An Infinite Series of Perfect Quadratic Forms and Big Delaunay Simplexes in $\mathbb{Z}^n$.

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Abstract

Georges Voronoi (1908-09) introduced two important reduction methods for positive quadratic forms: the reduction with perfect forms, and the reduction with L-type domains.

A form is *perfect* if it can be reconstructed from all representations of its arithmetic minimum. Two forms have the same L-type if Delaunay tilings of their lattices are affinely equivalent. Delaunay (1937-38) asked about possible relative volumes of lattice Delaunay simplexes. We construct an infinite series of Delaunay simplexes of relative volume $n-3$, the best known as of now. This series gives rise to an infintie series of perfect forms with remarkable properties: e.g. $\tau_5 \sim D_5$, $\tau_6 \sim E_6^*$, $\tau_7 \sim \varphi_{15}^7$; for all n the domain of τ_n is adjacent to the domain of D_n, the 2-nd perfect form . Perfect form τ_n is a direct n-dimensional generalization of Korkine and Zolotareff's 3-rd perfect form ϕ_2^5 in 5 variables. We prove that τ_n is equivalent to Anzin's (1991) form h_n.

1 [1]Introduction and main result

Positive definite quadratic forms (referred to as PDQFs) in n variables make an *open* cone $\mathfrak{P}_n$ of dimension $N = \frac{n(n+1)}{2}$ in $Sym(n,\mathbb{R}) \cong \mathbb{R}^N$, and this cone is the main object of study in our paper. The boundary of $\mathfrak{P}_n$ consists of positive semi-definite forms (referred to as PQFs). PDQFs serve as algebraic representations of *point lattices*. There is a natural one-to-one correspondence between isometry classes of n-dimensional lattices and integral equivalence classes (i.e. with respect to $GL(n,\mathbb{Z})$-conjugation) of PDQFs in n variables (see e.g. Conway, Sloane (1999)).

Conjugation by a fixed matrix from $GL(n,\mathbb{Z})$ is an invertible linear operator on $Sym(n,\mathbb{R})$. Therefore, conjugation defines a homomorphism $\mathcal{V}$ from $GL(n,\mathbb{Z})$ to $GL(N,\mathbb{Z})$, and $GL(n,\mathbb{Z})$ acts pointwise on the space of quadratic forms $Sym(n,\mathbb{R}) \cong \mathbb{R}^N$. Two subsets of $Sym(n,\mathbb{R})$ are called arithmetically equivalent if they are equivalent with respect to the action of $\mathcal{V}(GL(n,\mathbb{Z}))$.

[1]Dedicated to Sergei Ryshkov on the occasion of his 70^{th} birthday.

Definition 1.1 *A* $\mathfrak{R}$ *of* $\mathfrak{P}(n)$ *into* realtively open convex polyhedral *cones with apex at **0** is called a* reduction partition *if:*

1. *it is invariant with respect to* $GL(n,\mathbb{Z})$*;*
2. *there are only finitely many arithmetically inequivalent cones in this partition;*
3. *for each cone* C *of* $\mathfrak{R}$ *and any PQF* φ *in* n *indeterminates,* φ *can be* $GL(n,\mathbb{Z})$*-equivalent to at most finitely many forms lying in* C*.*

$(N-1)$-dimensional cones of such partition are called *walls*, while cones of dimension 1 are referred to as extreme rays.Voronoi defined two polyhedral reduction paritions of $\mathfrak{P}_n$: these are the tilings by *perfect domains* and *domains for lattice types,* also called *L-domains.* (Our usage of term *domain* is lax when we talk about perfect domains of forms corresponding to some well-known lattices, e.g. E_n or D_n; it should be clear from the context whether we mean the whole arithmetic class, or just one element of this class.)

Tiling the cone of metrical forms. Let φ be a PQF. The *arithmetic minimum* of φ is the minimum of φ on $\mathbb{Z}^n \backslash Ker(\varphi)$. The integral vectors on which this minimum is attained are called the representations of the minimum, or the *minimal vectors* of φ: these vectors have the minimal length among all vectors of $\mathbb{Z}^n \backslash Ker(\varphi)$, when φ is used as the metrical form. For each PDF φ with the set of minimal vectors $\mathsf{P} \subset \mathbb{Z}^n$ there is a P-domain Π_{P} defined by

$$\Pi_{\mathsf{P}} = \{\sum_{\mathbf{p}\in\mathsf{P}} \omega_{\mathbf{p}}(\mathbf{p}\cdot\mathbf{x})^2 \mid \omega_{\mathbf{p}} > 0\};$$

The P-domains form a partition of $\mathfrak{P}_n$ which is called the *perfect* partition. A PDQF φ with the Gramm matrix (φ_{ij}) is called *perfect* if it can be reconstructed up to scale from all representations of its arithmetic minimum; the corresponding P-domain is also then called perfect. In other words, a PDQF φ with the arithmetic minimum mar and $2s$ minimal vectors $\{\mathbf{v}_k = (v_k^1, ..., v_k^n) | \, k = 1, ..., 2s\}$, is called perfect if the system

$$\boxed{\sum_{i,j=1}^n \chi_{ij} v_k^i v_k^j = mar,}$$

where $k = 1, ..., 2s$, has a unique solution $(\chi_{ij}) = (\varphi_{ij})$ in $Sym(n,\mathbb{R})$ (indeed, uniquness requires the existence of at least $n(n+1)$ minimal vectors). The perfect domains are open polyhedral N-dimensional cones, which fit together facet-to-facet to tile $\mathfrak{P}_n$. The P-domains with lesser dimension are relatively open faces of this tiling. In fact, P-domains from a parition of the convex hull of all rational rank one forms:

$$\bigsqcup_{\mathsf{P}} \Pi_{\mathsf{P}} = \mathrm{conv}\{(\mathbf{p}\cdot\mathbf{x})^2 \mid \mathbf{p} \in \mathbb{Q}^n\} \subset \overline{\mathfrak{P}}_n,$$

where the union is taken over all sets of minimal vectors for PDQFs in n variables. Each extreme ray of this tiling lies on $\partial\mathfrak{P}_n$. Perfect forms play an important role in the problem of finding locally densest lattice sphere packings in $\mathbb{E}^n$.

Voronoi's second tiling for $\mathfrak{P}_n$ is the *tiling by domains for lattice types (* aka L-type domains), or, more simply, *L-domains*. Voronoi (1909) defined L-domains in terms of properties of Voronoi vectors, but we will follow a different treatment, which is due to Delaunay (1937-38).

Definition 1.2 *A convex polyhedron P in $\mathbb{R}^n$ is called a Delaunay cell of $\mathbb{Z}^n$ with respect to a positive (semidefinite, in general) quadratic form φ if:*

1. *for each face F of P we have* $\mathrm{conv}(L \cap F) = F$*;*
2. *there is a quadric $Q(P,\varphi)$, circumscribed about P, whose quadratic form is φ*
3. *no points of $\mathbb{Z}^n$ lie inside $Q(P,\varphi)$.*

When φ is positive definite, $Q(P,\varphi)$ is called the *empty ellipsoid* of P. If *rank* $\varphi < n$, Q_P is an elliptic cilinder. When $\varphi(\mathbf{x}) = \sum_{i=1}^{n} {x_i}^2$, our definition coincides with the classical definition of Delaunay cell (1937) for the point system $\mathbb{Z}^n \subset \mathbb{E}^n$. Delaunay cells for φ form a convex face-to-face tiling $\mathcal{D}(\varphi)$ of $\mathbb{R}^n$ with the vertex set $\mathbb{Z}^n$; $\mathcal{D}(\varphi)$ is uniquely defined by φ (Delaunay, 1937).

Definition 1.3 *PQFs f_1 and f_2 belong to the same L-domain if the Delaunay tilings of $\mathbb{Z}^n$ with respect to f_1 and f_2 are identical. f_1 and f_2 belong to the same L-type if these tilings are equivalent with respect to $GL(n,\mathbb{Z})$.*

Let $\mathcal{T}$ be the Delaunay tiling of $\mathbb{Z}^n$ for some positive semi-definite form. We denote by $\Lambda_{\mathcal{D}}$ the L-domain defined by $\mathcal{T}$:

$$\Lambda_{\mathcal{T}} := \{\varphi \in \mathfrak{P}_n | \mathcal{D}(\varphi) = \mathcal{T}\}.$$

The cells for Voronoi's L-type parition of $\mathfrak{P}_n$ are the L-domains. More precisely, L-domains partition the convex hull of all rational rank one forms:

$$\bigsqcup_{\mathcal{T}} \Lambda_{\mathcal{T}} = \mathrm{conv}\{(\mathbf{p}\cdot\mathbf{x})^2 \mid \mathbf{p} \in \mathbb{Q}^n\} \subset \overline{\mathfrak{P}}_n,$$

where the union if over all possible Delaunay tilings for positive semi-definite forms in n variables. The L-domains with full dimension N are open N-dimensional convex polyhedral cones (Voronoi, 1909). These domains are labeled by Delaunay triangulations—Delaunay tilings where each cell is a simplex. The L-domains with lesser dimension are relatively open faces, and correspond to Delaunay tilings where not all cells are simplexes. The notions of Delauny tiling and L-type are important in the study of extremal and group-theoretic properties of lattices. Understanding L-types is especially important for the problem of finding the sparest lattice coverings $\mathbb{E}^n$ with balls.

Theorem 1.4 *(Voronoi: Crelle's J.**133, 136**) The perfect and the L-type partition of $\mathfrak{P}_n$ are convex polyhedral reduction partitions. Moreover, both these paritions are face-to-face.*

Large Delaunay simplices in lattices Delaunay (1937) asked about possible volumes of Delaunay simplexes. Ryshkov (1973) showed that in every dimension $2r+1$ there is a lattice with a Delaunay simplex of relative volume r. Namely, Ryshkov proved that lattice A_n^k for $n \geq 2k+1$ has a Delaunay simplex of relative volume k. Ryshkov also noticed that in the case of A_n^k the existence of big Delaunay simplexes is closely related to another interesting phenomenon: for $n \geq 9$ perfect lattice A_n^k is not generated by its shortest vectors Earlier, Coxeter (1951) made a similar observation about the relevance of these two phenomena in case of A_n^k, but he did not know for sure if A_n^k had such big simplexes. In this paper we construct an infinite series of Delaunay simplexes of volume $n-3$ and link this series to Anzin's (1991) series of perfect froms h_n, and to the series D_n.

Infinite series of Delaunay simplices S_n It is well known that the Delaunay tiling of lattice E_6 consists of Gosset polytopes 2_{21} (e.g. see Baranovskii 1991). For properties of this polytope see Coxeter (1995). We refer to the Gosset polytope as the G-tope. All faces of the G-tope are regular simplices, except for 27 regular cross-polytopal facets. Let s_4 be a 4-face of the G-tope which is a common facet of two cross-polytopal facets. Since any pair of vertices of the G-tope defines either a diagonal or an edge of a cross-polytopal facet, a simple counting argument gives that there are only two vertices of the polytope which do not have common edges with vertices of s_4. The volume of the convex hull of s_4 and these two "distant" vertices is 3 times the volume of a fundamental simplex of E_6 . Using a computer program we checked that there are no other simplexes of volume 3, Delaunay or not, in the G-tope (in Section 4 we prove that all simplexes inscribed into the G-tope are Delaunay). There are exactly 216 4-faces that serve as common facets of pairs of cross-polytopal facets and all of them are equivivalent with respect to the group of the G-tope. Therefore, there are exactly 216 Delaunay simplexes of relative volume 3 in the G-tope. They are all equivalent with respect to the isometry group of the G-tope. According to Ryshkov and Baranovskii (1998) there is only one arithmetic type of triple Delaunay simplexes in 6-lattices, and there are no Delaunay simplexes of volume greater than 3 in 6-dimensional lattices.

In an appropriate coordinate system the vertices (here the column-vectors) of this simplex have the following form.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

A $\mathbb{Z}^n$-polytope P with $n+2$ vertices is called a repartitioning complex if there is a PDQF φ for with respect to which P is a Delaunay polytope. Voronoi (1909) showed that PDQFs for which P is a Delaunay polytope form a wall Λ_P between two Λ-domains.

There is a repartitioning complex R_6 one of whose triangulations includes the above simplex. The vertices of R_6 are the vertices of the above simplex plus the vertex $(0,0,0,0,0,0,1)^T$.

We generalized the construction of the above simplex to the following series of simplexes of volume $n-3$. Although, to our knowledge, this is the best infinite series of big Delaunay simplexes, in Leech lattice Λ_{24} all Delaunay simplexes are non-fundamental, and the biggest of them has volume 20480. Haase and Ziegler (2000) showed that for $n > 3$ there are empty lattice simplexes of arbitrary large volume (not Delaunay, indeed). A trivial upper bound on the relative volume of a Delaunay simplex is $\frac{n!}{2}$.

Definition 1.5

$$S_n := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 1 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & -(n-3) \end{pmatrix}$$

The corresponding repartitioning complex R_n is obtained by adjoining vertex $(0,...,0,1)^T$ to S_n .

1.1 Perfect domains and L-types: interpretable and non-interpretable perfect walls.

Voronoi (1908-09) proved that for $n = 2, 3$ the L-partition and the perfect partition of $\mathfrak{P}_n$ coincide. Voronoi also proved that for $n = 4$ the tiling of $\mathfrak{P}_n$ with L-type domains refines the partition of this cone into perfect domains (1909, vol. 136, §117). The perfect domain D_4 (the 2nd perfect form in 4 variables) exemplifies a new pattern in the relation of these partitions. Namely, a cone Π_{D_4} of type D_4 is decomposed into a number of simplicial L-domains like a pie: this decomposition consists of the cones with apex at the origin sharing the central ray of the cone Π_{D_4}. These simplicial cones are L-type domains of two arithmetic types: a cone of type I is adjacent to the a perfect domains of type A_4 (which is also happens to be an L-type domain, called the principal L-type domain Δ by Voronoi), a type II cone is adacent to an arithmetically equivalent L-type domain (also type II, indeed) from the L-subdivision of an adjacent D_4 domain (for details see *Delaunay et al.* (1963, 1970)).Voronoi conjectures implicitely and Dickson explicitely that the partition into L-domains refines the perfect partition. Ryshkov and Baranovskii (1975) proved the refinement hypothesis for $n = 5$. We have shown that this hypothesis fails for $n = 6$ (R.E., K.R 2001). In cases where L-type domains refine perfect ones, the L-type is changing on each perfect wall. We call such perfect walls *L-interpretable*. Notice that two perfect domains or two L-domains can share walls that are not $GL(n, \mathbb{Z})$-equivalent. It is not yet clear why some perfect walls are intepretable, while others, like the wall between domains of types E_6 and E_6^* (R.E., K.R 2001) are not.

Below, in Theorem 2.2, we construct an infinite in n series of $\mathbb{Z}^n$-polytopes R_n on $n+2$. We prove that they are repartitoning complexes. Therefore, PDQFs for which R_n is a Delaunay simplex form an L-wall Λ_{R_n}. One of the two triangulations of this polytope has a Delaunay simplex of relative volume $n-3$. This triangulation defines an N-dimensional L-type domain which is a subcone of the P-domain of a perfect form that we describe in Theorem 3.1. This perfect form turns out to be equivalent to Anzin's form h_n. The L-domain Λ_{R_n} is also a perfect wall separating perfect domains of types h_n and D_n. Thus, this perfect wall between h_n and D_n is *L-interpretable.*

2 Fat Symplexes

In this paper we use a short-hand notation for n-vectors that have few distinct integral coordinates and for families of such vectors obtained from some n-vector by all permutations of selected subsets of its coordinates. Here are the rules:

1. m^k stands for k consecutive positions filled with m's.

2. $[a_1, ...; ...; ..., a_n]$ denotes any vector that can be obtained from $a_1, ..., a_n$ by permutations in sequences of coordinates that are separated by commas and bordered on the sides by semicolons and/or brackets.

3. A family of vectors that are obtained from vector $[a_1, ...; ...; ..., a_n]$ by all admissible (see 2) permutations is denoted by $[a_1, ...; ...; ..., a_n]^k$, where k is the number of such vectors; if $k = 1$, we omit k.

Example: $[1^{n-3}, 0^2; 3]^{\binom{n-1}{2}}$ stands for all vectors with 3 at the last entry, two 0's and $n-3$ 1's among the first $n-1$ coordinates.

To prove that S_n is a lattice Delaunay simplex we need the theory of $(0,1)$-dual systems developed by Erdahl and Ryshkov (1990, 1991 a,b). Let S be a subset of $\mathbb{Z}^n$. The $(0,1)$-dual of S is the set of all elements of $\mathbb{Z}^n$ that have the scalar product of 0 or 1 with all vectors of S. We denote the $(0,1)$-dual of S by S^0. Erdahl and Ryshkov (1990) showed that if the double dual of an integral simplex has only $n+2$ points, then this simplex is a Delaunay simplex for some PDQFs.

Theorem 2.1 *(Erdahl, Ryshkov, 1990) Let S be an n-simplex in $\mathbb{Z}^n$. If $(S^0)^0$ consists of $n+2$ vectors, then there is an N-dimensional cone of PDQFs for which S is a Delaunay simplex.*

S_n is a simplex in $\mathbb{Z}^n$ whose vertices are the columns of the following matrix:

$$\begin{pmatrix} I_{n-1} & \mathbf{1}_{n-1} \\ \mathbf{0}_{n-1}^T & -(n-3) \end{pmatrix}.$$

Here I_{n-1} is the $(n-1)\times(n-1)$ identity matrix, 1_{n-1} is the column $(n-1)$-vector of ones, and 0_{n-1}^T is the row $(n-1)$-vector of zeros. R_n is obtained from S_n by adding vector $(0, ..., 0, 1)^T$. Using the Erdahl-Ryshkov theorem we verify in the following theorem that S_n is indeed a series of lattice Delaunay simplexes.

Theorem 2.2 *For any $n > 3$ S_n is a Delaunay simplex of relative volume $n-3$ and R_n is a repartitioning complex.*

Proof. S_n^0 cannot have vectors with negative numbers in positions 1 through $(n-1)$, for S_n contains an identity submatrix I_{n-1} (we use S_n to refer to both the simplex and its matrix) . Since S_n has a vector with $-(n-3)$ at the last coordinate, S_n^0 does not have vectors with the absolute value of the last coordinate different from 0 or 1; meanwhile, the last coordinate cannot be negative, since it would imply that one of the first $(n-1)$ coordinates is negative. Thus, 0 and 1 are the only choices for the last coordinate of a vector of S_n^0. If a vector of S_n^0 has 0 at the last position, it can have at most one 1 among the other coordinates. Evidently, if a vector of S_n^0 has 1 at the last position, it can have either $(n-2)$ or $(n-1)$ ones among the other coordinates. Therefore, the dual of S_n consists of the following $(0,1)$-vectors

$[1, 0^{n-2}; 0]^{n-1}$
$[1^{n-2}, 0; 1]^{n-1}$
$[1^{n-3}, 0^2; 1]^{\binom{n-1}{2}}$
$[0^n]$

It is easy to see that $(S_n^0)^0 = S_n \cup [0^{n-1}; 1]$, i.e., the double dual of S_n is obtained by adding a vector with zero coordinates, except for only 1 at the very last position.

Denote by D the matrix whose columns are the elements of $S_n^0 \backslash \mathbf{0}$. The images of vectors of $S_n^0 \backslash \mathbf{0}$ under the Voronoi-Veronese mapping ($\mathbf{p} \longmapsto (\mathbf{p}\cdot\mathbf{x})^2$) are linearly independent in $\mathbb{R}^N$, since there is an $N \times (N-1)$ matrix M such that $D^T M = I$ (we omit details here). Since $|S_n^0| = N$, rank one forms corresponding to the vectors of $S_n^0 \backslash \mathbf{0}$ span a cone of co-dimension 1 in $\mathfrak{P}_n$. This is the cone Λ_{R_n} of all PDQFs for which $(S_n^0)^0 = R_n$ is a Delaunay cell in $\mathbb{Z}^n$. By the preceeding theorem S_n is a Delaunay simplex in $\mathbb{Z}^n$ for some PDQFs. ■

3 Perfect Wall Tamed by Big Simplex

Let us prove that for any $n > 4$ Λ_{R_n} is a wall between the domain of second perfect form D_n and another perfect form; we refer to this form as τ_n. In lower dimensions: $\tau_5 \backsim \phi_2^5$ (III-rd perfect form of Korkine and Zolotareff), $\tau_6 \sim E_6^\star$, $\tau_7 \backsim \phi_{15}^7$ (from Stacey (1973, 1975)). We prove that τ_n is integrally equivalent to Anzin's h_n. Form $\tau_n \sim h_n$ exhibits a very interesting geometric behavior in higher dimensions, but this will be the subject of another paper.

Theorem 3.1 *For any $n > 4$ the cone Λ_{R_n} is a common wall of the perfect domain of type D_n and the domain of perfect form τ, where $= (\tau_{ij})_n$ is defined as follows. For even n :*

$$\tau_{ii} = 1 \text{ if } 1 \le i \le n-1;\ \tau_{nn} = \frac{1}{2}n^2 - \frac{7}{2}n + 7;\ \tau_{ij} = \frac{n-4}{2(n-2)} \text{ for } i \ne j,\ j \ne n$$

$$\tau_{in} = (-1)\frac{n^2 - 6n + 10}{2(n-2)} \text{ for } i < n$$

For odd n :

$$\tau_{ii} = 1 \text{ if } 1 \le i \le n-1;\ \tau_{nn} = \frac{n^3 - 8n^2 + 23n - 20}{2(n-1)};$$

$$\tau_{ij} = \frac{n-3}{2(n-1)} \text{ for } i \ne j,\ j < n;\quad \tau_{in} = (-1)\frac{n^2 - 5n + 8}{2(n-1)} \text{ for } i < n$$

Form τ is equivalent to h_n.

Proof. With respect to the standard scalar product on $\mathbb{R}^N$, Λ_{R_n} is defined by the equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{0}$, where $\mathbf{n}$ is given by formulae $n_{ii} = 0$ for $i < n$; $n_{in} = \frac{n-2}{2(n-4)}$ for $i < n$; $n_{ij} = 1$ for $i \ne j$, $j < n$ $n_{nn} = \frac{n^3 - 9n^2 + 24n - 19}{2(n-4)}$. We have to show how to complement the vectors of $R_n^0 \backslash \mathbf{0}$ (note: $R_n^0 = S_n^0$) to the set M_{D_n} of minimal vectors of D_n and to the set M_τ of minimal vectors of τ. We call elements of $\mathsf{M}_X \backslash \{R_n^0 \backslash \mathbf{0}\}$, ($X = D_n$, τ) *complimentary vectors.*

A) *Complimentary Vectors for D_n, the Second Perfect Form.*

In this case the complimentary vectors are all of type $[1,-1,0^{n-3};0]$; denote this set by $[1,-1,0^{n-3};0]^{\binom{n-1}{2}}$. Let (f_{ij}) be a form defined by the formulae: $f_{ii}=1$ for $i<n-1$, $f_{nn}=1+\binom{n-2}{2}$, $f_{ij}=\frac{1}{2}$ for $i\neq j$ and $i<n$, and $f_{in}=-\frac{(n-2)}{2}$. The minimum of (f_{ij}) is 1 and it is attained on all vectors of $R_n^0\backslash\mathbf{0}$ and all $\binom{n-1}{2}$ vectors of type $[1,-1,0^{n-3};0]$. Denote by $C[1,-1,0^{n-3};0]$ the set of all vectors whose coordinates are obtained by circular permutations of the first $n-1$ positions of $[1,-1,0^{n-3};0]$, except for the vector $[-1;0^{n-3};1;0]$. Let us show that (f_{ij}) is integrally equivalent to D_n.

With respect to the scalar product defined by (f_{ij}), the following integral vectors form a Coxeter diagram for $D_n:\mathcal{B}=\{C[1,-1,0^{n-3};0],[0^{n-1};1;0],[1^{n-2};0;1]\}$. In this diagram $[0^{n-3};1;-1;0]$ is the vertex of valence 3, and $[0^{n-1};1;0]$, $[1^{n-2};0;1]$ are the vertices of the two leaves of the diagram which are adjacent to the vertex of valence 3. If a vector of type $[1,-1,0^{n-3};0]$ has 1 at the k-th position, there are $n-2-k$ edges on the diagram between this vector and $[0^{n-3};1;-1;0]$. It is easy to see that $\mathcal{B}$ is a basis of $\mathbb{Z}^n$. Therefore (f_{ij}) is integrally equivalent to D_n.

Notice that $|\{R_n^0\backslash\mathbf{0}\}\cup[1,-1,0^{n-3};0]^{\binom{n-1}{2}}|=n(n-1)$, which is the number of non-collinear minimal vectors of D_n. Therefore $\{R_n^0\backslash\mathbf{0}\}\cup[1,-1,0^{n-3};0]^{\binom{n-1}{2}}$ are all of the minimal vectors of $(f_{ij})\backsim D_n$ up to inversion.

B) Generalization of the Third Perfect Form.

Notice that τ is 1 on all $\mathbf{v}\in R_n^0\backslash\mathbf{0}$. Set $w:=\lfloor\frac{n}{2}\rfloor$. The choice of complimentary vectors $Comp(\tau)$ depends on the parity of the dimension.

1) When n is even $Comp(\tau)$ is $[(w-3)^{n-1};w-2]$, $[w-2,(w-3)^{n-2};w-2]^{n-1}$, $[(w-2)^{n-1};w-1]$. The total number of minimal vectors of is $n(n+3)$ for $n>4$.

2) When n is odd $Comp(\tau)$ is $[(w-1)^{n-1};w]$. The total number of minimal vectors is $n(n+1)$

To prove that τ is a positive definite perfect form with minimal vectors $Comp(\tau)\cup R_n^0$, it is enough to present $\mathcal{T}\in GL(n,\mathbb{Z})$ that maps our vectors $Comp(\tau)\cup R_n^0$ to the minimal vectors of h_n provided by Anzin (1991). Indeed $\mathcal{T}=\mathcal{T}_2\mathcal{T}_1$, where $\mathcal{T}_1$ is given by $x_1'=x_n$, $x_n'=x_1$, $x_i'=x_i$, $i\neq 1,n$, and $\mathcal{T}_2$ is given by

$$\begin{pmatrix} -(n-3) & 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots & -1 \end{pmatrix}$$

The formulae for the coefficients of τ are found by solving the system of linear equations $\{\sum_{i,j=1}^n\tau_{ij}v^iv^j=1,\ \tau_{ij}=\tau_{ji}\}$, where $\mathbf{v}$ runs through $\{\mathbf{v}=(v^1,...,v^n)|\,\mathbf{v}\in Comp(\varphi)\cup R_n^0\backslash\mathbf{0}\}$.

To summarize:

Extreme rays of Λ_{R_n} :
$[1,0^{n-2};0]^{n-1}$; $[1^{n-2},0;1]^{n-1}$; $[1^{n-3},0^2;1]^{\binom{n-1}{2}}$
Complementary vectors for D_n :
$[1,-1,0^{n-3};0]^{\binom{n-1}{2}}$
Complementary vectors for τ_n $(n=2w)$:
$[(w-3)^{n-1};w-2]$, $[w-2,(w-3)^{n-2};w-2]^{n-1}$, $[(w-2)^{n-1};w-1]$
Complementary vector for τ_n $(n=2w+1)$:
$[(w-1)^{n-1};w]$

■

We refer to Λ_{R_n} as a *tame wall,* because it is both perfect and L-wall. For $n=6$ we know that $\tau \backsim E_6^*$ and Λ_{R_6} is one of the three (up to $GL(n,\mathbb{Z})$-equivalence) walls of the domain of E_6^* (see *Barnes* (1957)). The other two walls, called $W_2(24)$ and $W_3(21)$ by Barnes, are *wild,* as there is no change of L-type at almost all interior points of these perfect walls. We plan to publish the proof that $W_3(21)$ is not interpretable later.

4 The case of n=6: L-partition of E_6

In this subsection we discuss Delaunay tilings of lattices lying in a small neigbourhood of E_6 in the space of parameters. More specifically, we look at the L-partition of $\mathfrak{P}_n$ near the ray corresponding of E_6. The Delaunay tiling of lattice E_6 is formed by congruent copies of the Gosset polytope, which is the convex hull of a unique, up to a homothety, two-distance spherical set in $\mathbb{E}^6$. The G-topes of the Delaunay tiling of E_6 fall into two translation classes. The star of a lattice point is formed by 54 G-topes, 27 in each translation class.

The G-tope is quite remarkable. It has 27 vertices, 216 edges, 72 regular simplicial facets, and 36 regular cross-polytopal facets (e.g. Coxeter (1995)). Polytopes whose vertices form a spherical two distance sets are interesting combinatorial objects (see Deza and Laurent (1997), Deza, Grishukhin, Laurent (1992)). In the case of G-tope the two distance structure is realized so that for each vertex $\mathbf{v}$ of a G-tope G there is a vector $\mathbf{p_v}$ such that the vertex set of G can be represented as $\mathbf{v}\cup V_1\cup V_2$, where $V_1=\{\mathbf{u}\in G \mid (\mathbf{u}-\mathbf{v})\cdot\mathbf{p_v}=1\}$, and $V_2=\{\mathbf{u}\in G \mid (\mathbf{u}-\mathbf{v})\cdot\mathbf{p_v}=2\}$. If the edges of G have unit lengths, then $\mathbf{p_v}=2(\mathbf{c}-\mathbf{v})$, where $\mathbf{c}$ is the circumcenter of G. For a detailed description of geometric and group theoretic properties of the G-tope see (Coxeter (1995)).

Below, we show that for every subset of vertices of a Dealunay cell of E_6, E_6 can be perturbed so that this subset becomes a Delaunay cell for the perturbed lattice. In particular, this implies that there are perturbations of E_6 having a Delaunay simplex of volume 3, the maximal relative volume of a Delaunay lattice simplex in $\mathbb{E}^6$.

Theorem 4.1 *For every convex polytope D whose vertex set is a subset of the vertex set of the G-tope there is a perturbation of E_6 making D a Delaunay polytope for the perturbed lattice.*

Proof. Consider a linear transformation that maps lattice E_6 to $\mathbb{Z}^6$ and the standard metric to a quadratic form of type E_6. Let G be an affine copy of the Gosset polytope with integer vertices and P be a subset of vertices of G. Denote by $f_{E_6}(x)$ an inhomogenious quadratic function, such that its quadratic part is of arithmetic type E_6 and $f_{E_6}(x) = 0$ is an ellipsoid circumscribing G. Suppose $f_{E_6}(x)$ is scaled so that $f_{E_6}(\mathbf{v} - \mathbf{w}) = 1$ for any edge $\mathbf{vw}$ of G; in this case the edges of G are minimal vectors and the arithmetic minimum of $f_{E_6}(x)$ is 1. Let P be a subset of vertices of G. For some $\frac{1}{27} > \alpha > 0$ consider the inhomogeneous form

$$f(\mathbf{x}) = f_{E_6}(\mathbf{x}) + \alpha \sum_{\mathbf{v} \in V(G) \setminus P} (\mathbf{p_v} \cdot (\mathbf{x} - \mathbf{v}) - 1)(\mathbf{p_v} \cdot (\mathbf{x} - \mathbf{v}) - 2),$$

with $\mathbf{p_v} = 2(\mathbf{c} - \mathbf{v})$, where $\mathbf{c}$ is the circumcenter of G.

When α is sufficiently small the quadratic part of $f(\mathbf{x})$ is close to $f_{E_6}(x)$ in the space of parameters. If $\mathbf{x} \in P$, $f_{E_6}(\mathbf{x}) = 0$ and $(\mathbf{p_v} \cdot (\mathbf{x} - \mathbf{v}) - 1)(\mathbf{p_v} \cdot (\mathbf{x} - \mathbf{v}) - 2) = 0$ for any $\mathbf{v} \notin P$. If $\mathbf{x} \in V(G) \setminus P$, $f(\mathbf{x}) = f_{E_6}(\mathbf{x}) + 2\alpha > 0$. If $\mathbf{x} \in \mathbb{Z}^6 \setminus V(G)$, then $f(\mathbf{x}) > 1 - 27\alpha > 0$. Thus, $f(\mathbf{x}) = 0$ is a an empty ellipsoid circumscribing $\operatorname{conv} P$. Obviously, if $\dim \operatorname{conv} P = n$, P is a Delaunay polytope for $f(\mathbf{x})$. ∎

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