect Delaunay polytopes in dimensions 2, 3, and 4. These results were further extended by Erdahl (1992) by showing that the 7-dimensional Gosset polytope $C^7 = 3_{31}$ with 56 vertices is perfect, and that there are no perfect Delaunay polytopes of dimension d for $1 < d < 6$. Erdahl also proved that G^6 and C^7 are the only perfect Delaunay polytopes in the Delaunay tilings for the root lattices. Deza, Grishukhin, and Laurent (1992-1997, in various combinations) found more examples of perfect Delaunay polytopes in dimensions 15, 16, 22, and 23, but all of those seemed to be sporadic.

The first construction of an infinite sequence of perfect Delaunay polytopes was described at the Conference dedicated to the Seventieth Birthday of Sergei Ryshkov (Erdahl, 2001), and later reported by Rybnikov (2001) and Erdahl and Rybnikov (2002). During 2001-2004 Erdahl gave a number of talks where he described the constructions of infinite sequences of perfect Delaunay polytopes (see References). Perfect Delaunay polytopes have been classified through the dimension 6 – Dutour (2004) proved that $G^6 = 2_{21}$ is the only perfect polytope for $d = 6$. It is strongly suspected that the existing lists of seven and eight dimensional perfect Delaunay polytopes are complete (see Dutour, Erdahl, and Rybnikov, 2007). Later infinite series of perfect Delaunay polytopes were constructed by Dutour (2005) and Grishukhin (2006). Our 3-parametric series $P^d_{s,2k}$ coincides with the 1-parametric series of Dutour only in small dimensions ($d = 6, 7, 8$). Building on earlier work of Erdahl and Rybnikov (2002) Grishukhin (2006) constructed a 2-parametric series $P(d, t)$; Grishukhin's $P(d, 1)$ coincides with our G^d (see Tables 1 and 2). Since some of the Grishukhin subseries coincide with ours, it would be interesting to investigate the relationship between his $P(n, t)$ and our $P^d_{s,2k}$. The remarkable property of Dutour's series is that the size of the vertex set of Dutour's polytope grows exponentially with the dimension, for his d-polytope ED_d contains a section isometric to the $(d-1)$-halfcube for even d's (for odd d's it has two sections isometric to the

$(d-2)$-halfcube). Dutour conjectured that ED_d has the largest number of vertices among all perfect Delaunay polytopes of dimension d.

2. Quadratic polynomials on lattices

The Voronoi-Dirichlet and Delaunay tilings for point lattices are constructed using the Euclidean metric, but are most effectively studied by injectively mapping the lattice into $\mathbb{Z}^d$, and replacing the Euclidean metric by an equivalent metrical form. For a d-dimensional lattice Λ with a basis $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_d$ this is done as follows. A lattice vector with coordinates $z_1, z_2, ..., z_d$ relative to this basis can be written as $\mathbf{z} = \mathbf{B}\mathbf{z}$, where $\mathbf{B} = [\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_d}]$ is the basis matrix and $\mathbf{z}$ is the column vector given by $[z_1, z_2, ..., z_d]^t$. The squared Euclidean length is given by $|\mathbf{v}|^2 = \mathbf{z}^t\mathbf{B}^t\mathbf{B}\mathbf{z} =\varphi_{\mathbf{B}}(\mathbf{z})$. This squared length can equally well be interpreted as the squared length of the integer vector $\mathbf{z} \in \mathbb{Z}^d$ relative to the form $\varphi_{\mathbf{B}}$. Therefore, the Voronoi-Dirichlet and Delaunay tilings for Λ, constructed using the Euclidean metric, can be studied using the corresponding Voronoi-Dirichlet and Delaunay tilings for $\mathbb{Z}^d$ constructed using the form $\varphi_{\mathbf{B}}$ as metric. Moreover, variation of the Voronoi-Dirichlet and Delaunay tilings for Λ in response to variation of the lattice basis can be studied by varying the metric φ for the fixed lattice $\mathbb{Z}^d$. In this section we will keep the lattice fixed at $\mathbb{Z}^d$, and vary the quadratic form φ. We call a form φ positive if $\varphi[\mathbf{z}] \geq 0$ for any $\mathbf{z}$, and we call it positive definite if $\varphi[\mathbf{z}] > 0$ for any $\mathbf{z} \neq \mathbf{0}$. Same terminology is applied to all numbers and functions. Positive, but not positive definite forms are referred to as positive semidefinite, or just semidefinite.

2.1. The inhomogeneous domain of a Delaunay polytope. Let $\mathcal{P}^d$ be the cone of real quadratic polynomials defined by:

$$\mathcal{P}^d = \{\, f \in \mathbb{R}[x_1, \dots, x_d] \;\big|\; \deg f = 2,\ \ \forall \mathbf{z} \in \mathbb{Z}^d : f(\mathbf{z}) \geq 0 \,\}.$$

The condition "$\forall \mathbf{z} \in \mathbb{Z}^d : f(\mathbf{z}) \geq 0$" requires the quadratic part of f to be positive, and requires any subset of $\mathbb{R}^d$ where f assumes negative values to be free of $\mathbb{Z}^d$-elements. The real quadric determined by the equation $f(\mathbf{x}) = 0$, where $f \in \mathcal{P}^d$, might be the empty set or it might have the form

$$\mathcal{E}_f = \mathcal{E}_0 \times K,$$

where $\mathcal{E}_0$ is an ellipsoid of dimension $n \geq 0$ and K a complementary subspace of dimension $d-n$. When $d > n$, the degenerate "ellipsoid" $\mathcal{E}_0 \times K$ is an unbounded body whose interior is free of elements of $\mathbb{Z}^d$.

For any $f \in \mathcal{P}^d$ we denote by $\mathsf{V}(f)$ the set $\{\, \mathbf{z} \in \mathbb{Z}^d \,|\, f(\mathbf{z}) = 0\}$. In the case where the surface $\mathcal{E}_f = \{\, \mathbf{x} \in \mathbb{Z}^d \otimes \mathbb{R} \,|\, f(\mathbf{x}) = 0\}$ includes integer points and is bounded, $\mathsf{V}(f) = \mathcal{E}_f \cap \mathbb{Z}^d$ is the vertex set for the corresponding Delaunay polytope $D_f = \operatorname{conv} \mathsf{V}(f)$. Conversely, if D is a Delaunay polytope in $\mathbb{Z}^d \otimes \mathbb{R}$, there is a circumscribing empty ellipsoid determined by a function $f_D \in \mathcal{P}^d$. More precisely, since D is Delaunay, there is a form φ_D, a center $\mathbf{c}$, and a radius R, so that $f_D(\mathbf{x}) = \varphi_D[\mathbf{x} - \mathbf{c}_D] - R_D^2 = 0$ defines a circumscribing empty ellipsoid. Since f_D is positive on $\mathbb{Z}^d$, it is an element of $\mathcal{P}^d$.

In the case where $\mathcal{E}_f$ is an empty cylinder, there is an infinite number of integer points lying on this surface. When this happens $\mathsf{V}(f)$ is the set of all lattice points on the boundary of a non-bounded body $D_f = \operatorname{conv} \mathsf{V}(f)$, which we call a *Delaunay polyhedron* (we follow a recent convention that a polyhedron deserves to be called polytope only when it is bounded). *Although in the case of unbounded D_f the elements of $\mathsf{V}(f)$ are not vertices of D_f in the* *sense of real polyhedral geometry, we still refer to them as such and denote them by* $\mathsf{vert}\, D_f$*.* With this definition of Delaunay polyhedron, for a given form $\varphi \geq 0$ Delaunay polyhedra need not form a tiling of $\mathbb{Z}^d \otimes \mathbb{R}$. They do form a face-to-face tiling if and only if the real kernel of the quadratic part of f intersects with $\mathbb{Z}^d$ over a sublattice of the same dimension, i.e., when $\dim \mathfrak{Ker}_{\mathbb{R}}\, \varphi = \dim\{\mathfrak{Ker}_{\mathbb{R}}\, \varphi \cap \mathbb{Z}^d\}$; we will refer to the latter condition by saying that φ has fully rational kernel.

Definition 2.1. A polyhedron $D \subset \mathbb{Z}^d \otimes \mathbb{R}$ is called a Delaunay polyhedron, if there is $f \in \mathcal{P}^d$ such that $D = \operatorname{conv} \mathsf{V}(f)$.

Definition 2.2. Let D be a Delaunay polyhedron in $\mathbb{Z}^d \otimes \mathbb{R}$. Then, the (inhomogeneous) domain $\mathcal{P}^d_D$ for D is:

$$\mathcal{P}^d_D = \{\, f \in \mathcal{P}^d \;\big|\; D_f = D.\}$$

Such domains are relatively open convex cones that partition $\mathcal{P}^d$.

Each element $f \in \mathcal{P}^d_D$ satisfies the linear equations $f(\mathbf{z}) = 0$, $\mathbf{z} \in D \cap \mathbb{Z}^d$. When D is a single lattice point and $d > 0$, $\mathcal{P}^d_D$ is a relatively open cone of dimension $\binom{d+2}{2} = \dim \mathcal{P}^d$. When D is a 1-polytope, $\mathcal{P}^d_D$ is a relatively open facet of the partition with dimension $\dim \mathcal{P}^d - 1 = \binom{d+2}{2} - 1$. When the rank of the system $\{f(\mathbf{z}) = 0 \,\big|\, \mathbf{z} \in D \cap \mathbb{Z}^d\}$ is full, i.e. $\binom{d+2}{2}$, the only possible f is 0 and $D = \mathbb{Z}^d \otimes \mathbb{R}$. If D is a d-dimensional ($d > 0$) Delaunay simplex then $\dim \mathcal{P}^d_D = \binom{d+1}{2}$, but if D is a perfect Delaunay polyhedron and $d > 0$, then $\dim \mathcal{P}^d_D = 1$.

Definition 2.3. A function $p \in \mathcal{P}^d$ is perfect if any solution f to the system of equations

$$\{f(\mathbf{z}) = 0 \mid \mathbf{z} \in \mathsf{V}(p)\}$$

is a scalar multiple of p. In this case we also call $\operatorname{conv} \mathsf{V}(p)$ a perfect polyhedron.

The elements of 0- and 1-dimensional inhomogeneous domains are perfect, and the Delaunay polyhedra that determine such domains are perfect. The domain that consists of strictly positive constant functions corresponds to the empty Delaunay polytope and the 0-dimensional domain, which consists of zero function, corresponds to $D = \mathbb{R}^d$. The perfect subsets $\mathsf{V}(p)$ must be maximal among the subsets $\{\, \mathsf{V}(f) \mid f \in \mathcal{P}^d\,\}$. With the exception of the origin, all inhomogeneous domains $\mathcal{P}^d_D$ have proper faces that are inhomogeneous domains of lesser dimensions. Perfect quadratic functions are analogues of perfect forms – both achieve their arithmetic minimum at a sufficient number of points so that the representations of the minimum uniquely determine the polynomial.

The following theorem and the next subsection show the important role played by perfect Delaunay polytopes in the theory of lattice Delaunay polytopes and Voronoi's L-types.

Theorem 2.4. *(Erdahl, 1992) Let D be a Delaunay polyhedron in $\mathbb{Z}^d \otimes \mathbb{R}$. Then*

$$\mathcal{P}^d_D = \left\{ \sum_{\{p \,\mid\, p \text{ is perfect},\, \mathsf{vert}\, D \subseteq \mathsf{V}(p)\}} \omega_p p \;\middle|\; \omega_p \in \mathbb{R}_{>0} \right\},$$

This theorem shows that an arbitrary element $f \in \mathcal{P}^d_D$ has the following representation:

$$f = \sum_{\{p \,\mid\, p \text{ is perfect},\, \mathsf{vert}\, D \subseteq \mathsf{V}(p)\}} \omega_{f,p} p,$$

where $\omega_{f,p} > 0$.

2.2. The homogeneous domain of a Delaunay tiling: Voronoi's classification theory for lattices, his *theory of lattice types* (L-types), was formulated using positive definite forms and the fixed lattice $\mathbb{Z}^d$. In this theory two lattices are considered to be the same type if their Delaunay tilings are affinely equivalent. This characterization is not apparent in Voronoi's memoirs and was brought to light by Delaunay. Here is a brief reminder of Delaunay's approach to Voronoi's theory of L-types. Consider a positive definite quadratic form φ. A $\mathbb{Z}^d$-polytope D is Delaunay relative to φ if (1) it can be circumscribed by *empty ellipsoid* $\mathcal{E}$ defined by the equation of the form

$$\varphi[\mathbf{x}-\mathbf{c}] = R^2 \text{ with } R > 0,$$

(2) conv $\mathcal{E}$ has no interior $\mathbb{Z}^d$-elements, and (3) $\mathsf{vert}\, D = \mathcal{E} \cap (\mathbb{Z}^d \cap \text{aff } D)$. The relative interiors of $\mathbb{Z}^d$-polytopes that are Delaunay with respect to φ form a partition of $\mathbb{R}^d$. Furthermore, the d-dimensional Delaunay polytopes relative to φ fit together facet-to-facet to tile $\mathbb{R}^d$, a tiling that is uniquely determined by φ. The resulting tiling is the *Delaunay tiling* $\mathcal{D}_\varphi$ of $\mathbb{Z}^d \otimes \mathbb{R}$ relative to the form φ. If another form ϑ has Delaunay tiling $\mathcal{D}_\vartheta$ and this tiling is $GL(n,\mathbb{Z})$-equivalent to $\mathcal{D}_\varphi$, then φ and ϑ are *forms of the same L-type.*

The description that we give below requires that certain degenerate positive forms be admitted into the discussion, namely, those forms φ with $\operatorname{rank}\varphi < d$ for which $\mathfrak{Ker}_{\mathbb{R}}\,\varphi$ is fully rational. The Delaunay polyhedra for such a form are themselves degenerate – they are cylinders with axis $\mathfrak{Ker}_{\mathbb{R}}\,\varphi$ and Delaunay polytopes as bases. These cylinders fit together to form the (degenerate) Delaunay tiling $\mathcal{D}_\varphi$.

Example 2.5. If $\mathbf{p} \in \mathbb{Z}^d$ is primitive, i.e. $\text{GCD}(p_1,\ldots,p_d) = 1$, then $\varphi[\mathbf{x}] = (\mathbf{p}\cdot\mathbf{x})^2$ is such a form – the kernel $\mathfrak{Ker}_{\mathbb{R}}\,\varphi$ is the solution set for $\varphi[\mathbf{x}] = 0$, and given by $\mathbf{p}^\perp$, which is fully rational. The Delaunay tiles are infinite slabs, each squeezed in between a pair of hyperplanes $\mathbf{p}\cdot\mathbf{x} = k, \mathbf{p}\cdot\mathbf{x} = k+1$, $k \in \mathbb{Z}$. These fit together to tile $\mathbb{Z}^d \otimes \mathbb{R}$.

Let Φ^d be the cone of positive definite forms and semidefinite forms with fully rational kernels (Φ^d is sometimes referred to as the rational closure of the cone of positive definite forms). For each form $\varphi \in \Phi^d$ there is a Delaunay tiling $\mathcal{D}_\varphi$ of $\mathbb{Z}^d \otimes \mathbb{R}$.

Definition 2.6. If $\mathcal{D}$ is a Delaunay tiling for $\mathbb{Z}^d$, then the following cone of positive definite quadratic forms

$$\Phi_{\mathcal{D}} = \left\{ \varphi \in \Phi^d \;\middle|\; \mathcal{D}_\varphi = \mathcal{D} \right\}$$

is called an L-type domain.

For this definition the Delaunay tilings can be the usual ones, where the tiles are Delaunay polytopes – or they could be degenerate Delaunay tilings where the tiles are cylinders with a common axis K.

The relatively open faces of an L-type domain, are L-type domains. If $\mathcal{D}$ is a triangulation, $\Phi_{\mathcal{D}}$ has full dimension $\binom{d+1}{2}$; this is the generic case. If $\mathcal{D}$ is not a triangulation, $\dim \Phi_{\mathcal{D}}$ is less than $\binom{d+1}{2}$, and $\Phi_{\mathcal{D}}$ is a boundary cell of a full-dimensional L-type domain. In the case where $\dim \Phi_{\mathcal{D}} = 1$, the elements $\varphi \in \Phi_{\mathcal{D}}$ are called edge forms – Voronoi showed that such domains are exactly extreme rays of full-dimensional L-type domains.

Let π_Φ be the projection operator from $\mathcal{P}^d$ onto its quadratic part. If $D \in \mathcal{D}$ and $\dim D = d$, then $\Phi_{\mathcal{D}} \subset \pi_\Phi(\mathcal{P}^d_D)$. Since this containment holds for all Delaunay d-tiles $D \in \mathcal{D}$, there is the following description of $\Phi_{\mathcal{D}}$ in terms of inhomogeneous domains:

$$\Phi_{\mathcal{D}} = \bigcap_{D \in \mathcal{D}} \pi_\Phi(\mathcal{P}^d_D),$$

where the intersection is over all d-dimensional Delaunay polyhedra in $\mathcal{D}$. (It is also true that the intersection can be taken over all Delaunay polyhedra in $\mathcal{D}$.) Since the containment $\Phi_{\mathcal{D}} \subset \pi_\Phi(\mathcal{P}^d_D)$ also holds for all Delaunay tilings $\mathcal{D}$ that contain D, there is the following description of $\pi_\Phi(\mathcal{P}^d_D)$ in terms of homogeneous domains:

$$\pi_\Phi(\mathcal{P}^d_D) = \bigsqcup_{\mathcal{D} \ni D} \Phi_{\mathcal{D}},$$

where the disjoint union is over all Delaunay tilings of $\mathbb{Z}^d \otimes \mathbb{R}$ that contain D. This holds not only for full-dimensional cells of $\mathcal{P}^d_D$, but for cells of all dimensions. The last equality shows that $\pi_\Phi(\mathcal{P}^d_D)$ is tiled by L-type domains. It also establishes the following

Proposition 2.7. If $p \in \mathcal{P}^d$ is perfect and, therefore, an edge form for an inhomogeneous domain, then $\pi_\Phi p$ is an edge form for an L-type domain.

This result can also be established in a more direct way by appealing to the definition of L-type domain: if D is a perfect Delaunay polyhedron, then $\pi_\Phi(\mathcal{P}^d_D)$ is a one-dimensional L-type domain. By definition, the elements of $\pi_\Phi(\mathcal{P}^d_D)$ are then edge forms.

The converse of theorem does not hold - there are edge forms for L-type domains that are not inherited from perfect elements in $\mathcal{P}^d$. Evidence has accumulated recently that the growth of numbers of types of edge forms with dimension is very rapid starting in six dimensions (see Dutour and Vallentin, 2003), but the growth of perfect inhomogeneous forms is much less rapid - there is some hope that a complete classification can be made through dimension nine.

3. Structure of Perfect Delaunay Polyhedra

In this section we will describe how to construct new arithmetic types of perfect Delaunay polytopes in dimension $d+1$ from known types of perfect Delaunay polytopes in dimension d for $d > 1$. Since there are no perfect Delaunay d-polytopes for $1 < d < 6$, this construction can be used starting from $d = 6$.

In the previous section we mostly treated the case of "ground lattice" $\mathbb{Z}^d$ and tilings of $\mathbb{Z}^d \otimes \mathbb{R}$. Often, as in the case of the following sections, it is convenient to allow for more general lattices than $\mathbb{Z}^n$. If Λ is a lattice and φ is a positive quadratic form on Λ, then $\mathcal{D}_\varphi(\Lambda)$ denotes the Delaunay tiling on Λ with respect to φ. Our starting point is the following structural characterization.

Theorem 3.1. *(Erdahl, 1992) A polyhedron $P \in \mathcal{D}_\varphi(\Lambda)$ is perfect if and only if there is a quadratic* function *p on Λ with $\pi_\Phi p = \varphi$ such that $P \cap \Lambda = \mathsf{V}(p) = \{\mathbf{v}+\mathbf{z} \mid \mathbf{v} \in \mathsf{vert}\, D, \mathbf{z} \in \Gamma\}$, where D is a perfect* polytope *in $\mathcal{D}_\varphi(\Lambda \cap \operatorname{aff} D)$ and Γ is a submodule of Λ such that Λ is the direct sum of modules $(\Lambda \cap \operatorname{aff} D) - (\Lambda \cap \operatorname{aff} D)$ and Γ.*

Example 3.2. Recalling Example 2.5, let $\mathbf{p} \in \mathbb{Z}^d$ be primitive, and let $\varphi[\mathbf{x}] = (\mathbf{p}\cdot\mathbf{x})^2$. The perfect Delaunay polyhedra are infinite slabs, each squeezed in between a pair of hyperplanes $\mathbf{p}\cdot\mathbf{x} = k, \mathbf{p}\cdot\mathbf{x} = k+1$, $k \in \mathbb{Z}$. These fit together to tile $\mathbb{Z}^d \otimes \mathbb{R}$. If $P = \{\mathbf{x} \mid 0 \leq$

$\mathbf{p} \cdot \mathbf{x} \leq 1\}$, then the polytope D can be chosen as $\operatorname{conv}\{\mathbf{0}, \mathbf{q}\}$, where $\mathbf{q}$ is primitive and $\mathbf{p} \cdot \mathbf{q} = 1$. In this case $\Gamma = \mathbf{p}^{\perp}$.

By the above theorem, for each perfect $P \in \mathcal{D}_{\varphi}(\Lambda)$ the set $\mathsf{vert}\, P$ of lattice points on P can be written as $\mathsf{vert}\, D \oplus \Gamma$, where D is a Delaunay polytope and Γ is a (vector) sublattice of Λ. The direct sum $\oplus$ sign means that (i) each vertex of P can be represented as a vertex of D plus a vector of Γ and (ii) *for any fixed choice of the pair* (D, Γ) such representation is unique. We call this a direct decomposition of P.

Let P be a perfect Delaunay d-polytope in $\mathbb{Z}^d$ with respect to a form $\varphi : \mathbb{Z}^d \to \mathbb{R}$. Obviously, $\mathsf{vert}\, P \oplus \mathbb{Z}^1$ is the vertex set of a perfect Delaunay polyhedron in $\mathbb{Z}^{d+1}$ relative to $\varphi_1[z_1, \ldots, z_d, z_{d+1}] \triangleq \varphi[z_1, \ldots, z_d]$. The following theorem gives a universal construction of a perfect Delaunay polyhedron P' of dimension $d+1$ from P such that $\mathsf{vert}\, P'$ is not arithmetically equivalent to $\mathsf{vert}\, P \oplus \mathbb{Z}^1$.

THEOREM 3.3. *Let P be a perfect* polytope *in $\mathcal{D}_{\varphi}(\Lambda)$ and let p be its* perfect *quadratic function, i.e.* $\mathsf{vert}\, P = \mathsf{V}(p)$ *and $\varphi = \pi_{\Phi} p$. Suppose $D \in \mathcal{D}_{\varphi}(\Lambda)$ is another Delaunay cell of full dimension, which is not a Λ-translate of P. If $\mathbf{e} \notin \Lambda \otimes \mathbb{R}$, then there is a positive form ψ on $\Lambda \oplus \mathbb{Z}\mathbf{e}$, which is arithmetically distinct from $\varphi_1[z_1, \ldots, z_d, z_{d+1}] \triangleq \varphi[z_1, \ldots, z_d]$, and a perfect* polyhedron *$P' \in \mathcal{D}_{\psi}(\Lambda \oplus \mathbb{Z}\mathbf{e})$ such that $P = P' \cap \operatorname{aff} \Lambda$ and $D + \mathbf{e} = P' \cap \{\operatorname{aff} \Lambda + \mathbf{e}\}$*

PROOF. We can safely assume $\mathbf{0} \in \mathsf{vert}\, P$ and $\mathbf{0} \in \mathsf{vert}\, D$. Let then $f(\mathbf{y}) = \psi[\mathbf{y}] + l(\mathbf{y})$ be a quadratic function with linear part l such that: `(i)` $f(\mathbf{y}) = 0$ is the equation of a quadric passing through the vertices of P and $D + \mathbf{e}$, `(ii)` $f|_{\Lambda} = p$. Since any $\mathbf{y} \in \Lambda \oplus \mathbb{Z}\mathbf{e}$ can be uniquely written as $\mathbf{x} + k\mathbf{e}$ with $\mathbf{x} \in \Lambda$ and $k \in \mathbb{Z}$, we have $\psi[\mathbf{y}] = \psi[\mathbf{x}] + \psi[k\mathbf{e}] + 2\psi(\mathbf{x}, k\mathbf{e}) = \varphi[\mathbf{x}] + k^2\psi[\mathbf{e}] + 2k\psi(\mathbf{x}, \mathbf{e})$ and $l(\mathbf{y}) = l(\mathbf{x}) + kl(\mathbf{e})$. $l|_{\Lambda}$ is known from p. So, f is determined uniquely if and only if $\psi[\mathbf{e}]$, $\psi(\mathbf{e}, \mathbf{e}_1), \ldots, \psi(\mathbf{e}, \mathbf{e}_d)$, $l(\mathbf{e})$ are fixed.

We will first show that for any positive (≥ 0) value of $\psi[\mathbf{e}]$, function f, that satisfies the conditions `(i)`, `(ii)` is unique, and $f(\mathbf{y}) \leq 0$ defines a quadric, circumscribed about $P \cup \{D + \mathbf{e}\}$, such that no points of $\Lambda \oplus \mathbb{Z}\mathbf{e}$ satisfy $f(\mathbf{y}) < 0$. Let us fix some $\psi[\mathbf{e}] \geq 0$. Since $\dim D = d$, it has $d+1$ affinely independent vertices. For each non-zero $\mathbf{v} \in \mathsf{vert}\, D$ consider equation $f(\mathbf{v} + \mathbf{e}) = 0$. If $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are independent, then the resulting system of equations is independent. (The origin has already been used to set the constant term of f to 0.) We know the line joining the circumcenters of $D + \mathbf{e}$ and P is perpendicular to Λ with respect to ψ. This condition can be written as $\psi(\mathbf{c}_D - (\mathbf{c}_P + \mathbf{e}), \mathbf{v}) = 0$, where $\mathbf{v} \in \mathsf{vert}\, D \setminus \mathbf{0}$, and $\mathbf{c}_D$, $\mathbf{c}_P$ are the circumcenters of D and P. The resulting system of $d+1$ linear equations has full rank. It is inhomogeneous because $\varphi[\mathbf{v}] + \psi[\mathbf{e}] > 0$. So, the solution is unique. The positivity of ψ follows from the positivity of φ, the fact that $\psi[\mathbf{e}] \geq 0$, and elementary geometric considerations. For a sufficiently small value of $\psi[\mathbf{e}] \geq 0$ the interior of $\{\mathbf{z} \in (\Lambda \oplus \mathbb{Z}\mathbf{e}) \otimes \mathbb{R} \mid f(\mathbf{z}) \leq 0\}$ is free of lattice points. This is easy to establish by a calculation, or just by observing that large values of $\psi[\mathbf{e}]$ correspond to ellipsoids that lie inside of an infinite slab of $(\Lambda \oplus \mathbb{Z}\mathbf{e}) \otimes \mathbb{R}$, which is squeezed between hyperplanes corresponding to $k = -1$ and $k = 2$.

Thus, under the assumptions of the theorem, for a fixed $\psi[\mathbf{e}] \geq 0$, the quadric circumscribed about $P \cup \{D + \mathbf{e}\}$ is uniquely defined, has positive quadratic part, and is Delaunay if $\psi[\mathbf{e}]$ is sufficiently large. We can now start continuously decreasing $\psi[\mathbf{e}]$, at a constant rate, until the ellipsoid hits a new lattice point, i.e. a point of $\Lambda \oplus \mathbb{Z}\mathbf{e} \ \setminus \ (P \cup \{D + \mathbf{e}\})$. If it hits a new lattice element $\mathbf{a}$ in a finite time, we are done, since $f(\mathbf{a}) = 0$ will give us a linear equation independent of those that we used to determine $\psi(\mathbf{e}, \mathbf{e}_1), \ldots, \psi(\mathbf{e}, \mathbf{e}_d)$ and

$l(\mathbf{e})$. This new equation will determine $\psi[\mathbf{e}]$; the resulting Delaunay ellipsoid $f(\mathbf{z}) = 0$ will be perfect. Note that $f(\mathbf{z}) = 0$ may contain other new lattice elements besides $\mathbf{a}$. Set

$$\alpha = \inf\{\lambda \in \mathbb{R} \,|\, \psi[\mathbf{e}] = \lambda,\ \psi \text{ is positive definite}\}.$$

If such a collision does not happen in finite time, then $\psi[\mathbf{e}] = \alpha$ determines a perfect Delaunay surface, which is a cylinder with elliptic base. By Theorem 3.1 $\mathsf{vert}\, P' = \mathsf{vert}\, P_0 + \mathbb{Z}k(\mathbf{e} + \mathbf{c}_D - \mathbf{c}_P)$, where k is the minimal natural number such that $k(\mathbf{e} + \mathbf{c}_D - \mathbf{c}_P) \in \Lambda \oplus \mathbb{Z}\mathbf{e}$ and P_0 is a perfect Delaunay polytope in a d-dimensional sublattice of $\Lambda \otimes \mathbb{Z}\mathbf{e}$. Obviously, $k > 1$, which implies that $\mathsf{vert}\, P'$ is not arithmetically equivalent to $\mathsf{vert}\, P \oplus \mathbb{Z}\mathbf{e}$. The projection of $\operatorname{aff} P_0 \cap \Lambda'$ on $\operatorname{aff} P$ along $\mathbb{Z}(\mathbf{e} + \mathbf{c}_D - \mathbf{c}_P)$ is a superlattice Λ_0 of Λ of index k. The Delaunay ellipsoid of P with quadratic form φ serves also as a Delaunay ellipsoid for the projection of P_0 onto $\operatorname{aff} P$ in the lattice Λ_0. We conclude that in this case the lattice Λ can be extended to a superlattice Λ_0 so that the perfect Delaunay polytope P for Λ becomes a proper subset of a perfect Delaunay polytope in Λ_0. We should remark that we do not know of any such examples. $\square$

Corollary 3.4. *Suppose the conditions of Theorem 3.3 hold. If the circumradius of P is different from the circumradius of D, then the perfect Delaunay polyhedron P' is a polytope.*

This corollary follows from the proof of Theorem 3.3.

Since P is perfect in $\mathcal{D}_\varphi(\Lambda)$, it determines φ (up to a scale factor) and the corresponding Delaunay tiling of Λ uniquely. However, there can be different choices of D. For example, the 35-tope of Erdahl and Rybnikov lives in a 7-dimensional Delaunay tiling which has many inequivalent (even combinatorially!) types of Delaunay polytopes of full dimension. Applying this theorem with different choices of D results, in general but not always, in arithmetically distinct perfect polytopes in the next dimension. For example, the Delaunay tiling determined by $P = C^7$ (this is the Delaunay tiling of $[E_7, \|\ \|]$) consists of translates of a single copy of C^7, and many (more than two – one always has at least 2 for a simplex) translation classes of Delaunay simplices of index 2 (see Erdahl 1992 for an exact description). All these simplices are arithmetically equivalent, in particular, they are isometric relative to the form determined by C^7; however, there is more than one orbit of such simplices with respect to the subgroup generated by lattice translations and inversion with respect to the origin. But no matter which copy of such a simplex we use for D, the resulting perfect Delaunay polytope and quadratic lattice in dimension 8 is the same. On the other hand, different choices of translationally-inequivalent D's in the case of the Delaunay tiling determined by the 35-tope sometimes lead to inequivalent perfect Delaunay polytopes in dimension 8. The only known cases where one cannot find a D which is not a translate of P, are those of $d = 0$ and $d = 1$. A surprising fact is that for $d = 0$ the construction works even without such a D, since $[0, 1]$ is perfect Delaunay in $\mathbb{Z}^{d+1}$ for $d = 0$. We do not know of any other example of a perfect Delaunay polytope which can be translated by a lattice vector to every Delaunay polytope of full dimension in the Delaunay tiling! Very likely that such an example does not exist. If we drop the perfection condition, the only example that we know of is a d-parallelepiped. There should be a way to show that no other examples exist, but, embarrassingly, we do not have a proof at the moment.

4. Infinite sequences of perfect Delaunay polytopes

It is more convenient to present our construction of the infinite sequences using the language of *point lattices.* A *point lattice* in a real affine space is a set $\mathbb{L}$ such that for any $x \in \mathbb{L}$ the set of vectors $\{y - x \,|\, y \in \mathbb{L}\}$ is isomorphic to $\mathbb{Z}^d$, for some $d \geq 0$, as a $\mathbb{Z}$-module. For example, $\mathbb{Z}^d$ is a $\mathbb{Z}$-module, i.e. a "vector" lattice, but the "ends of vectors" of $\mathbb{Z}^d$ form a point lattice in the affine space $\mathbb{Z}^d \otimes \mathbb{R} \cong \mathbb{R}^d$ – it is customary to use the symbol $\mathbb{Z}^d$ in both senses. If L is a subset of $\mathbb{Z}^d$ defined by an affine equation, where $\mathbb{Z}^d$ is considered as a $\mathbb{Z}$-module, then the set $L - L = \{\mathbf{z} - \mathbf{z}' \mid \mathbf{z}, \mathbf{z}' \in \Lambda\}$ is a $\mathbb{Z}$-module; if $\mathbb{L}$ is an affine section of the point lattice $\mathbb{Z}^d$, then $\mathbb{L}$ is a point lattice. The point lattice setup is more intuitive in the study of quadratic *functions* and Delaunay polytopes, while the "vector" ($\mathbb{Z}$-module) setup is more natural for studying quadratic *forms* and, perhaps, Voronoi-Dirichlet cells.

A pair $[\Lambda, \varphi]$, where Λ is a free $\mathbb{Z}$-module of finite rank and φ is a quadratic form on Λ is called a *quadratic lattice*; a pair $(\mathbb{L}, \psi)$, where $\mathbb{L}$ is a point lattice and ψ is a quadratic form on $\mathbb{L} - \mathbb{L}$ is called a *quadratic point lattice.* We use $\mathcal{D}_\varphi(\Lambda)$ to denote the Delaunay tiling of lattice Λ with respect to φ and $\mathcal{D}_\psi(\mathbb{L})$ to denote the Delaunay tiling of *point lattice* $\mathbb{L}$ with respect to ψ, if ψ is a form defined on $\mathbb{L} - \mathbb{L}$. To simplify the notation we will denote a *point* sublattice spanned by all affine integral combinations of a set of lattice points S by $\operatorname{aff}_{\mathbb{Z}} S$.

Denote by $\mathbf{j}$ the vector of all ones, i.e. $[1, \ldots, 1]$, in $\mathbb{Z}^d$, and denote by $\mathbf{V}^d_{s,2k}$ the following set of points in $\mathbb{R}^d$:

$$\mathbf{V}^d_{s,2k} \triangleq \left\{ [1^s, 0^{d-s}] - \frac{s-1}{d-2k}\mathbf{j} \right\} \times \binom{d}{s},$$

where $s, k \in \mathbb{N} = \{0, 1, \ldots\}$. All permutations of entries are taken, so that $|\mathbf{V}^d_{s,2k}| = \binom{d}{s}$. In general, whenever we use a shorthand notation for vectors or points such as $[1^s, 0^{d-s}]$, *it is understood that all permutations of the components are taken, except for those components that are separated from others by semicolumns on both sides,or by a semicolumn and a bracket.* Also, let us set

$$P^d_{s,2k} \triangleq \operatorname{conv}\left\{ \mathbf{V}^d_{s,2k} \cup -\mathbf{V}^d_{s,2k} \cup \mathbf{V}^d_{s+1,2k} \cup -\mathbf{V}^d_{s+1,2k} \right\}.$$

The following is the main constructive theorem of this paper.

THEOREM 4.1. *Let $d, s, k \in \mathbb{N}$. If $2 \leq k$ and $1 \leq s \leq \frac{d}{2k}$, then $P^d_{s,2k}$ is a symmetric perfect Delaunay polytope for the quadratic* point *lattice $(\Lambda^d_{s,2k}, \varphi^d_{s,2k})$, where $\Lambda^d_{s,2k} = \operatorname{aff}_{\mathbb{Z}} \mathsf{vert}\, P^d_{s,2k}$. The circumscribing empty ellipsoid is defined as $\{\mathbf{x} \in \mathbb{R}^d \,|\, \varphi^d_{s,2k}[\mathbf{x}] = R^2\}$, where*

$$\varphi^d_{s,2k}[\mathbf{x}] = 4k(d - 2sk - k)|\mathbf{x}|^2 + \left(d^2 - (4k + 2s + 1)d + 4k(2s + k)\right)(\mathbf{j} \cdot \mathbf{x})^2. \tag{4.1}$$

The origin is the center of symmetry for $P^d_{s,2k}$ and does not belong to $\Lambda^d_{s,2k}$. Thus, each pair of integers (s, k), for $s \geq 1, k \geq 2$ and $s \leq \frac{d}{2k}$, determines an infinite sequence of symmetric perfect Delaunay polytopes, one in each dimension. For $s = 1, k = 2$ the infinite sequence is the one described in the opening commentary, i.e., $G^d_7, d \geq 7$, where the initial term is the Gosset polytope $C^7 = 3_{21}$ with 56 vertices. Let us set

$$\mathbf{D}^d_{s,2k} \triangleq \left\{ \operatorname{conv}\{\mathbf{v}, -\mathbf{v}\} \;\middle|\; \mathbf{v} \in \mathbf{V}^d_{s,2k} \right\}$$

The $\binom{d+1}{s+1}$ diagonals $\mathbf{D}^d_{s,2k} \cup \mathbf{D}^d_{s+1,2k}$ for $P^d_{s,2k}$ have the origin as a common mid-point, forming a segment arrangement that generalizes the cross formed by the diagonals of a cross-polytope. Moreover, these $\binom{d+1}{s+1}$ diagonals are *primitive* as vectors and belong to the same *parity class* for $\Lambda^d_{s,2k}$, namely, they are equivalent modulo $2\Lambda^d_{s,2k}$. More generally, primitive vectors $\mathbf{u}, \mathbf{v}$ in some lattice Λ, with mid-points equivalent modulo Λ, are necessarily equivalent modulo 2Λ and thus belong to the same parity class. And conversely, the mid-points of lattice vectors $\mathbf{u}, \mathbf{v} \in \Lambda$ belonging to the same parity class are equivalent modulo Λ. By analogy with the case of cross-polytopes, we call any such arrangement of segments or vectors a cross. The convex hulls of such crosses often appear as cells in Delaunay tilings – cross polytopes are examples, as are the more spectacular symmetric perfect Delaunay polytopes. There is a criterion, essentially due to Voronoi (1908), but first formulated and formally proved by Baranovskii (1991), that determines whether a cross is Delaunay: *Let Λ be a lattice, φ a positive definite form, C the convex hull of a cross of primitive vectors belonging to the same parity class. Then C is Delaunay relative to φ if and only if the set of vectors forming the cross is the complete set of vectors of minimal length, relative to φ, in their parity class.* This is the criterion we have used to establish the Delaunay property for the symmetric perfect Delaunay polytopes $P^d_{s,2k}$.

The following result shows that asymmetric perfect Delaunay polytopes can appear as sections of symmetric ones.

THEOREM 4.2. *For $d \geq 6$ let $\mathbf{u} = [-1^2; 1^{d-1}] \in \mathbb{Z}^{d+1}$. Then*

$$G^d = \mathrm{conv}\{\, \mathbf{v} \in \mathsf{vert}\, P^{d+1}_{1,4} \;\mid\; \mathbf{v}\cdot\mathbf{u} = \frac{1}{2}\,\}$$

is an asymmetric perfect Delaunay polytope for the point *lattice* $\mathrm{M}^d_{1,4} = \mathrm{aff}_{\mathbb{Z}}\, \mathsf{vert}\, G^6_d$ *with respect to the restriction of the form*

$$8(d-5)|\mathbf{x}|^2 + (d^2 - 9d + 22)(\mathbf{j}\cdot\mathbf{x})^2$$

to $\mathrm{M}^d_{1,4} - \mathrm{M}^d_{1,4}$.

PROOF. By Theorem 4.1 $P^{d+1}_{1,4}$ is Delaunay in $\mathrm{aff}_{\mathbb{Z}}\, \mathsf{vert}\, P^{d+1}_{1,4}$ with respect to $\varphi^{d+1}_{1,4}[\mathbf{x}] = 8(d-5)|\mathbf{x}|^2 + (d^2 - 9d + 22)(\mathbf{j}\cdot\mathbf{x})^2$. Since G^d is a section of $P^{d+1}_{1,4}$, it is also Delaunay. The verification of the perfection property is a straightforward exercise (but see Theorem 4.6 for the general method). □

This theorem describes the infinite sequences of G^d-topes and C^d-topes (where $C^{d+1} = P^{d+1}_{1,4}$) described in the introduction, with Gossett polytope $G^6 = 2_{21}$ as the initial term for $d = 6$. Note that the formula for the form on $\mathrm{M}^d_{1,4}$ is given with respect to $\mathbb{Z}^{d+1}$.

The terms in this sequence have similar combinatorial properties. For example, the lattice vectors running between vertices all lie on the boundary, in all cases. These lattice vectors are either edges of simplicial facets, or diagonals of cross polytope facets–there are two types of facets, simplexes and cross polytopes. The Gossett polytope G^6 has 27 cross-polytopal facets, but for other members of the series G^d, the number of cross-polytopal facets is equal to $2d$. G^6 can be found as a section of G^7, but G^8 and G^9 do not have sections arithmetically equivalent to G^6.

We summarize the properties of both G-series in Table 1. Table 2 (note that entries separated by semicolumns are fixed) gives coordinates of the vertices of G^d-topes (named Υ^d-polytopes there), discovered in 2001 by Erdahl and Rybnikov.

Polytope	$\dim P$	$\lvert\mathsf{vert}\, P\rvert$	*Symmetry*
$P^{d+1}_{1,4}$	$d+1$	$2\left(\binom{d+1}{1}+\binom{d+2}{1+1}\right)=2\binom{d+2}{2}$	centrally-symmetric
$\Upsilon^d = G^d$	d	$\frac{d(d+2)}{2}-1$	*asymmetric*

Table 1. Properties of constructed perfect Delaunay polytopes

$[\mathbf{0^d}]\times 1$	$[-\mathbf{1},\mathbf{0^{d-2}};\mathbf{1}]\times(d-1)$	$[\mathbf{1^{d-1}};-(\mathbf{d-3})]\times 1$
$[\mathbf{0},\mathbf{1^{d-2}};-(\mathbf{d-4})]\times(d-1)$	$[\mathbf{1^2},\mathbf{0^{d-3}};-\mathbf{1}]\times\frac{(d-1)(d-2)}{2}$	$[\mathbf{1},\mathbf{0^{d-2}};\mathbf{0}]\times(d-1)$

Table 2. Vertices of G^d-topes (same as Υ^d) generalizing Gosset's G^6 on 27 vertices (see the beginning of Section 4 for notational conventions.

4.1. Proof of Delaunay property. Let $\mathbf{e}_1,\dots,\mathbf{e}_d$ stand for the canonical basis of $\mathbb{Z}^d$. We will use the following notation: $\varphi_1[\mathbf{x}] = \left(\sum_{i=1}^d x_i\right)^2$, $\varphi_2[\mathbf{x}] = \left|\mathbf{x} - \frac{\sum_{i=1}^d x_i}{d}\mathbf{j}\right|^2$; $\varphi_2[\mathbf{x}]$ is the squared Euclidean distance from $\mathbf{x}$ to the line $\mathbb{R}\mathbf{j}$. The following theorem addresses the Delaunay property of polytopes $P^d_{s,2k}$ asserted by Theorem 4.1.

Theorem 4.3. *Let $d,s,k\in\mathbb{N}$, and let $2\le k$ and $1\le s\le\frac{d}{2k}$. Let also*

$$L^1 = \{[(-1)^k, 1^{d-k}]\}\times\binom{d}{k}$$

$$\Lambda = \mathbb{Z}\left\langle \mathbf{e}_1,\dots,\mathbf{e}_d, \frac{\mathbf{j}}{d-2k}\right\rangle$$

$$\Lambda_1 = \{\mathbf{z}\in\Lambda \mid \mathbf{z}\cdot L^1 \equiv 1 \mod 2\}$$

Then there is a positive definite quadratic form of the type

$$\varphi[\mathbf{x}] = \alpha\varphi_1[\mathbf{x}] + \beta\varphi_2[\mathbf{x}], \tag{4.2}$$

with $\alpha,\beta\in\mathbb{R}_{>0}$, such that $P^d_{s,2k}$ is a Delaunay polytope for the quadratic point lattice (Λ_1,φ).

The proof of Theorem 4.3 will be based on the following two lemmas. For the rest of the paper let $n = d-2k$.

Lemma 4.4. *Suppose $\varphi[\mathbf{x}] = \alpha\varphi_1[\mathbf{x}] + \beta\varphi_2[\mathbf{x}]$ where $\alpha,\beta>0$. Then all points $\mathbf{z}\in\Lambda_1$ which are closest to $\mathbf{0}\notin\Lambda_1$ with respect to φ, ie.,*

$$\varphi[\mathbf{z}] = \min\{\varphi(u) : u\in\Lambda^0\}$$

are, up to permutations of components, of the type

$$\operatorname{sgn}(l)[1^{|l|};0^{d-|l|}] + \frac{a}{n}\mathbf{j},$$

with $-\frac{d}{2}\le l<\frac{d}{2}$, for some $a\in\mathbb{Z}$. Furthermore, each point of Λ_1, closest to the origin, has only one representation of the described type.

PROOF. Suppose $\mathbf{z} \in \Lambda_1$ is such that $\varphi[\mathbf{z}] = \min_{\mathbf{x}\in\Lambda_1} \varphi[\mathbf{x}]$ and let $\mathbf{z} = a_1\mathbf{e}_1 + \cdots + a_d\mathbf{e}_d + a\frac{\mathbf{j}}{n}$, where $a_1, \ldots, a_d, a \in \mathbb{Z}$ and $A = a_1 + \cdots + a_d$. We have

$$\varphi_2[\mathbf{z}] = \varphi_2\left[\sum_{i=1}^{d} a_i\mathbf{e}_i + a\frac{\mathbf{j}}{n}\right] = \left|\sum_{i=1}^{d} a_i\mathbf{e}_i - A\frac{\mathbf{j}}{d}\right|^2 = \sum_{i=1}^{d} a_i^2 - \frac{A^2}{d}. \tag{4.3}$$

Let us prove that the coefficients $\{a_i\}$ are of two consecutive integer values. Suppose, to the contrary, there are a_i and a_j such that $a_i - a_j \geq 2$. Consider the vector $\mathbf{z}' = a_1\mathbf{e}_1 + \cdots + (a_i - 1)\mathbf{e}_i + \cdots + (a_j + 1)\mathbf{e}_j + \cdots + a_d\mathbf{e}_d + a\frac{\mathbf{j}}{n}$. We have $\varphi_2[\mathbf{z}'] - \varphi_2[\mathbf{z}] = (a_i - 1)^2 + (a_j + 1)^2 - a_i^2 - a_j^2 = 2(a_j - a_i) + 2 \leq -2$ and $\mathbf{j} \cdot \mathbf{z}' = \mathbf{j} \cdot \mathbf{z}$. Since $\mathbf{z}' \in \Lambda_1$ and $\varphi[\mathbf{z}'] < \varphi[\mathbf{z}]$, it follows that the vector $\mathbf{z}$ is not closest to $\mathbf{z}$ which is a contradiction.

Now, let b be the smallest of the values of the coefficients $\{a_i\}$. Subtract $b\mathbf{j}$ from the first part and add an equal value of $bk\frac{\mathbf{j}}{n}$ to the second part of the existing representation of $\mathbf{z}$ as an integral linear combination of $\mathbf{e}_i$'s and $\frac{\mathbf{j}}{n}$. After a permutation of the components $\mathbf{z}$ is equal to $\operatorname{sgn}(l)[1^{|l|}, 0^{d-|l|}] + (a + bk)\frac{\mathbf{j}}{n}$ where $0 \leq l < d$. If $l \geq \frac{d}{2}$, again subtract $[1^d]$ from the first summand and add $\mathbf{j}$ to the second summand to get the required representation.

Note that although $\dim \Lambda = d$, we represent points of Λ_1 as integral linear combinations of $d+1$ vectors (plus the origin); thus, we have to understand if the representation described in the statement of the lemma is unique for each of the points of Λ_1 that are closest to the origin. To prove that each point of Λ_1 where φ reaches its minimum has only one encoding of the described type, note that the components of the vector $\operatorname{sgn}(l)[1^{|l|}; 0^{d-|l|}] + a\frac{\mathbf{j}}{n}$ are of at most two values. In $\operatorname{sgn}(l)[1^{|l|}; 0^{d-|l|}]$, either 1's fill the positions of the largest value, or -1's fill the positions of the smallest value. Since $-\frac{d}{2} \leq l < \frac{d}{2}$, the choice is unique. $\square$

LEMMA 4.5. *All elements of the point lattice Λ_1, which are closest to $\mathbf{0} \notin \Lambda_1$ with respect to a form $\varphi = \alpha\varphi_1 + \beta\varphi_2$, where $\alpha, \beta > 0$, belong to the set $\{\mathbf{z} \in \Lambda_1 \mid |\mathbf{z} \cdot \mathbf{j}| \leq \frac{d}{n}\}$. If $\mathbf{z}$ is closest to $\mathbf{0}$ with respect to φ and $|\mathbf{z} \cdot \mathbf{j}| = \frac{d}{n}$, then $\mathbf{z} \in \{\pm\frac{\mathbf{j}}{n}\}$.*

PROOF. By multiplying both α and β by the same positive number, we can assume that the minimal value of φ on Λ_1 is 1. Since $\frac{\mathbf{j}}{n} \in \Lambda_1$, $\varphi[\frac{\mathbf{j}}{n}] = \alpha(\frac{d}{n})^2 \geq 1$. Let $\mathbf{z} \in \mathbb{R}^d$ be a point with $\varphi[\mathbf{z}] = 1$. Represent $\mathbf{z} = \frac{\gamma}{n}\mathbf{j} + \mathbf{u}$ where $\gamma \in \mathbb{R}$, $\mathbf{u} \cdot \mathbf{j} = 0$. We have $\varphi[\mathbf{z}] = \alpha\frac{\gamma^2 d^2}{n^2} + \beta|\mathbf{u}|^2 = 1$, therefore $\alpha\frac{\gamma^2 d^2}{n^2} \leq 1$. Since $\alpha(\frac{d}{n})^2 \geq 1$, we have proven that $|\gamma| \leq 1$, so $|\mathbf{z} \cdot \mathbf{j}| = |\gamma|\frac{d}{n} \leq \frac{d}{n}$.

If the latter inequality holds strictly, then $|\gamma| = 1$ and $\varphi[\mathbf{z}] = \alpha\frac{d^2}{n^2} + \beta|\mathbf{u}|^2 = 1$. Since $\alpha(\frac{d}{n})^2 \geq 1$, we necessarily have $\beta = 0$ so $\mathbf{z} = \pm\frac{\mathbf{j}}{n}$. $\square$

PROOF. (of Theorem 4.3) Let $\varphi_{\alpha,\beta}$ be the type (4.2), i.e. $\varphi_{\alpha,\beta}[\mathbf{x}] = \alpha\varphi_1[\mathbf{x}] + \beta\varphi_2[\mathbf{x}]$. Let us first describe a "reduced" set of points M, such that any point of Λ_1 where the minimum of $\varphi_{\alpha,\beta}$ over Λ_1 is attained, belongs to this set, possibly after a permutation of the components. Then we "handpick" the form $\varphi = \varphi_{\alpha,\beta}$ so that the vertices of $P^d_{s,2k}$ are the minimal *points* and other points of Λ_1 are not. Below is an implementation of this plan.

Consider the set

$$M = \left\{\mathbf{z} \in \Lambda_1 \;\middle|\; \mathbf{z} = \operatorname{sgn}(l)[1^{|l|}; 0^{d-|l|}] + a\frac{\mathbf{j}}{n}, 0 \leq |\mathbf{z} \cdot \mathbf{j}| < \frac{d}{n}, \; -\frac{d}{2} \leq l < \frac{d}{2}\right\} \cup \{\frac{\mathbf{j}}{n}\}. \tag{4.4}$$

By the two previous lemmas, every $\mathbf{z} \in \Lambda_1$, which is closest to the origin of Λ with respect to φ, perhaps after a permutation the components, belongs to this set.

For $\mathbf{z} = \operatorname{sgn}(l)[1^{|l|}; 0^{d-|l|}] + a\frac{\mathbf{j}}{n}$, we have

$$\begin{aligned} \mathbf{z} \cdot \mathbf{j} &= l + \frac{ad}{n}, \\ \varphi_1[\mathbf{z}] &= \left(l + \frac{ad}{n}\right)^2, \\ \varphi_2[\mathbf{z}] &= |l| - \frac{l^2}{d}, \\ L^1 \cdot \mathbf{z} &\equiv a + l \mod 2. \end{aligned} \tag{4.5}$$

From the calculations above, we get that M

(4.6)

$$M = \left\{ \operatorname{sgn}(l)[1^{|l|}; 0^{d-|l|}] + a\frac{\mathbf{j}}{n} \;\middle|\; l + a \equiv 1 \mod 2,\ 0 \le ln + ad < d,\ -\frac{d}{2} \le l < \frac{d}{2} \right\} \cup \{\frac{\mathbf{j}}{n}\}.$$

Thus, we have shown that

(A) *For any point of Λ_1, minimal with respect to $\varphi_{\alpha,\beta}$, there is a permutation of the components that turns it into a member of M given by (4.6).*

We define a mapping from $\mathbb{R}^d$ to $\mathbb{R}^2$ by $m(\mathbf{x}) = (\varphi_1[\mathbf{x}], \varphi_2[\mathbf{x}])$. We will call the image of M the *φ-diagram.* Lines parallel to the φ_1-axis in $\mathbb{R}^2$ will be called horizontal. Next, we will show that

(B) *If $\mathbf{z}_1, \mathbf{z}_2 \in M$, then $m(\mathbf{z}_1)$ and $m(\mathbf{z}_2)$ on the φ-diagram belong to the same horizontal line if and only if $\mathbf{z}_1 \in \pm\mathbf{z}_2$.*

To prove this, take $\mathbf{z} = \operatorname{sgn}(l)[1^{|l|}; 0^{d-|l|}] + a\frac{\mathbf{j}}{n} \in M$. If $l \neq 0$, then the condition $0 \le ln + ad < d$ uniquely determines a for a given value of l, and if $l = 0$, then $a = 1$. Therefore l uniquely defines a.

The function $l \to |l| - \frac{l^2}{d}$ is even and increasing on $[0, \frac{d}{2}]$, so two different points of the φ-diagram may belong to the same horizontal line if and only if their preimages are $\mathbf{z}_1 = [1^{|l|}, 0^{d-|l|}] + a_1\frac{\mathbf{j}}{n}$ and $\mathbf{z}_2 = -[1^{|l|}, 0^{d-|l|}] + a_2\frac{\mathbf{j}}{n}$ for some l,a_1,a_2. If $l \neq 0$, then $0 \le lk + a_1 d < d$, $0 \le -lk + a_2 d < d$. Adding these inequalities, we get $0 \le (a_1 + a_2)d < 2d$, so $a_1 + a_2 \in \{0, 1\}$. If $a_1 + a_2 = 0$, then $\mathbf{z}_1 = -\mathbf{z}_2$ so $m(\mathbf{z}_1) = m(\mathbf{z}_2)$. If $a_1 + a_2 = 1$, then the numbers $l + a_1$ and $-l + a_2$ have different parity which contradicts conditions $l + a_1 \equiv -l + a_2 \equiv 1 \mod 2$. If $l = 0$, then $a_1 = a_2 = 1$. This proves claim (B).

(C) The proof of Delaunay property is based on that each line $\alpha x_1 + \beta x_2 = 1$, where $\alpha, \beta > 0$, which contains an edge of the convex hull of the φ-diagram, gives rise to a quadratic form $\varphi[\mathbf{x}] = \alpha\varphi_1[\mathbf{x}] + \beta\varphi_2[\mathbf{x}]$ such that the ellipsoid $\varphi[\mathbf{x}] \le 1$ contains some points of the point lattice Λ_1 and does not have any points of Λ_1 in its interior.

We are going to show that the following set is the vertex set for a centrally-symmetric Delaunay polytope in Λ_1 with center at the origin of Λ.

$$\mathbf{V}^d_{s,2k} \cup \mathbf{V}^d_{s+1,2k} \cup -\mathbf{V}^d_{s,2k} \cup -\mathbf{V}^d_{s+1,2k} \tag{4.7}$$

A point from $\mathbf{V}^d_{s,2k}$ will be denoted by $\mathbf{v}_{s,2k}$.

(D) First consider the case of $d = 7$, $2k = 4$, $s = 1$. The φ-diagram for these values of parameters is shown in Figure 1 (left). We see that the line $\frac{3}{7}x_1 + \frac{2}{3}x_2 = 1$ passes through points $m(\mathbf{v}_{1,4}) = m([1, 0^6])$ and $m(\mathbf{v}_{2,4}) = m([-1^2, 0^5] + \frac{\mathbf{j}}{3})$, and all other points of the φ-diagram are contained in the open half-plane $\frac{3}{7}x_1 + \frac{2}{3}x_2 > 1$. This means that polytope $P^7_{1,4}$ is a Delaunay polytope with respect to quadratic form $\frac{3}{7}\varphi_1[\mathbf{x}] + \frac{2}{3}\varphi_2[\mathbf{x}] = \frac{2}{3}\mathbf{x}\cdot\mathbf{x} + \frac{1}{3}(\mathbf{j}\cdot\mathbf{x})^2$.

(E) Now suppose that $d \geq 8$, $2 \leq k$, and $1 \leq s \leq \frac{d}{2k}$. Suppose $\mathbf{z} = \operatorname{sgn}(l)[1^{|l|}, 0^{d-|l|}] + a\frac{\mathbf{j}}{n} \in M$ and $0 \leq \varphi_2[\mathbf{z}] \leq \frac{d}{2k} - \frac{d}{4k^2}$ (or, equivalently, $|l| \leq \frac{d}{2k}$). We want to prove the following two statements (E1) and (E2).

(E1) *If $|l| = \frac{d}{2k}$ (clearly, in this case $\frac{d}{2k}$ is an integer), then*

$$\mathbf{z} = \pm\left([1^{\frac{d}{2k}}, 0^{d-\frac{d}{2k}}] - (\frac{d}{2k} - 1)\frac{\mathbf{j}}{n}\right). \tag{4.8}$$

A direct check shows these points belong to set M. Then we use claim (B) that there are at most two points that can be written as

$$[1^{|l|}, 0^{d-|l|}] + a\frac{\mathbf{j}}{n} \in M$$

for any fixed value of $|l|$, and there are exactly two if and only if $lk + ad = 0$, in which case the points are $\pm\mathbf{z}$.

(E2) *If $0 \leq l < \frac{d}{2k}$, then*

$$\mathbf{z} = [1^{|l|}, 0^{d-|l|}] - (l-1)\frac{\mathbf{j}}{n}. \tag{4.9}$$

Recall that $\mathbf{v}_{s,2k}$ stands for $[1^s, 0^{d-s}] - (s-1)\frac{\mathbf{j}}{n}$. We have

$$\varphi_1[\mathbf{v}_{s,2k}] = \left(s + (1-s)\frac{d}{n}\right)^2, \quad \varphi_2[\mathbf{v}_{s,2k}] = s - \frac{s^2}{d}. \tag{4.10}$$

All points $m(\mathbf{v}_{s,2k})$ therefore belong to the parabola

$$t \mapsto \left(\left(t + (1-t)\frac{d}{n}\right)^2, t - \frac{t^2}{d}\right) \tag{4.11}$$

which touches the vertical axis in the point $(0, \frac{d}{2k} - \frac{d}{4k^2})$ when $t = \frac{d}{2k}$. The parabola is shown in dash on Figure 1. The portion of the parabola for $0 \leq t \leq \frac{d}{2k}$ is the graph of a convex function.

Therefore, for each $1 \leq s \leq \frac{d}{2k}$ we can find a line with equation $\alpha x_1 + \beta x_2 = 1$ which passes through points $m(\mathbf{v}_{s,2k})$, $m(\mathbf{v}_{s+1,2k})$, supports the convex hull of the φ-diagram and does not contain points $m(\mathbf{z})$ for $\mathbf{z} \in M \setminus \{\pm\mathbf{v}_{s,2k} \cup \pm\mathbf{v}_{s+1,2k}\}$. Quadratic form $\varphi[\mathbf{x}] = \alpha\varphi_1[\mathbf{x}] + \beta\varphi_2[\mathbf{x}]$ defines an empty ellipsoid centered at $\mathbf{0}$, which contains the vertices of $P^d_{s,2k}$ on its boundary, and does not contain any other points of Λ_1. An example of the φ-diagram for $d = 19$, $2k = 6$ is shown in Figure 2 (right-hand image). Note that not all points of $m(M)$ belong to the parabola (4.11).

We have proven that $P^d_{s,2k}$ is a Delaunay polytope in the point lattice Λ_1 with respect to quadratic form $\varphi = \varphi^d_{s,2k}$ for $d \geq 7$. Explicit formula (4.1) for $\varphi^d_{s,2k}$ is established by a direct calculation. □

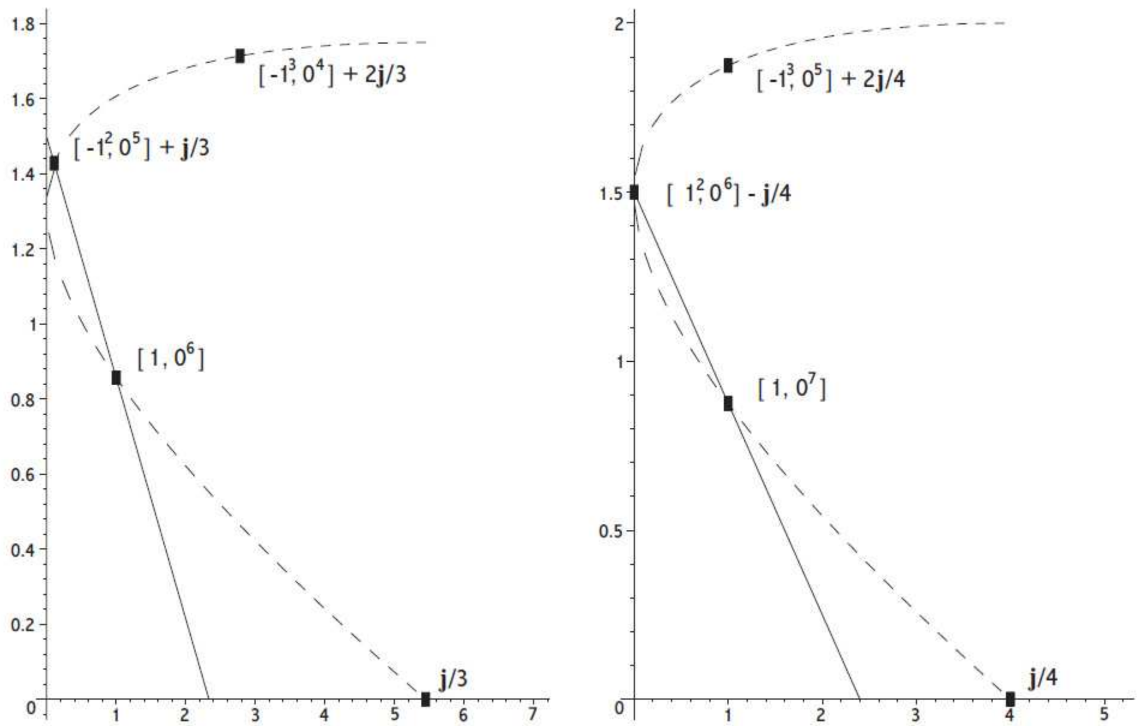

FIGURE 1. φ-diagrams for $d = 7$ (left) and $d = 8$ (right); $2k = 4$

4.2. Proof of perfection property.

THEOREM 4.6. *Let d, k, $n = d - 2k$, s, $\alpha\varphi_1 + \beta\varphi_2$ be as in Theorem 11. Then, there is at most one pair consisting of a quadratic form $\varphi = \alpha\varphi_1 + \beta\varphi_2$ and $\mathbf{c} \in \mathbb{R}^d$ such that the quadric $\varphi[\mathbf{x} - \mathbf{c}] = 1$ circumscribes the polytope $P^d_{s,2k}$.*

PROOF. Since $\dim P^d_{s,2k} = d$ and $P^d_{s,2k}$ has $\mathbf{0}$ as the center of symmetry, $\mathbf{0}$ is the center of the circumscribing quadric and $\mathbf{c} = \mathbf{0}$. Therefore, if φ and ψ are two quadratic forms such that the corresponding quadrics circumscribe $P^d_{s,2k}$, then $(\varphi - \psi)|_{\mathbf{V}^d_{s,2k} \cup \mathbf{V}^d_{s+1,2k}} = 0$. Let us consider an arbitrary quadratic form ξ such that $\xi|_{\mathbf{V}^d_{s,2k} \cup \mathbf{V}^d_{s+1,2k}} = 0$ and prove that $\xi \equiv 0$: this will prove the theorem.

We will use the following symmetrization technique. Let G be a subgroup of the group S_d of permutations on the d coordinates. If $\tau \in G$, then we write $\tau : i \mapsto i\tau$. The symmetrization of the form ξ by G is defined as

$$\mathtt{Sym}_G\,\xi[\mathbf{x}] = \sum_{\tau \in G} \xi(x_{1\tau}, \dots, x_{d\tau}) = \sum_{\tau \in G} \xi[\mathbf{x}\tau] \qquad (4.12)$$

Since the polytope $P^d_{s,2k} \subset \mathbb{R}^d$ is invariant under permutations of the coordinates of $\mathbb{R}^d$, all components in the above sum are 0 on $\mathsf{vert}\, P^d_{s,2k}$. Therefore $\xi|_{\mathbf{V}^d_{s,2k} \cup \mathbf{V}^d_{s+1,2k}} = 0$.

First we prove that if a form ξ satisfies the condition $\xi|_{\mathbf{V}^d_{s,2k} \cup \mathbf{V}^d_{s+1,2k}} = 0$, then the sum of the diagonal elements of ξ, and the sum of off-diagonal elements are 0. We symmetrize ξ by the group $G = S_d$ and get a form $\mathtt{Sym}_G\,\xi$ with diagonal coefficients proportional to the sum

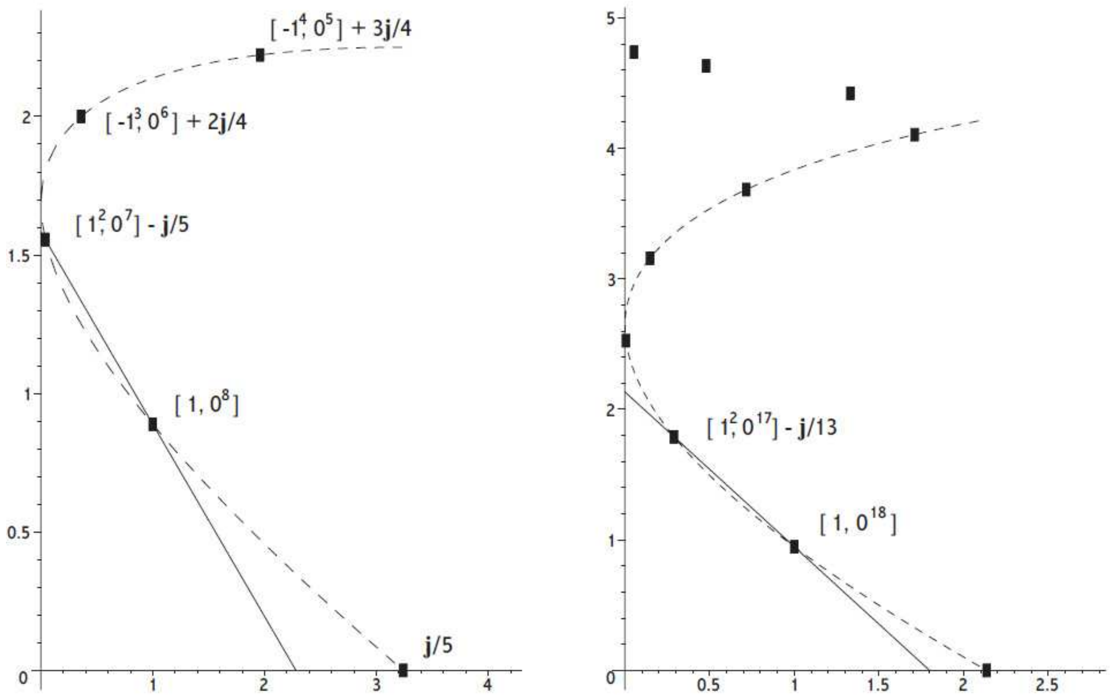

FIGURE 2. φ-diagrams for $d = 9$ (left) and $d = 19$ (right); $2k = 4$

of diagonal coefficients of ξ, and non-diagonal coefficients proportional to the sum of non-diagonal coefficients of ξ. Therefore $\mathtt{Sym}_G\,\xi[\mathbf{x}]$ can be written as $\alpha\varphi_1[\mathbf{x}] + \beta\varphi_2[\mathbf{x}]$. Suppose that α and β are not both equal to 0, i.e. $\alpha^2 + \beta^2 > 0$. Since $\mathtt{Sym}_G\,\xi|_{\mathbf{V}^d_{s,2k}\cup\mathbf{V}^d_{s+1,2k}} = 0$, the line $\alpha x_1 + \beta x_2 = 0$ passes through the points $m(\mathbf{V}^d_{s,2k})$ and $m(\mathbf{V}^d_{s+1,2k})$, where $m(\mathbf{x}) = (\varphi_1[\mathbf{x}], \varphi_2[\mathbf{x}])$. This is impossible because the line that passes through these points is uniquely defined and does not contain 0. This contradiction proves our claim.

Next we prove that all diagonal elements of the form ξ are equal to 0. Suppose that one of them is nonzero. Without a limitation of generality we may assume that $t = \Xi_{11} \neq 0$, where Ξ is the Gram matrix of ξ. Let $St(1)$ be the subgroup of permutations which leave 1 fixed. The Gram matrix of $\mathtt{Sym}_{St(1)}\,\xi$ is

$$\begin{bmatrix} t & \beta & \beta & . & . & \beta \\ \beta & \alpha & \delta & . & . & \delta \\ \beta & \delta & \alpha & \delta & . & \delta \\ \dots & & & & & \\ \beta & \delta & . & \delta & \alpha & \delta \\ \beta & \delta & . & . & \delta & \alpha \end{bmatrix} \tag{4.13}$$

Consider the following vectors:

$$\mathbf{v}_1 = [1^s; 0^{d-s}] - (s-1)\frac{\mathbf{j}}{n} \in \mathbf{V}^d_{s,2k} \quad \mathbf{v}_2 = [1^{s+1}; 0^{d-s-1}] - s\frac{\mathbf{j}}{n} \in \mathbf{V}^d_{s+1,2k}.$$

The conditions $\mathtt{Sym}_{St(1)}\,\xi(\mathbf{v}_1) = 0$, $\mathtt{Sym}_{St(1)}\,\xi(\mathbf{v}_2) = 0$ yield

$$t + 2(s-1)\beta + (s-1)\alpha + (s-1)(s-2)\delta - \frac{2(s-1)(t+(s+d-2)\beta + (s-1)\alpha + (s-1)(d-2)\delta)}{n} + \frac{(s-1)^2(t+2(d-1)\beta + (d-1)\alpha + (d-1)(d-2)\delta)}{n^2} = 0, \tag{4.14}$$

$$t + 2s\beta + s\alpha + s(s-1)\delta - \frac{2s(t+(s+d-1)\beta + s\alpha + s(d-2)\delta)}{n} + \frac{s^2(t+2(d-1)\beta + (d-1)\alpha + (d-1)(d-2)\delta)}{n^2} = 0. \tag{4.15}$$

We also have the conditions that the sums of the diagonal and the off-diagonal elements are equal to 0:

$$t + (d-1)\alpha = 0, (d-1)(d-2)\delta + 2(d-1)\beta = 0. \tag{4.16}$$

The determinant of the system of these 4 equations in variables α, β, δ and t is equal to

$$\frac{2}{n}(d-1)(s-d+1)(s-d)(d-2-n) \tag{4.17}$$

and it is not equal to 0 for the specified values of d, n and s. Hence $t = 0$.

We have shown that the diagonal coefficients of ξ are all zero. Next we prove that all off-diagonal elements of ξ are also equal to 0. Without loss of generality we can assume that $\Xi_{12} \neq 0$. Let $\mathtt{Sym}_{St(1)}\,\xi$ be the symmetrization of ξ by the group of permutations which map the set $\{1, 2\}$ onto itself. The Gram matrix of $\mathtt{Sym}_{St(1)}\,\xi$ is then

$$\begin{bmatrix} 0 & \alpha & \beta & \dots & \dots & \beta \\ \alpha & 0 & \beta & \dots & \dots & \beta \\ \beta & \beta & 0 & \delta & \dots & \delta \\ \dots & \dots & \delta & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta & \beta & \dots & \dots & 0 & \delta \\ \beta & \beta & \delta & \dots & \delta & 0 \end{bmatrix}, \tag{4.18}$$

where $\alpha \neq 0$. We consider the following vectors:

$$\mathbf{u}_1 = [1^{s+1}; 0^{d-s-1}] - s\frac{\mathbf{j}}{n} \in \mathbf{V}^d_{s+1,2k} \quad \mathbf{u}_2 = [0^{d-s-1}; 1^{s+1}] - s\frac{\mathbf{j}}{n} \in \mathbf{V}^d_{s+1,2k}$$

The equations $\mathtt{Sym}_{St(1)}\,\xi[\mathbf{u}_1] = 0$, $\mathtt{Sym}_{St(1)}\,\xi[\mathbf{u}_2] = 0$ yield

$$4(s-1)\beta+2\alpha+(s-1)(s-2)\delta-2\frac{s}{n}(2\alpha+2(s-1+d-2)\beta+(s-1)(d-3)\delta)+\frac{(s-1)^2(2\alpha+4(d-2)\beta+(d-2)(d-3)\delta)}{n^2}=0, \tag{4.19}$$

$$(s+1)s\delta-\frac{2(s+1)s(2\beta+(d-3)\delta)}{n}+\frac{s^2(2\alpha+4(d-2)\beta+(d-2)(d-3)\delta)}{n^2}=0. \tag{4.20}$$

We also know that the sum of off-diagonal elements of the matrix of $\mathtt{Sym}_{St(1)}\,\xi$ is equal to 0:

$$2\alpha+4(d-2)\beta+(d-2)(d-3)\delta=0 \tag{4.21}$$

so, with some simplifications, the previous two equations can be rewritten as

$$\begin{aligned} 4(s-1)\beta+2\alpha+(s-1)(s-2)\delta-2\frac{s}{n}(2\alpha+2(s+d-3)\beta+(s-1)(d-3)\delta)&=0,\\ \delta-\frac{2(2\beta+(d-3)\delta)}{n}&=0. \end{aligned} \tag{4.22}$$

The systems of equations (4.21) and (4.22) in variables α, β and δ has determinant $8(d-2-n)(-d+s+1)$ which is not equal to 0 for the specified parameters d, n and s which proves that the system has only zero solution. In particular, $\alpha=0$. We have proven that all off-diagonal elements of the matrix of form ξ are equal to 0. □

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