On Voronoi's two tilings of the cone of metrical forms

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Abstract

In his last two papers Georges Voronoi introduced two tilings for the cone of metrical forms for lattices, the tiling by perfect domains and the tiling by lattice type domains. Both are facet-to-facet tilings by polyhedral subcones, and invariant with respect to the natural action of $GL(n, \mathbb{Z})$ on $Sym(n, \mathbb{R})$. In working out the details of his theory of lattice types, his most important legacy for geometers, Voronoi observed that the tiling by lattice types refines the tiling by perfect domains in the case of 2-, 3- and 4-dimensional lattices. The technical advantage gained in this way allowed him to work out the details of his classification theory for four-dimensional lattices. The work of all researchers following Voronoi relied heavily on this hypothesis, that the tiling by lattice types refines the tiling by perfect domains. In particular, the classification of the 5-dimensional generic lattice types by Ryshkov and Baranovski would not have been possible without invoking this hypothesis. In this note we show that this hypothesis does not hold in six dimensions, and therefore in all higher dimensions.

This paper is an announcement of results and will be followed by a more complete treatment where all proofs are given.

This paper is dedicated to Peter Gruber on the occassion of his sixtieth birthday.

1 Introduction

Let Φ be the cone of metrical forms. That is, Φ is the cone of non-negative quadratic forms in n variables, so that if $\varphi \in \Phi$, then

$$\varphi(\mathbf{x}) \geq 0, \mathbf{x} \in \mathbb{R}^n.$$

The interior points of Φ are positive definite forms, and each boundary form is semi-definite with the rank of the coefficient matrix less than n.

Throughout our discussion we will be considering the fixed lattice $\mathbb{Z}^n$. Each positive definite metrical form $\varphi \in \Phi$ determines a Voronoi and Delaunay tilings for $\mathbb{Z}^n$, which is an affine variant of the usual Voronoi and Delaunay tilings for lattices, and can be described as follows. The Voronoi polytope V_φ, centered at the origin and determined by φ, is the set of points in $\mathbb{R}^n$ no further from the origin than from any other point in $\mathbb{Z}^n$, with distance measured using φ. The Voronoi tiling relative to φ is generated by taking all $\mathbb{Z}^n$ translates of V_φ. The tiles for Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^n$ are constructed using "empty ellipsoids" determined by φ; the details are given below.

More general lattices can be studied using metrical forms and the lattice $\mathbb{Z}^n$. An arbitrary geometric lattice $\Lambda \subset \mathbb{R}^n$ can be mapped onto $\mathbb{Z}^n$ by an affine transformation. This same transformation maps the unit ball at the origin determined by the Euclidean metric to an ellipsoid, which in turn determines a metrical form φ up to a scale factor. Consequently, the Voronoi and Delaunay tilings for Λ determined by the standard Euclidean metric are mapped by this affinity to Voronoi and Delaunay tilings for $\mathbb{Z}^n$ determined by φ.

The correspondence we have just described, between lattices and metrical forms, was used in the 1831 commentary by Gauss on the work of Seeber [7], where the notion of a geometric lattice was linked for the first time to the integral theory of quadratic forms. This is the central correspondence on which Minkowski's geometry of numbers is based, and is also central for Voronoi's classification theory for lattices, his theory of *lattice types* [11]. More precisely, there is a natural one-to-one correspondence between the isometry classes of $n-$dimensional lattices and integral equivalence classes of positive metrical forms in n variables - metrical forms φ_1 and φ_2 are said to be integally equivalent if there is an element $\mathbf{U} \in GL(n, \mathbb{Z})$ such that $\varphi_1(\mathbf{x}) = \varphi_2(\mathbf{Ux})$.

The correspondence between geometric lattices and metrical forms give alternate ways to study the geometric properties of lattices. For a general lattice the lengths of a basis and the angles between basis vectors can be varied and changes in the structures of the Voronoi and Delaunay tilings tracked, these tilings being determined by the standard Euclidean metric. Equivalently, the lattice can be fixed at $\mathbb{Z}^n$ and the metrical from varied. The changes in the Voronoi and Delaunay tilings relative to φ are then tracked as the metrical form varies over Φ. We have adopted this second perspective. More specifically, we consider metrical forms for the root lattices E_6 and E_6^* and the associated Delaunay tilings, and determine how the tilings vary as the forms are perturbed in certain directions.

Voronoi's two tilings for Φ. Voronoi studied two facet-to-facet tilings of Φ, where the individual tiles are polyhedral subcones. The first is the tiling by *perfect domains* [10].

Consider a positive definite form $\pi \in \Phi$. Then the minimal vectors for π are given by

$$\mathsf{P}_\pi = \{\mathbf{p} \in \mathbb{Z}^n | \pi(\mathbf{p}) = m\},$$

where m is the arithmetic minimum for π - the minimal value on the non-zero elements of $\mathbb{Z}^n$. For each set of minimal vectors $\mathsf{P} \subset \mathbb{Z}^n$ there is a P-domain defined by

$$\Phi_{\mathsf{P}} = \{\sum_{\mathbf{p}\in\mathsf{P}} \omega_{\mathbf{p}}\varphi_{\mathbf{p}} | \omega_{\mathbf{p}} > 0\},$$

where $\varphi_{\mathbf{p}}(\mathbf{x}) = (\mathbf{p}\cdot\mathbf{x})^2$. The P-domains are the relatively open cells for the *tiling by perfect domains*. The union of the P-domains of all dimensions gives a (disjunctive) partition

$$\bigsqcup_{\mathsf{P}} \Pi_{\mathsf{P}} = \mathrm{conv}\{(\mathbf{p}\cdot\mathbf{x})^2 \mid \mathbf{p} \in \mathbb{Q}^n\} \subset \Phi,$$

where the union is taken over all sets of minimal vectors for non-negative forms in n variables.. The P-domains with full dimension are relatively open polyhedral cones that fit together facet-to-facet to tile Φ, and the P-domains with lesser dimension are relatively open faces of this tiling. In particular, the one-dimensional faces are the semi-infinite rays generated by taking all positive multiples of the edge forms $\varphi_{\mathbf{p}}(\mathbf{x}) = (\mathbf{p}\cdot\mathbf{x})^2, \mathbf{p} \in \mathbb{Z}^n$.

If Φ_{P} has full dimension $\binom{n+1}{2}$, the edge forms $\varphi_{\mathbf{p}}(\mathbf{x}) = (\mathbf{p}\cdot\mathbf{x})^2, \mathbf{p} \in \mathsf{P}$ span the space of metrical forms. Equivalently, if π_{P} is a positive form with arithmetic minimum equal to m, and with minimal vectors equal to P, then π_{P} is the unique quadratic form satisfying the system of equations $\pi(\mathbf{p}) = m, \mathbf{p} \in \mathsf{P}$. Positive forms with this property, and the corresponding domains, are called *perfect*.

Perfect forms play an important role in lattice sphere packings. For example, extreme forms are those giving local maxima for the packing density, and Korkine and Zolotareff showed [8] that extreme forms must also be perfect. Voronoi improved on this result by showing that a form is extreme if and only if it is both perfect and eutactic [10] (for a definition and a discussion of the eutaxy condition in the contex of the material presented in this paper, see [6]).

Voronoi's second tiling for Φ is the *tiling by lattice type domains*, or more simply, *L-domains* [11]. Each form $\varphi \in \Phi$ determines a Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^n$, and each Delaunay tiling $\mathcal{D}$ for $\mathbb{Z}^n$ determines an *L-domain* $\Phi_{\mathcal{D}}$ of forms, which is defined by

$$\Phi_{\mathcal{D}} = \{\varphi \in \Phi | \mathcal{D}_\varphi = \mathcal{D}\}.$$

The L-domains are the relatively open cells for the *tiling by lattice type domains*. The union of the L-domains of all dimensions gives a (disjunctive) partition

$$\bigsqcup_{\mathcal{D}} \Phi_{\mathcal{D}} = \mathrm{conv}\{(\mathbf{p}\cdot\mathbf{x})^2 \mid \mathbf{p} \in \mathbb{Q}^n\} \subset \Phi,$$

where the union if over all possible Delaunay tilings for positive semi-definite forms in n variables.

The L-domains with full dimension $\binom{n+1}{2}$ are relatively open polyhedral cones, which fit together facet-to-facet to form the second tiling of Φ. These domains are labeled by Delaunay "triangulations" - Delaunay tilings where each cell is a relatively open simplex. The L-domains with lesser dimension are relatively open faces, and correspond to Delaunay tilings with cells that are not simplexes.

The Delaunay tiling for $\mathbb{Z}^n$, relative to $\varphi \in \operatorname{int}\Phi$, is a tiling where the cells are lattice polytopes characterized by the Delaunay property: *the lattice polytope P is Delaunay if and only if there is a circumscribing empty ellipsoid $\mathcal{E}_P$, with equation $\varphi(\mathbf{x}-\mathbf{c}_P) = R_P^2$, such that the vertices of P are given by $\mathcal{E}_P \cap \mathbb{Z}^n$, and no elements of $\mathbb{Z}^n$ are interior to $\mathcal{E}_P$.* The $n-$dimensional cells are relatively open lattice polytopes that fit together facet-to-facet to tile space, and the cells of lesser dimension are relatively open faces of these tiles. If φ is positive semi-definite and has rational kernel (i.e. $\varphi \in \operatorname{conv}\{(\mathbf{p}\cdot\mathbf{x})^2 \mid \mathbf{p} \in \mathbb{Q}^n\} \setminus \operatorname{int}\Phi$), then the Delaunay tiling consists of lattice *polyhedra* characterized by the Delaunay property: *the lattice polyhedron P is Delaunay if and only if there is a circumscribing empty elliptic cylinder $\mathcal{E}_P$, with equation $\varphi(\mathbf{x}-\mathbf{c}_P) = R_P^2$, such that the vertices of P are given by $\mathcal{E}_P \cap \mathbb{Z}^n$, and no elements of $\mathbb{Z}^n$ are interior to $\mathcal{E}_P$.*

Delaunay tilings and L-domains play an important role in the lattice covering problem. Barnes and Dickson [1], Dickson [4], and then Delaunay's group at Steklov Institute [3], investigated extreme lattice coverings by spheres. The most important result achieved in these investigations was that the closure of any L-domain corresponding to a Delaunay triangulation contains at most one local minimum of the sphere covering density.

Voronoi's hypothesis and his theory of lattice types. Voronoi's geometric program was to classify lattices by using the structure of their Delaunay tilings, a program now refered to as his *Theory of Lattice Types* [11]. The key step in executing this program was to describe the second tiling by L-domains for lattices in the dimension under consideration. For this he needed the following finiteness result for P- and L-domains, which he proved:

1. The tilings by P-domains, and by L-domains, are invariant with respect to the natural $GL(n,\mathbb{Z})$ action on Φ;

2. There are finitely many integrally inequivalent P-domains and L-domains.

He further tacitly invoked the following hypothesis.

Voronoi's Hypothesis *The tiling by L- domains refines the tiling P- domains.*

Under conditions where this hypothesis holds the enumeration problem for L-domains is reduced to more manageable subproblems: the tiling by P-domains is first described, and then, the tiling of each P-domain by L-domains is described. In the hands of Voronoi this *reduction via perfect forms* was employed with great force, and resulted in a complete description of all lattice types in four dimensions. This reduction provides a fundamental link between Voronoi's research on perfect forms [10] and his work on lattice types [11], and attracted the attention of all geometers that followed. Many speculated that Voronoi's Hypothesis held in all dimensions, and others only hoped this to be the case. Voronoi gave no indication what his beliefs in this regard were.

Voronoi proved that for $n = 2, 3$ the tiling by perfect and lattice type domains coincide. Voronoi also proved that for $n = 4$ the tiling by lattice type domains strictly refines the tiling by perfect domains. The perfect domain D_4 (the 2nd perfect form in 4 variables) introduces a new phenomena, namely, that 64 simplicial L-domains tile D_4 much in the way that the pieces of a pie fill the pie plate. In all the work on Voronoi's program that followed geometers relied heavily on Voronoi's Hypothesis. Delaunay [2] reworked the four-dimensional case, giving a more vivid description by introducing his "empty sphere" method. Ryshkov and Baranovski [9] showed that Voronoi's Hypothesis holds in five dimensions, and for this reason were able to classify the 5-dimensional simplicial L-domains and solve the lattice covering problem in this dimension. In his paper of 1972 T.J. Dickson proved that the perfect domain of A_n, called the *first perfect form*, is the only P-domain with full dimension that is also an L-domain [5]. In this paper he also explicitly formulated the Voronoi Hypothesis as a conjecture, and was the first to do so.

The main result. We show that Voronoi's hypothesis does not hold in six dimensions by constructing an explicit counter-example. This shows that Voronoi's strategy of "reduction via perfect domains" is of limited use for lattices in six dimensions or higher. This article is intended to only announce this result, and proofs are often sketched or even absent. A complete treatment will be available in the future [6].

Commensurate and incommensurate lattice tilings. In order to construct the counter-example we introduce the notion of *commensurate* and *incommensurate lattice tilings*. We will call a polytope $P \subset \mathbb{R}^n$ a lattice polytope if its vertices belong to the fixed lattice $\mathbb{Z}^n$, and are given by $P \cap \mathbb{Z}^n$; we do not require that P has full dimension. Similarly we will call a facet-to-facet tiling $\mathcal{T}$ of $\mathbb{R}^n$ a lattice tiling if the tiles are lattice polytopes; the tiles for a lattice tiling are full-dimensioned lattice polytopes.

If P_1 and P_2 are lattice polytopes then the intersection polytope $P_1 \cap P_2$ need not be a lattice polytope, and similarly, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are lattice tilings then the intersection tiling $\mathcal{T}_1 \cap \mathcal{T}_2$ need not be a lattice tiling.

Definition *Lattice polytopes P_1, P_2 are commensurate if and only if the intersection polytope $P_1 \cap P_2$ is a lattice polytope. Similarly, lattice tilings $\mathcal{T}_1, \mathcal{T}_2$ are commensurate if and only if the intersection tiling $\mathcal{T}_1 \cap \mathcal{T}_2$ is a lattice tiling.*

If lattice tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ are incommensurate, then $\mathcal{T}_1 \cap \mathcal{T}_2$ has a non-integral vertex, say $\mathbf{v}$. Since the relatively open cells of the intersection tiling $\mathcal{T}_1 \cap \mathcal{T}_2$ are the intersections of relatively open cells of $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathbf{v}$ can be prepresented as $\mathbf{v} = C_1 \cap C_2$, where the cells $C_1 \in \mathcal{T}_1, C_2 \in \mathcal{T}_2$ are relatively open. This establishes the following criteria for incommensurability: lattice tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ are incommensurate if and only if there are relatively open cells $C_1 \in \mathcal{T}_1, C_2 \in \mathcal{T}_2$ so that $C_1 \cap C_2$ is a non-integral point.

The Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^n$ relative to the form φ is an example of a lattice tiling. Such a lattice tiling is invariant with respect to arbitrary $\mathbb{Z}^n$-translates, and inversion through the origin. Therefore, if P is a tile in $\mathcal{D}_\varphi$, then all $\mathbb{Z}^n$-translates of P are tiles, and all $\mathbb{Z}^n$-translates of $-P$ are tiles.

Definition *Lattice polytopes P_1 and P_2 are homologous if P_2 is a $\mathbb{Z}^n$-translate of either P or $-P$. The homology class of a lattice polytope P is the set of all lattice translates of P, and all lattice translates of $-P$.*

The notion of a homology class of lattice polytopes will prove convenient when we enumerate the tiles for some particular Delaunay tilings below.

2 A pair of adjacent perfect domains

Consider the two sets of vectors $\mathsf{E}_6^* = \mathsf{P}_1 \cup \mathsf{P}_2$ and $\mathsf{E}_6 = \mathsf{P}_2 \cup \mathsf{P}_3$, where

$$\begin{aligned}
\mathsf{P}_1 &= \{\pm[-3, 2^5], \pm[2, -2, -1^4], \pm[1, 0, -1^4]\},\\
\mathsf{P}_2 &= \{\pm[2, -1^5], \pm[-2, 1; 2, 1^3] \times 4, \pm[-1, 1; 0, 1^3] \times 4, \pm[0; 1, 0^4] \times 5,\\
&\quad \pm [1; 0^2, -1^3] \times 10, \pm[2, -1^5]\},\\
\mathsf{P}_3 &= \{\pm[0, 0; 1, -1, 0^2] \times 6, \pm[-1, 0; 1^2, 0^2] \times 6\}.
\end{aligned}$$

A compact notation is used where entries following a semi-colon should be permuted to form a set of vectors. For example, $\pm[1; 0^2, -1^3] \times 10$ is an $\mathcal{A}_5-$invariant set of ten vectors, where $\mathcal{A}_5$ is permutation of the last five coordinates, and similarly, $\pm[0, 0; 1, -1, 0^2] \times 6$ is an $\mathcal{A}_4-$invariant set of 12 vectors. These are the sets of minimal vectors for the *perfect forms* $\pi_{\mathsf{E}_6^*}$ and π_{E_6}, with coefficient matrices

$$\mathbf{P}_{\mathsf{E}_6^*} = \frac{m}{4}\begin{bmatrix} 16 & 5 & 5 & 5 & 5 & 5\\ 5 & 4 & 1 & 1 & 1 & 1\\ 5 & 1 & 4 & 1 & 1 & 1\\ 5 & 1 & 1 & 4 & 1 & 1\\ 5 & 1 & 1 & 1 & 4 & 1\\ 5 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}, \qquad \mathbf{P}_{\mathsf{E}_6} = \frac{m}{2}\begin{bmatrix} 8 & 1 & 3 & 3 & 3 & 3\\ 1 & 2 & 0 & 0 & 0 & 0\\ 3 & 0 & 2 & 1 & 1 & 1\\ 3 & 0 & 1 & 2 & 1 & 1\\ 3 & 0 & 1 & 1 & 2 & 1\\ 3 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

As is indicated by the notation, these are metrical forms for the root lattice E_6 and its dual E_6^*. This follows from the observation that $|\mathsf{E}_6| = 72$ and $|\mathsf{E}_6^*| = 54$.

The domain $\Phi_{\mathsf{E}_6^* \cap \mathsf{E}_6}$. The perfect domains $\Phi_{\mathsf{E}_6^*}$ and Φ_{E_6} have full dimension 21, and $\Phi_{\mathsf{E}_6^* \cap \mathsf{E}_6}$ is a boundary cell lying between them. The forms $(1-t)\pi_{\mathsf{E}_6^*} + t\pi_{\mathsf{E}_6}, 0 \le t \le 1$

have arithmetic minimum m, and the vectors $\mathsf{E}_6^* \cap \mathsf{E}_6 = \mathsf{P}_2$ are minimal vectors for the intermediate forms on this line segment.

Proposition 1 *The P-domain $\Phi_{\mathsf{E}_6^* \cap \mathsf{E}_6}$ is a facet in the tiling by perfect domains.*

Proof. This follows because the linear span of the forms $\varphi_{\mathbf{p}}(\mathbf{x}) = (\mathbf{p} \cdot \mathbf{x})^2$, $\mathbf{p} \in \mathsf{E}_6 \cap \mathsf{E}_6^*$ has co-dimension one. ■

3 G-topes and G*-topes

Let $\mathcal{G}$ and $\mathcal{G}^*$ be the Delaunay tilings determined by $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}^*}$, where

$$\varphi_{\mathcal{G}}(\mathbf{x}) = \frac{m}{12} \sum_{\mathbf{p} \in \mathsf{E}_6^*} (\mathbf{p} \cdot \mathbf{x})^2, \qquad \varphi_{\mathcal{G}^*}(\mathbf{x}) = \frac{m}{16} \sum_{\mathbf{p} \in \mathsf{E}_6} (\mathbf{p} \cdot \mathbf{x})^2.$$

These are the tilings by G-topes and G*-topes, which are described below.

The forms $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}^*}$ lie on the central axes of the perfect domains $\oplus_{\mathsf{E}_6^*}$ and Φ_{E_6}, have arithmetic minimum equal to m, and are edge forms for the tiling by lattice type domains. Any form, φ', that is close to $\varphi_{\mathcal{G}}$, but not a scalar multiple of it, has a Delaunay tiling that refines $\mathcal{G}$, and similarly, any form close to $\varphi_{\mathcal{G}}$, other than a scalar multiple, has a Delaunay tiling that refines $\mathcal{G}^*$. That is, $\Phi_{\mathcal{G}} = \{\alpha\varphi_{\mathcal{G}} | \alpha \in \mathbb{R}^+\}$, and $\Phi_{\mathcal{G}^*} = \{\alpha\varphi_{\mathcal{G}^*} | \alpha \in \mathbb{R}^+\}$, and $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}^*}$ are edge forms for the tiling by lattice type domains.

The short and long vectors. The vectors in $\mathbb{Z}^6$ that fit in the tiles of $\mathcal{G}$ or $\mathcal{G}^*$, running between vertices, are either *short* or *long vectors* relative to the corresponding metrical forms. The short vectors are the minimal vectors for $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}^*}$, and given by $\mathsf{S}_{\mathcal{G}} = \mathsf{S}_2 \cup \mathsf{S}_3$, $\mathsf{S}_{\mathcal{G}^*} = \mathsf{S}_1 \cup \mathsf{S}_2$, where

$$\begin{aligned}
\mathsf{S}_1 &= \{\pm[2,-1,1^4], \pm[-2,0,-1^4], \pm[0,1,0^4]\} \\
\mathsf{S}_2 &= \{\pm[3;1^5], \pm[0,1;-1,0^3] \times 4, \pm[2,0;0,1^3] \times 4, \pm[1;1,0^4] \times 5, \\
&\quad \pm[1;1^2,0^3] \times 10\} \\
\mathsf{S}_3 &= \{\pm[0,0;1,-1,0^2] \times 6, \pm[2,1;1^2,0^2] \times 6\}.
\end{aligned}$$

The vectors in $\mathsf{S}_{\mathcal{G}}$ are edge vectors for G-topes, which are the tiles in $\mathcal{G}$, and the vectors in $\mathsf{S}_{\mathcal{G}^*}$ are edge vectors for G*-topes, which are the tiles in $\mathcal{G}^*$.

The set of 270 long vectors for $\varphi_{\mathcal{G}}$ is defined by

$$\mathsf{L}_{\mathcal{G}} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{\mathcal{G}}(\mathbf{z}) = 2m\},$$

where $2m$ is the *second minimum* for $\varphi_{\mathcal{G}}$, the minimal value assumed on the non-zero elements of $\mathbb{Z}^6$ not belonging to $\mathsf{S}_{\mathcal{G}}$. As described below, these appear as diagonals of G-topes - they run between vertices of a G-tope, but are not edges.

The 72 long vectors for $\varphi_{\mathcal{G}^*}$ are defined by

$$\mathsf{L}_{\mathcal{G}^*} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{\mathcal{G}^*}(\mathbf{z}) = \frac{3}{2}m\},$$

where $\frac{3}{2}m$ is the second minimum for $\varphi_{\mathcal{G}^*}$. These are edge vectors of G*-topes.

These short vectors for $\varphi_{\mathcal{G}}, \varphi_{\mathcal{G}^*}$ follow the same pattern as the minimal vectors for π_{E_6} and $\pi_{\mathsf{E}_6^*}$, and are related by the arithmetical equivalences $\mathsf{S}_{\mathcal{G}} = \mathbf{U}_*^{-1}\mathsf{E}_6$, $\mathsf{S}_{\mathcal{G}^*} = \mathbf{U}_*^{-1}\mathsf{E}_6^*$, where

$$\mathbf{U}_* = \begin{bmatrix} 2 & -2 & 0 & -1 & -1 & -1 \\ -2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \mathbf{U}_*^{-1} = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}.$$

The corresponding forms are related by the formulas $\varphi_{\mathcal{G}}(\mathbf{x}) = \pi_{\mathsf{E}_6}(\mathbf{U}_*\mathbf{x})$, $\varphi_{\mathcal{G}^*}(\mathbf{x}) = \pi_{\mathsf{E}_6^*}(\mathbf{U}_*\mathbf{x})$.

Being arithmetically equivalent, $\varphi_{\mathcal{G}}$ and π_{E_6} are alternate metrical forms for the same geometric lattice E_6, and similarly, $\varphi_{\mathcal{G}^*}$ and $\pi_{\mathsf{E}_6^*}$ are metrical forms for the geometric lattice E_6^*. The Delaunay tilings for $\mathbb{Z}^6$, relative to either $\varphi_{\mathcal{G}}$ or π_{E_6} are affinely equivalent the Delaunay tiling for the root lattice E_6, and the Delaunay tilings relative to either $\varphi_{\mathcal{G}^*}$ or $\pi_{\mathsf{E}_6^*}$ are affinelty equivalent to the Delaunay tiling for the dual lattice E_6^*.

Tiling space with G-topes. Let G be the convex hull of the 45 $\mathsf{L}_{\mathcal{G}}-$*triangles*

$$\begin{aligned}
\Delta_G^1 &= conv\{[0;0^5],[0;1,0^4],[2;0,1^4]\} \times 5, \\
\Delta_G^2 &= conv\{[3;1^5],[-1;-1,0^4],[0;1,0^4]\} \times 5, \\
\Delta_G^3 &= conv\{[1;0,1^2,0^2],[1;0^3,1^2],[0;1,0^4]\} \times 15, \\
\Delta_G^4 &= conv\{[1;1^2,0^3],[-1;0,-1,0^3],[2;0,1^4]\} \times 20,
\end{aligned}$$

which have a common centroid $\mathbf{c}_G = \frac{1}{3}[2;1^5]$. As is indicated by the notation, these inscribed triangles belong to four $\mathcal{A}_5-$classes of size $5, 5, 15$, and 20. There are 135 edges, and 135 pairs of opposite edge vectors, which belong to $\mathsf{L}_{\mathcal{G}}$. Therefore, the 45 long triangles inscribed in G give a geometric accounting for the $270 = 45 \times 6$ long vectors for $\varphi_{\mathcal{G}}$.

We will refer to G as the reference G-tope. More generally: *a lattice polytope P is a G-tope if and only if it is the convex hull of forty five $\mathsf{L}_{\mathcal{G}}-$triangles with a common centroid.*

Proposition 2 *The collection of all G-topes gives a facet-to-facet lattice tiling, which is the Delaunay tiling $\mathcal{G}$ for $\mathbb{Z}^6$ relative to the form $\varphi_{\mathcal{G}}$. The G-topes belong to a single homology class.*

The lattice polytope G is the famous *Gosset polytope* with 27 vertices. Fifty-four homologous copies fit around the origin, facet-to-facet, to form the star of G-topes at the origin.

There are ten vertices in G that are at a distance $2m$ from the origin relative to $\varphi_{\mathcal{G}}$. These are the vertices $\{\mathbf{r}_1, ..., \mathbf{r}_5; \mathbf{b}_1, ..., \mathbf{b}_5\} = \{[0; 1, 0^4] \times 5; [2; 0, 1^4] \times 5\}$, which belong to the five inscribed triangles with a vertex at the origin. They are also the vertices of a five-dimensional cross-polytope. The five diagonals intersect at $\frac{1}{2}[2; 1^5]$, and the ten diagonal vectors are given by

$$\{\pm(\mathbf{b}_1 - \mathbf{r}_1), \pm(\mathbf{b}_2 - \mathbf{r}_2), ..., \pm(\mathbf{b}_5 - \mathbf{r}_5)\} = \{\pm[2; -1, 1^4] \times 5\}.$$

This cross-polytope is a facet of G.

The situation is identicle for each of the 27 vertices of G. Each has five long triangles attached to it, and the ten vertices at distance $2m$ are the vertices of a cross-polytope facet of G. Each cross-polytope has 10 diagonal vectors, so there are a total of $27 \times 10 = 270$ diagonal vectors. Since these diagonals belong to $\mathsf{L}_{\mathcal{G}}$, this gives a second accounting for the 270 elements of $\mathsf{L}_{\mathcal{G}}$. Besides the cross-polytope facets there are 72 simplicial facets. The edges of the cross-polytope, and simplicial facets are short relative to the form $\varphi_{\mathcal{G}}$.

Tiling space with G*-topes. Let G^* be the convex hull of the three $\mathsf{L}_{\mathcal{G}^*}-$ *triangles*

$$\begin{aligned}
\Delta^1_{G^*} &= conv\{[0^6], [0^4, 1, 0], [0^5, 1]\}, \\
\Delta^2_{G^*} &= conv\{[2, 0, 1^4], [-1, 0^2, -1, 0^2], [-1, 0, -1, 0^3]\}, \\
\Delta^3_{G^*} &= conv\{[0, 1, 0^4], [1, 0^3, 1^2], [-1^2, 0^4]\},
\end{aligned}$$

with the common centroid $\mathbf{c}_T = \frac{1}{3}[0^4, 1^2]$. These long triangles belong to complementary $2-$spaces, so G^* has full dimension. The edge vectors belong to $\mathcal{L}_{E_6^*}$.

We will refer to G^* as the reference G*-tope. More generally: *a lattice polytope P is a G^*-tope if and only if it is the convex hull of three $\mathsf{L}_{\mathcal{G}^*}-$triangles with a common centroid.*

Proposition 3 *The collection of all G^*-topes gives a facet-to-facet lattice tiling, which is the Delaunay tiling $\mathcal{G}^*$ for $\mathbb{Z}^6$ relative to the form $\varphi_{\mathcal{G}^*}$. The G^*-topes belong to forty homology classes.*

Since each G*-tope has nine vertices, each homology class accounts for 18 G*-topes with a vertex at the origin. In total, there are $40 \times 18 = 720$ G*-topes that fit around the origin, facet-to-facet, to form the star at the origin.

The facets of G^* are easily visualized. They are the convex hull of one edge from each of the long triangles inscribed in G^*, and are simplicial. Accordingly there are 27 such facets, each with three long and twelve short edges relative to the form $\varphi_{\mathcal{G}^*}$. All facets of G*-topes have identicle descriptions.

4 T-topes, Q-topes and R-topes

We now initiate a discussion on how the two Delaunay tilings $\mathcal{G}$ and $\mathcal{G}^*$ intersect. This requires distinguishing two subspecies of G*-tope, the *T-tope* and the *Q-tope*, which differ in their intersection properties with G-topes. This also requires introducing a new species of lattice polytope, the *R-tope*.

T-topes and Q-topes. The reference T-tope is G^*, which is contained in G, and therefore commensurate with $\mathcal{G}$. For this reason it is convenient to switch notation from G^* to T, and rename the inscribed triangles as Δ_T^1, Δ_T^2, Δ_T^3. The reference Q-tope is Q, which is the convex hull of the three $\mathsf{L}_{\mathcal{G}^*}-$triangles

$$\begin{aligned}
\Delta_Q^1 &= conv\{[0^3,1,0^2],[0^4,1,0],[0^5,1]\},\\
\Delta_Q^2 &= conv\{[-1,0,-1,0^3],[-1,0^5],[2,0,1^4]\},\\
\Delta_Q^3 &= conv\{[1,0^2,1^3],[-1^2,0^4],[0,1,0^4]\}.
\end{aligned}$$

These three triangles have edges that are long relative to $\varphi_{\mathcal{G}^*}$, and have the common centroid $\mathbf{c}_Q = \frac{1}{3}[0^3,1^3]$; these properties establish that Q is a G*-tope.

The edges of triangles inscribed in G*-topes are long relative to $\varphi_{\mathcal{G}^*}$, but their lengths vary relative to the form $\varphi_{\mathcal{G}}$. Each of the triangles Δ_T^1, Δ_T^2, Δ_T^3 in T has two edges that are long relative to $\varphi_{\mathcal{G}}$, and a single edge that is short. On the other hand, the triangle Δ_Q^1 in Q has edges that are short, and Δ_Q^2, Δ_Q^3 have edges that are long relative to $\varphi_{\mathcal{G}}$. This observation leads to the following definition: *A G*-tope: (1) is a T-tope if and only if each inscribed triangle has one edge that is short relative to $\varphi_{\mathcal{G}}$, and two that are long; (2) is a Q-tope if and only if one inscribed triangle has edges that are short relative to $\varphi_{\mathcal{G}}$, and two inscribed triangles have edges that are long.*

A straight forward counting exercise shows that there are 120 homology classes of lattice triangles with edges that are long relative to $\varphi_{\mathcal{G}^*}$. Of these triangles there are 72 homology classes with one short edge relative to $\varphi_{\mathcal{G}}$, and two long edges; there are 16 homology classes of triangles where all edges are short relative to $\varphi_{\mathcal{G}}$, and 32 where all edges are long. Since $72+16+32=120=3\times 40$ these classes account for all the long triangles inscribed in G*-topes.

Since there are 40 homology classes of G*-tope, the centroids of the 120 homology classes of $\mathsf{L}_{\mathcal{G}^*}-$triangles must belong to only 40 homology classes. Straighforward enumeration shows that there are two classes of centroids: (1) 24 homology classes are the centroids of three $\mathsf{L}_{\mathcal{G}^*}-$triangles with one edge that is short relative to $\varphi_{\mathcal{G}}$, and two that are long; (2) 16 homology classes are the centroids of a single $\mathsf{L}_{\mathcal{G}^*}-$triangle that has short edges relative to $\varphi_{\mathcal{G}}$, and two $\mathsf{L}_{\mathcal{G}^*}-$triangles with edges that are long relative to $\varphi_{\mathcal{G}}$. That is, the 120 homology classes of $\mathsf{L}_{\mathcal{G}^*}-$triangles form 24 homology clases of T-topes, and 16 homology classes of Q-topes.

Proposition 4 *The* 40 *homology classes of G^*-topes are divided between* 24 *homology classes of T-topes, and* 16 *homology classes of Q-topes. Each T-tope is commensurate with the tiling by G-topes, and each Q-tope in incommensurate.*

Proof. The first assertion follows from the enumerations exercised that was sketched immediately before the statement of the Proposition. Regarding the statement of commensurability we only argue for the case for the reference T-tope and Q-tope, T and Q. The details for arbirtrary T-topes and Q-topes can be found in [[6]].

We have already observed that $T \subset G$, so is commensurate with $\mathcal{G}$. A more intricate argument is required to show that Q is incommensurate with $\mathcal{G}$. Consider the simplicial 3−faces $S_Q \subset Q$ and $S_G \subset G$. The simplex $S_Q = conv\{[-1, 0^5], [2, 0, 1^4], [-1^2, 0^4], [0, 1, 0^4]\} \subset Q$ is the convex hull of an edge from each of the triangles Δ^2_Q, Δ^3_Q, so is a 3−face of Q. The centroid $\frac{1}{4}[0^2, 1^4]$ is also the centroid of the simplex $S_G = conv\{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\}$, which is a 3−face of G. These simplexes lie in complementary 3−spaces, so satisfy the condition that $S_Q \cap S_G = \frac{1}{4}[0^2; 1^4]$. The relatively open cells of the intesection polytope $Q \cap G$ are the intersections of relatively open cells of Q and G. Therefore, the common centroid is a vertex of $Q \cap G$. Since this vertex is non-integral, Q is incommensurate with G, and incomensurate with $\mathcal{G}$. ■

This Proposition established that the two lattice tilings $\mathcal{G}$ and $\mathcal{G}^*$ are incommensurate.

R-topes. The intersection of the 3−cells $S_G = conv\{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\} \subset G$ and $S_Q = conv\{[-1, 0^5], [2, 0, 1^4], [-1^2, 0^4], [0, 1, 0^4]\} \subset Q$ illustrates the action that results when the lattice tilings $\mathcal{G}$ and $\mathcal{G}^*$ are intersected. Since $S_G \cap S_Q = \frac{1}{4}[0^2; 1^4]$, this point is a non-integral vertex of both $Q \cap G$ and the intersection tiling $\mathcal{G} \cap \mathcal{G}^*$. This was the key step in the proof of *Proposition 3* showing that Q is incommensurate with G, and therefore that $\mathcal{G}$ is incommensurate with $\mathcal{G}^*$.

In order to describe the intersection tiling $\mathcal{G} \cap \mathcal{G}^*$, and in particular to enumerate the non-integral vertices of this tiling, we introduce a new species of lattice polytope, the *R-tope*. We define the reference R-tope to be $R = conv(S_Q \cup S_G)$, which has the non-integral vertex $\frac{1}{4}[0^2; 1^4]$ as centroid. More generally: *a lattice polytope P is an R-tope if and only if it is isometrically equivalent to R with repect to both $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}^*}$.*

Proposition 5 *All non-integral vertices of the intersection tiling $\mathcal{G} \cap \mathcal{G}^*$ are centroids of R-topes. There are* 12 *homology classes of R-topes, and therefore* 12 *homology classes of non-integral vertices.*

The proof of this proposition is lengthy and given in [6].

5 Repackaging Q-topes

By dissecting each Q-tope into simplexes, and reassembling a portion of these into R-topes, we construct a lattice tiling $\mathcal{R}$ that is commensurate with both $\mathcal{G}$ and $\mathcal{G}^*$. This construction requires a detailed description on how R-topes intersect with both $\mathcal{G}$ and $\mathcal{G}^*$, and on how Q-topes intersect with $\mathcal{G}$. This is achieved using G-stars and G*-stars: *for $C^* \in \mathcal{G}^*$, let $star_{\mathcal{G}^*}(C^*)$ be the collection of G*-topes incident at C^*, and for $C \in \mathcal{G}$, let $star_{\mathcal{G}}(C)$ be the collection of G-topes incident at C.*

The geometry of R-topes. Since there are eight vertices, $R = conv(S_Q \cup S_G)$ can be triangulated in just two ways. By taking the convex hull of S_Q with each facet of S_G , four *blue simplexes* are formed, which tile R; by taking the convex hull of S_G with each facet of S_Q , four *yellow simplexes* are formed, which also tile R. The first triangulation is the star of simplexes in R incident at S_Q, and the second is the star of simplexes in R incident at S_G. Lattice polytopes that can be triangulated in just two ways, such as R, are refered to as *repartitioning complexes* in the literature on lattice Delaunay tilings.

More explicitely, the four facets of S_G are given by

$$\begin{aligned}
F_G^1 &= conv\{[0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},\\
F_G^2 &= conv\{[0^2, 1, 0^3], [0^4, 1, 0], [0^5, 1]\},\\
F_G^3 &= conv\{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^5, 1]\},\\
F_G^4 &= conv\{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^4, 1, 0]\},
\end{aligned}$$

and the blue simplexes are given by $B^i = conv(S_Q \cup F_G^i), 1 \le i \le 4$. The four facets of S_Q are given by

$$\begin{aligned}
F_Q^1 &= conv\{[2, 0, 1^4], [-1^2, 0^4], [0, 1, 0^4]\},\\
F_Q^2 &= conv\{[-1, 0^5], [-1^2, 0^4], [0, 1, 0^4]\},\\
F_Q^3 &= conv\{[-1, 0^5], [2, 0, 1^4], [0, 1, 0^4]\},\\
F_Q^4 &= conv\{[-1, 0^5], [2, 0, 1^4], [-1^2, 0^4]\},
\end{aligned}$$

and the yellow simplexes are given by $Y^i = conv(F_Q^i \cup S_G), 1 \le i \le 4$.

Notice that $B^1 = conv(S_Q \cup F_G^1) \subset Q \in star_{\mathcal{G}^*}(S_Q)$. Permutation of the last four coordinates generates the other three blue simplexes, and three Q-topes containing them that belong to $star_{\mathcal{G}^*}(S_Q)$. Therefore, $star_{\mathcal{G}^*}(S_Q) \cap R = \{B^1, ..., B^4\}$. Similarly, $Y^1 = conv(F_Q^1 \cup S_G) \subset G \in star_{\mathcal{G}}(S_G)$. There are four homologous copies of S_G on the boundary of G, and therefore four G-topes in $star_{\mathcal{G}}(S_G)$. Each of these G-topes contains a yellow simplex, and therefore $star_{\mathcal{G}}(S_G) \cap R = \{Y^1, ..., Y^4\}$. These formulas show that R is commensurate with both $\mathcal{G}$ and $\mathcal{G}^*$.

The intersection properties described for R hold for all R-topes: *If R' is an arbitrary R-tope, then R' can be tiled by four blue simplexes, $B'^1, ..., B'^4$, so that*

$R \cap \mathcal{G}^* = \{B'^1, ..., B'^4\}$, and can be tiled by four yellow simplexes, $Y'^1, ..., Y'^4$, so that $R \cap \mathcal{G}^* = \{Y'^1, ..., Y'^4\}$. Therefore, R' is commensurate with both $\mathcal{G}^*$ and $\mathcal{G}$.

The geometry of Q-topes. There are three ways to tile Q by lattice simplexes. The star of simplexes in Q that are incident at Δ_Q^1 is such a tiling; the simplexes are formed by taking the convex hull of Δ_Q^1 with each of the nine facets of $conv\{\Delta_Q^2 \cup \Delta_Q^3\}$. The other tilings are the stars of simplexes incident at either Δ_Q^2 or Δ_Q^3. Of the inscribed triangles it is only Δ_Q^1 that is a closed $2-$cell in $\mathcal{G}$, so it is $star_{\mathcal{G}}(\Delta_Q^1)$ that is useful in determining how Q intersects with $\mathcal{G}$.

Each facet of $conv\{\Delta_Q^2 \cup \Delta_Q^3\}$ can be represented as the convex hull of an edge of Δ_Q^2, and an edge of Δ_Q^3. These facets are given by

$$
\begin{aligned}
F_Q^{11} &= conv\{[-1, 0^5], [2, 0, 1^4], [-1^2, 0^4], [0, 1, 0^4]\},\\
F_Q^{22} &= conv\{[-1, 0, -1, 0^3], [2, 0, 1^4], [1, 0^2, 1^3], [0, 1, 0^4]\},\\
F_Q^{33} &= conv\{[-1, 0, -1, 0^3], [-1, 0^5], [1, 0^2, 1^3], [-1^2, 0^4]\},\\
F_Q^{21} &= conv\{[-1, 0, -1, 0^3], [2, 0, 1^4], [-1^2, 0^4], [0, 1, 0^4]\},\\
F_Q^{31} &= conv\{[-1, 0, -1, 0^3], [-1, 0^5], [-1^2, 0^4], [0, 1, 0^4]\},\\
F_Q^{12} &= conv\{[-1, 0^5], [2, 0, 1^4], [1, 0^2, 1^3], [0, 1, 0^4]\},\\
F_Q^{32} &= conv\{[-1, 0, -1, 0^3], [-1, 0^5], [1, 0^2, 1^3], [0, 1, 0^4]\},\\
F_Q^{13} &= conv\{[-1, 0^5], [2, 0, 1^4], [1, 0^2, 1^3], [-1^2, 0^4]\},\\
F_Q^{23} &= conv\{[-1, 0, -1, 0^3], [2, 0, 1^4], [1, 0^2, 1^3], [-1^2, 0^4]\};
\end{aligned}
$$

the two superscripts index the edges of the triangles Δ_Q^2 and Δ_Q^3. Define three *blue simplexes* by $B^{ii} = conv\{\Delta_Q^1 \cup F_Q^{ii}\}, 1 \leq i \leq 3$, and six *white simplexes* by $W^{ij} = conv\{\Delta_Q^1 \cup F_Q^{ij}\}, 1 \leq i \neq j \leq 3$. These are the simplexes incident at Δ_Q^1 that tile Q.

The three simplexes

$$
\begin{aligned}
S_G^{11} &= conv\{[0^2, 1, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},\\
S_G^{22} &= conv\{[2, 1, 0, 1^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},\\
S_G^{33} &= conv\{[-2, -1^2, 0^3], [0^3, 1, 0^2], [0^4, 1, 0], [0^5, 1]\},
\end{aligned}
$$

are homologous to $3-$faces of G. The centroids $\mathbf{c}^{11}, \mathbf{c}^{22}, \mathbf{c}^{33}$, are also the centroids of the faces F_Q^{11}, F_Q^{22}, F_Q^{33}, and satisfy the equalities $\mathbf{c}^{11} = \frac{1}{4}[0^2, 1^4] = S_G^{11} \cap F_Q^{11}$, $\mathbf{c}^{22} = \frac{1}{4}[2, 1, 0, 2^3] = S_G^{22} \cap F_Q^{22}$, $\mathbf{c}^{33} = \frac{1}{4}[-2, -1^2, 1^3] = S_G^{33} \cap F_Q^{33}$. Using the definition, $R^{ii} = conv(S_G^{ii} \cup F_Q^{ii}), 1 \leq i \leq 3$, can be identifies as R-topes. It follows that $B^{ii} = R^{ii} \cap Q, 1 \leq i \leq 3$.

The simplex $W^{21} = conv\{\Delta_Q^1 \cup F_Q^{21}\} \subset G \in star_{\mathcal{G}}(\Delta_Q^1)$. There are six homologous copies of Δ_Q^1 on the boundary of G, and therefore six G-topes in $star_{\mathcal{G}}(S_G)$. By inspection, each of these G-topes contains one of the white simplexes $W^{ij}, 1 \leq i \neq$

$j \leq 3$. Therefore, each of the six white simplexes is commensurate with both the tiling by G-topes, and by G*-topes.

The intersection properties for Q are characteristic of any of the Q-topes: *If Q' is an arbitrary Q-tope, then Q' can be tiled by three blue and six white simplexes. The blue simplexes are the intersections of Q' with R-topes, and the white simplexes are commensurate with both the tiling by G-topes, and tiling by G*-topes.*

The commensurate tiling $\mathcal{R}$. Since there are 16 homology classes of Q-topes, and each contains 3 blue simplexes, there are 48 homology classes of blue simplexes. The 12 homology classes of R-topes gives a second accounting, since each R-tope is tiled by four blue simplexes. This allows a repackaging of the blue simplexes contained in Q-topes, into R-topes. Besides the blue simplexes, each Q-tope contains 6 white simplexes, so there are a total of 96 homology classes of white simplexes. The portion of space tiled by the 16 classes of Q-topes, can equally well be tiled by the 96 homology classes of white simplexes, and 12 homology classes of R-topes.

Proposition 6 *The* 24 *homology classes of T-topes, the* 12 *homology classes of R-topes, and the* 96 *homology classes of white simplexes fit together facet-to-facet to tile space. This lattice tiling $\mathcal{R}$ is commensurate with both $\mathcal{G}$ and $\mathcal{G}^*$.*

Proof. Since the T-topes and Q-topes tile space, repackaging the Q-topes into R-topes and white simplexes gives the required tiling $\mathcal{R}$. That this tiling is commensurate with both the tilings $\mathcal{G}$ and $\mathcal{G}^*$ follows from the intersection properties of the T-topes, the R-topes and the white simplexes. ■

By the intersection properties of the R-topes it follows that in the intersection tiling $\mathcal{G} \cap \mathcal{R}$ the 12 homology classes of R-topes are replaced by 48 homology classes of yellow simplexes, and in the intersection tiling $\mathcal{R} \cap \mathcal{G}^*$ these R-topes are replaced by 48 homology classes of blue simplexes.

6 Counter-example for Voronoi's Hypothesis

Consider the line segment $\varphi_t = (1-t)\varphi_{\mathcal{G}} + t\varphi_{\mathcal{G}^*}$, $0 \leq t \leq 1$, running between the forms $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{G}^*}$. The counter-example to Voronoi's Hypothesis requires a description of the Delaunay tilings along the line segment φ_t, $0 < t < 1$.

The form $\varphi_{\mathcal{R}}$. Suppose that the reference $R = conv(S_G \cup S_Q)$ is Delaunay with respect to the metrical form φ. Then there is a scalar c and vector $\mathbf{c}$ so that $f_R(\mathbf{x}) = c + \mathbf{c} \cdot \mathbf{x} + \varphi(\mathbf{x}) = 0$ is the equation of an empty ellipsoid circumscribing R. If V_G, V_Q are the vertex sets for the component triangles S_G and S_Q, then the centroid of R, and of each of the component triangles, is given by $\mathbf{c}_R = \frac{1}{4}\sum_{\mathbf{v}\in V_G} \mathbf{v} =$

$\frac{1}{4}\sum_{\mathbf{v}\in V_Q}\mathbf{v} = \frac{1}{4}[0^2, 1^4]$. Since f_R is zero on each vertex of R, the metrical form φ satisfies the condition

$$\begin{aligned}
0 &= \sum_{\mathbf{v}\in V_G} f_R(\mathbf{v}) - \sum_{\mathbf{v}\in V_Q} f_R(\mathbf{v}) \\
&= \mathbf{p}\cdot\left(\sum_{\mathbf{v}\in V_G}\mathbf{v} - \sum_{\mathbf{v}\in V_Q}\mathbf{v}\right) + \sum_{\mathbf{v}\in V_G}\varphi(\mathbf{v}) - \sum_{\mathbf{v}\in V_Q}\varphi(\mathbf{v}) \\
&= \sum_{\mathbf{v}\in V_G}\varphi(\mathbf{v}) - \sum_{\mathbf{v}\in V_Q}\varphi(\mathbf{v}).
\end{aligned}$$

This is the equation of a hyperplane H_R, and positive definite forms φ' on this hyperplane determine ellipsoids $\mathcal{E}_{\varphi'}$, which contain the vertices of R. All positive definite metrical forms for which R is Delaunay lie on this hyperplane, but the converse is not true. For an arbitrary positive definite metrical form $\varphi' \in H_R$ additional elements of $\mathbb{Z}^6$ might lie on $\mathcal{E}_{\varphi'}$, or even be interior to $\mathcal{E}_{\varphi'}$.

Explicit form for this hyperplane can be given by introducing a scalar product: if forms π, φ have coefficient matrices, $\mathbf{P}_\pi$ and $\mathbf{F}_\varphi$, then let $\langle\pi,\varphi\rangle = trace\mathbf{P}_\pi\mathbf{F}_\varphi$. If π_R is the form with matrix

$$\begin{aligned}
\mathbf{P}_R &= \sum_{\mathbf{v}\in V_G}\mathbf{v}\mathbf{v}^T - \sum_{\mathbf{v}\in V_Q}\mathbf{v}\mathbf{v}^T \\
&= -\begin{bmatrix} 6 & 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},
\end{aligned}$$

then the equation for the hyperplane is given by

$$\begin{aligned}
\langle\pi_R,\varphi\rangle &= trace\mathbf{P}_R\mathbf{F}_\varphi = \sum_{\mathbf{v}\in V_G}\mathbf{v}^T\mathbf{F}_\varphi\mathbf{v} - \sum_{\mathbf{v}\in V_Q}\mathbf{v}^T\mathbf{F}_\varphi\mathbf{v} \\
&= \sum_{\mathbf{v}\in V_G}\varphi(\mathbf{v}) - \sum_{\mathbf{v}\in V_Q}\varphi(\mathbf{v}) \\
&= 0.
\end{aligned}$$

The line segment $\varphi_t = (1-t)\varphi_{\mathcal{G}} + t\varphi_{\mathcal{G}^*}$, $0 \le t \le 1$, pierces H_R when t satisfies the equation $< \pi_R, \varphi_t >= 0$. Since

$$\begin{aligned}
\langle \pi_R, \varphi_t \rangle &= \langle \pi_R, (1-t)\varphi_{\mathcal{G}} + t\varphi_{\mathcal{G}^*} \rangle \\
&= (1-t) \times trace \mathbf{P}_R \mathbf{F}_{\varphi_{\mathcal{G}}} + t \times trace \mathbf{P}_R \mathbf{F}_{\varphi_{\mathcal{G}^*}} \\
&= (1-t)(-\frac{1}{2}m) + t(\frac{1}{2}m) \\
&= -\frac{1}{2}m + mt,
\end{aligned}$$

it follows that $\varphi_{\frac{1}{2}} \in H_R$.

Similarly, each R-tope R' determines a form $\pi_{R'}$, which in turn corresponds to a hyperplane $H_{R'}$ with equation $< \pi_{R'}, \varphi >= 0$. Hyperplanes corresponding to homologous R-topes are equal, so there are twelve such hyperplanes in total. Moreover, each of these hyperplanes intersects the line segment $\varphi_t = (1-t)\varphi_{\mathcal{G}} + t\varphi_{G^*}$, $0 \le t \le 1$, at the same point $\varphi_{\frac{1}{2}}$.

Proposition 7 *The commensurate tiling $\mathcal{R}$ is Delaunay relative to the form $\varphi_{\mathcal{R}} = \varphi_{\frac{1}{2}}$. That is, $\mathcal{R} = \mathcal{D}_{\varphi_R}$.*

The proof of this Proposition requires some additional material not included in this announcement, but will be published in [[6]].

Commensurate Delaunay tilings. The construction of the counter-example requires the following result on pairs of commensurate Delaunay tilings.

Lemma 8 *Consider forms $\varphi, \vartheta \in \Phi$, and the correspoding Delaunay tilings $\mathcal{D}_\varphi$ and $\mathcal{D}_\vartheta$. If $\mathcal{D}_\varphi$ is commensurate with $\mathcal{D}_\vartheta$, then $\mathcal{D}_{(1-t)\varphi+t\vartheta} = \mathcal{D}_\varphi \cap \mathcal{D}_\vartheta$, for $0 < t < 1$.*

Proof. Consider two $d-$dimensional Delaunay polytopes $P_\varphi \in \mathcal{D}_\varphi$, $P_\vartheta \in \mathcal{D}_\vartheta$, with a common interior point. Since the two tilings are commensurate, $P_\varphi \cap P_\vartheta$ is a lattice polytope with vertices belonging to $\mathbb{Z}^n$. Since P_φ, P_ϑ, are Delaunay, there are scalars c_φ, c_ϑ, and vectors $\mathbf{c}_\varphi, \mathbf{c}_\vartheta \in \mathbb{R}^d$, so that $f_\varphi(\mathbf{x}) = c_\varphi - \mathbf{c}_\varphi \cdot \mathbf{x} + \boldsymbol{\varphi}(\mathbf{x}) = 0$ is the equation of an ellipsoid circumscribing P_φ, and $f_\vartheta(\mathbf{x}) = c_\vartheta - \mathbf{c}_\vartheta \cdot \mathbf{x} + \boldsymbol{\vartheta}(\mathbf{x}) = 0$ is the equation of an ellipsoid circumscribing P_ϑ. These functions are non-negative on $\mathbb{Z}^n$, and therefore the same holds for $(1-t)f_\varphi + tf_\vartheta, 0 < t < 1$. Since the only elements of $\mathbb{Z}^n$ where f_φ is zero valued are the vertices of P_φ, and where f_ϑ is zero valued are the vertices of P_ϑ, the only elements of $\mathbb{Z}^n$ where f_t is zero valued are the vertices of $P_\varphi \cap P_\vartheta$. Hence, $P_\varphi \cap P_\vartheta$ is Delaunay relative to the each of the forms $(1-t)\varphi + t\vartheta, 0 < t < 1$.

Similarly, each intersection polytope in $\mathcal{D}_\varphi \cap \mathcal{D}_\vartheta$ is Delaunay relative to $(1-t)\varphi + t\vartheta$. Therefore, $\mathcal{D}_\varphi \cap \mathcal{D}_\vartheta$ is Delaunay relative to each of the forms $(1-t)\varphi + t\vartheta, 0 < t < 1$. ∎

This Lemma allows a complete description of the Delaunay tilings relative to the forms $\varphi_t = (1-t)\varphi_{\mathcal{G}} + t\varphi_{\mathcal{G}^*}, 0 \leq t \leq 1$. Since *Proposition 7* establishes that $\mathcal{R}$ is Delaunay relative to the form $\varphi_{\frac{1}{2}} = \varphi_{\mathcal{R}}$, and since $\mathcal{R}$ is commensurate with both $\mathcal{G}$ and $\mathcal{G}^*$, it follows that: *For* $0 < t < \frac{1}{2}$, $\mathcal{D}_{\varphi_t} = \mathcal{G} \cap \mathcal{R}$, *and for* $\frac{1}{2} < t < 1$, $\mathcal{D}_{\varphi_t} = \mathcal{R} \cap \mathcal{G}^*$. More explicitly, the tiling for $\varphi_{\frac{1}{2}}$ has 24 homology classes of T-topes, 96 classes of white simplexes, and 12 homology classes of R-topes. For $0 < t < \frac{1}{2}$ this tiling is modified by replacing the 12 homology classes of R-topes by 48 classes of yellow simplexes, and for $\frac{1}{2} < t < 1$ the R-topes are replaced by 48 homology classes of blue simplexes.

This action along the line can be interpreted in terms of the L-domains that partition Φ - five separate L-domains are crossed. In order, they are $\Phi_{\mathcal{G}}, \Phi_{\mathcal{G}\cap\mathcal{R}}, \Phi_{\mathcal{R}}, \Phi_{\mathcal{R}\cap\mathcal{G}^*}, \Phi_{\mathcal{G}^*}$. The L-domains $\Phi_{\mathcal{G}}, \Phi_{\mathcal{R}}$ are boundary cells for $\Phi_{\mathcal{G}\cap\mathcal{R}}$, which is traversed by the open segment $\varphi_t, 0 < t < \frac{1}{2}$, and the L-domains $\Phi_{\mathcal{R}}, \Phi_{\mathcal{G}^*}$ are boundary cells for $\Phi_{\mathcal{R}\cap\mathcal{G}^*}$, which is traversed by the open segment $\varphi_t, \frac{1}{2} < t < 1$.

Counter-example to the Voronoi Hypothesis. Consider the facet $\Phi_{\mathsf{E}_6^*\cap\mathsf{E}_6}$, the form $\pi_0 = \pi_{\mathsf{E}_6^*} - \pi_{\mathsf{E}_6}$, and the rank one forms $\varphi_{\mathbf{p}}(\mathbf{x}) = (\mathbf{p}\cdot\mathbf{x})^2, \mathbf{p} \in \mathbb{Z}^n$. Then, $\langle \pi_0, \varphi_{\mathbf{p}} \rangle = \langle \pi_{\mathsf{E}_6^*} - \pi_{\mathsf{E}_6}, \varphi_{\mathbf{p}} \rangle = trace(\mathbf{P}_{\mathsf{E}_6^*} - \mathbf{P}_{\mathsf{E}_6})\mathbf{p}\mathbf{p}^T = \mathbf{p}^T(\mathbf{P}_{\mathsf{E}_6^*} - \mathbf{P}_{\mathsf{E}_6})\mathbf{p} = \pi_{\mathsf{E}_6^*}(\mathbf{p}) - \pi_{\mathsf{E}_6}(\mathbf{p})$. If $\mathbf{p} \in \mathsf{E}_6^*$, then $\pi_0(\mathbf{p}) = \pi_{\mathsf{E}_6^*}(\mathbf{p}) - \pi_{\mathsf{E}_6}(\mathbf{p}) = m - \pi_{\mathsf{E}_6}(\mathbf{p}) \leq 0$, with equality if an only if $\mathbf{p} \in \mathsf{E}_6 \cap \mathsf{E}_6^*$, and, if $\mathbf{p} \in \mathsf{E}_6$, then $\pi_0(\mathbf{p}) = \pi_{\mathsf{E}_6^*}(\mathbf{p}) - \pi_{\mathsf{E}_6}(\mathbf{p}) = \pi_{\mathsf{E}_6^*}(\mathbf{p}) - m \geq 0$, with equality if an only if $\mathbf{p} \in \mathsf{E}_6 \cap \mathsf{E}_6^*$. It follows that the hyperplane H_0, with equation $\langle \pi_0, \varphi \rangle = 0$, separates $\Phi_{\mathsf{E}_6^*}$ and Φ_{E_6}, and contains the facet $\Phi_{\mathsf{E}_6^*\cap\mathsf{E}_6}$.

The point where the line segment $\varphi_t = (1-t)\varphi_{\mathcal{G}} + t\varphi_{\mathcal{G}^*}$, $0 \leq t \leq 1$, pierces $\Phi_{\mathsf{E}_6^*\cap\mathsf{E}_6}$ is determined by solving the equation

$$\begin{aligned}
\langle \pi_0, \varphi_t \rangle &= \langle \pi_{\mathsf{E}_6^*} - \pi_{\mathsf{E}_6}, (1-t)\varphi_{\mathcal{G}} + t\varphi_{\mathcal{G}^*} \rangle \\
&= (1-t)trace\mathbf{P}_0\mathbf{F}_{\varphi_{\mathcal{G}}} + (t)trace\mathbf{P}_0\mathbf{F}_{\varphi_{\mathcal{G}^*}} \\
&= (1-t)(-\frac{2}{8}m^2) + t(\frac{3}{8}m^2) \\
&= -\frac{2}{8}m^2 + \frac{5}{8}m^2 t \\
&= 0,
\end{aligned}$$

where $\mathbf{P}_0 = \mathbf{P}_{\mathsf{E}_6^*} - \mathbf{P}_{\mathsf{E}_6}$. Since $t = \frac{2}{5}$ satisfies this equation, $\varphi_{\frac{2}{5}} \in \Phi_{\mathsf{E}_6^*\cap\mathsf{E}_6}$. The form $\varphi_{\frac{2}{5}}$ is also interior to the open segment $\varphi_t, 0 < t < \frac{1}{2}$, which belongs to $\Phi_{\mathcal{G}\cap\mathcal{R}}$. Therefore, $\varphi_{\frac{2}{5}} \in \Phi_{\mathcal{G}\cap\mathcal{R}} \cap \Phi_{\mathsf{E}_6^*\cap\mathsf{E}_6}$, yet $\Phi_{\mathcal{G}\cap\mathcal{R}}$ is not entirely contained in $\Phi_{\mathsf{E}_6^*\cap\mathsf{E}_6}$. *This shows that the lattice type partition does not refine the perfect partition in six dimensions, and Voronoi's hypothesis does not hold in this dimension.*

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