Fast verification of convexity of piecewise-linear hypersurfaces

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We show that a piecewise-linear (PL) complete immersion of a connected manifold of dimension \( n - 1 \) into \( n \)-dimensional Euclidean space \( (n > 2) \) is the boundary of a convex polyhedron, bounded or unbounded, if and only if the interior of each \( (n - 3) \)-face has a point with a neighborhood (on the surface) that lies on the boundary of a convex body. No initial assumptions about the topology or orientability of the input surface are made, but if the surface is unbounded, the existence of a point of strict convexity is required. The theorem is derived from a refinement and generalization of Van Heijenoort’s (1952) theorem on locally convex manifolds to spherical spaces. Our convexity criterion for PL-manifolds implies an easy-to-implement polynomial-time algorithm for checking convexity of a given PL-hypersurface in \( n \)-dimensional Euclidean space. For \( n = 3 \) the number of arithmetic operations used by the algorithm is linear in the number of vertices; in the general dimension it is \( O(f_{n-2}n-3) \), where \( f_{n-2}n-3 \) is the number of incidences between \( (n - 2) \)- and \( (n - 3) \)-faces. The algorithm is optimal with respect to the highest degree of evaluated polynomial predicates. The algorithm works under significantly weaker assumptions and is easier to implement than convexity verification algorithms suggested by Mehlhorn et al. (1996a,b; 1999), and Devillers et al. (1998).

1 Introduction

Blum and Kannan (1989) suggested the paradigm of output verification. Since a complete check of a program is often difficult or not possible (e.g. because the code has not been made public), it is important to have algorithms that verify the important properties of objects “constructed by computers”. Instead of code verification we can try to verify the essential properties of the output. In computational geometry this paradigm was developed, among others, by the Melhorn’s LEDA team (1996a,b; 1999) and Devillers et al (1998). LEDA C++ library even contains programs verifying some geometric properties such as convexity. Devillers et al. argue that it is easier to evaluate the quality of the output of a geometric algorithm, than the correctness of the algorithm or program producing it. This paper contributes to the problem of verification of convexity of a large class of PL-hypersurfaces in \( \mathbb{R}^n \) for \( n > 2 \).

Let \( \mathbb{X}^n \) denote \( \mathbb{R}^n \) or \( \mathbb{S}^n \). By a subspace of \( \mathbb{X}^n \) we mean an affine subspace in the case of \( \mathbb{R}^n \), and the intersection of \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \) with a linear subspace of \( \mathbb{R}^{n+1} \) in the case of \( \mathbb{R}^n \). A hyperplane is a subspace of codimension 1. A set \( K \subset \mathbb{X}^n \) is convex if for any \( \mathbf{x}, \mathbf{y} \in K \) there is a minimal geodesic segment that lies in \( K \) with end-points \( \mathbf{x} \) and \( \mathbf{y} \); the dimension of \( \dim K \), is the dimension of a minimal subspace containing \( K \) (since such a subspace is unique in \( \mathbb{R}^n \) and \( \mathbb{S}^n \), we denote it by \( \text{aff } K \), even in the spherical case). \( \text{int } K \) denotes the interior of \( K \) in \( \text{aff } K \). A hypersurface in \( \mathbb{X}^n \) is a pair \( (M, \tau) \) where \( M \) is a manifold of dimension \( n - 1 \) and \( \tau : M \rightarrow \mathbb{X}^n \) is continuous. \( (M, \tau) \) is called an immersion if \( \tau \) is a local homeomorphism. We call a \( k \)-submanifold (with or without boundary) \( S \) of \( M \) flat if \( \tau : S \rightarrow \tau(S) \) is an embedding into a \( k \)-dimensional subspace of \( \mathbb{X}^n \). A submanifold is open if it is non-compact and does not contain any boundary points. If for a connected subset \( S \subset M \) we have \( \tau(S) \subset S \), we denote by \( \tau^{-1}_S(S) \) the connected component of \( S \) in the preimage of \( S \).

A convex body is a closed convex set of full dimension. \( (M, \tau) \) is called locally convex at \( p \) if \( p \) has a neighborhood \( U_p \subset M \) such that, restricted to \( U_p \), is a homeomorphism onto \( \tau(U_p) \), and \( \tau(U_p) \) lies on the boundary of a convex body \( K_p \); if \( K_p \) can be chosen so that \( K_p \setminus \tau(p) \) lies in an open half-space defined by a hyperplane containing \( \tau(p) \), hypersurface \( (M, \tau) \) is called strictly convex at \( \tau(p) \). By a well-known theorem of Busemann [4] \( K_p \) and \( U_p \) can always be chosen so that there is a support hyperplane \( H \) at \( \tau(p) \) such that the orthogonal projection of \( \tau(U_p) \) onto \( H \) is an open \( (n - 1) \)-ball, centered at \( \tau(p) \). We will always assume that \( K_p \) and \( U_p \) satisfy this assumption. We refer to \( K_p \) as a convex witness for \( p \).

Recall that a point \( p \) on the boundary of a convex set \( K \) is called exposed if \( K \) has a support hyperplane that intersects \( K \) only at \( p \); \( p \) is called extreme if it does not belong to the interior of any interval contained in \( \partial K \). Thus, an exposed point on a convex body \( B \) is a point of strict convexity on the hypersurface \( \partial B \). Local convexity can be defined in many other, non-equivalent, ways (see van Heijenoort, 1952).
This paper is mainly focused on convexity of piecewise-linear (PL) hypersurfaces, in particular, boundaries of polytopes. Denote by $B^k$ the closed unit ball at the origin in $\mathbb{R}^k$. A (disjoint) countable partition $P$ of a manifold $M$ is called a regular cell-partition (our definition is slightly more general than the common one, e.g. Ziegler (2002)) if (1) each element $C \in P$, called a cell of $M$, is homeomorphic to $int B^{\dim C}$, where $0 \leq \dim C \leq \dim M$; (2) the closure $\overline{C}$ of each $C \in P$ is the inion of $C$ and cells of smaller dimensions; (3) for each $C \in P$ there is a mapping $i_C : \overline{C} \to B^{\dim C}$ which is a homeomorphism onto $i_C(\overline{C})$ and whose restriction to $C$ is a homeomorphism onto $i_C(C)$ (our definition is slightly more general than the common one, e.g. Ziegler (2002)); (4) each point of $M$ belongs to the closure of finitely many cells. Since any regular cell-partition can be subdivided into a triangulation, any manifold admitting a regular cell-partition belong to the category of PL-manifolds.

A PL-surface in $\mathbb{R}^n$ is a triple $(M, P, \tau)$, where $M$ is a manifold with a regular cell-partition $P$, and $\tau : M \to \mathbb{R}^n$ is a continuous realization map satisfying the following conditions:
1) $\tau$ is a homeomorphism from $\overline{C}$ onto $\tau(\overline{C})$ for each $C \in P$;
2) for each $C \in P$ the image $\tau(C)$ lies on a subspace of $\mathbb{R}^n$ of dimension $\dim C$.

In this case we say that $\tau$ is a PL-realization of $M$ in $\mathbb{R}^n$. Note that although $\tau$ need not even be an immersion, the restriction of $\tau$ to the closure of any cell $C$ of $M$ must be an embedding. If $\dim C = k$, then $\tau(C)$ is called a $k$-face of $(M, P, \tau)$. Throughout the paper all faces, just as all cells in topological partitions, are assumed to be relatively open.

We say that $(M, \tau)$ is the boundary of a convex body $K \subset \mathbb{R}^n$ if $\tau$ is a homeomorphism from $M$ onto $\partial P$. Hence, we exclude the cases when $\tau(M)$ coincides with the boundary of a convex body, but $\tau$ is not injective. The algorithm will always detect non-immersion, but in this case it will produce negative answer without trying to determine if $\tau(M)$ is the boundary of a convex body. Of course, the algorithmic and topological aspects of this case may be important for certain areas of geometry, such as e.g., origami. Note that for $n > 2$ a closed $(n-1)$-manifold $M$ cannot be immersed into $\mathbb{R}^n$ by a non-injective map $\tau$ so that $\tau(M)$ is the boundary of a convex set, since any convex hypersurface in $\mathbb{R}^n$ is simply-connected and any covering map onto a simply-connected manifold must be a homeomorphism. However, such immersions might be possible in the hyperbolic space $\mathbb{H}^n$ – there are infinitely many topological types of convex hypersurfaces in $\mathbb{H}^n$ for $n > 2$ (Kuzmynykh, 2005). Note that all local geometric arguments for $\mathbb{R}^n$ are applicable to the sphere (and hyperbolic space) without changes.

Our main Theorem 9 asserts that any complete PL-realization $(M, P, \tau)$ in $\mathbb{R}^n$ $(n > 2)$ of a connected manifold $M$ (dim $M = n - 1$) with at least one point of strict convexity, and such that each $(n-3)$-cell has a point at which $\tau$ is locally convex, is the boundary of a convex body. Notice that if the last condition holds for some point on an $(n-3)$-face, it holds for all points of this face. This theorem implies a test for global convexity of a PL-hypersurface: check local convexity on each of the $(n - 3)$-faces. The algorithm implicitly checks if the given realization $\tau$ is an immersion. The pseudo-code for the algorithm is given in Section 4. The complexity of this test depends not only on the model of computation, but also on the way the surface is given as input data. Let us adopt the algebraic complexity model with the basis consisting of comparison, addition, subtraction, and multiplication. Assuming the input consists of the poset of faces of dimensions $n - 1$, $n - 2$, $n - 3$, equipped with the equations of $(n-1)$-face, the algebraic complexity of the algorithm for a closed PL-manifold is $O(f_{n-3}n-2) = O(f_{n-3}n-1)$, where $f_{k,l}$ is the number of incidences between cells of dimensions $k$ and $l$. If the surface is given as the poset of faces of dimensions $n - 1$, $n - 2$, $n - 3$, and 0, together with the coordinates of the vertices, the complexity is $O(f_{0n-3} + f_{n-2}n-2)$. The complexity of our algorithm is asymptotically equal to that of Devillers et al (1998) and Mehlhorn et al (1999) for simplicial 2-dimensional surfaces; for $n > 3$ our algorithm is asymptotically faster than theirs. If the vertices of the manifold are assumed to be in a sufficiently general position, then the dimension of the space does not affect the complexity at all. Another advantage of this algorithm is that it consist of $f_{n-3}$ independent subroutines corresponding to the $(n - 3)$-faces, each with complexity not exceeding $O$ in the number of $(n-1)$-cells incident to the $(n-3)$-face. Besides the algorithmic implications, our generalization implies that any $(n-3)$-simple compact PL-hypersurface in $\mathbb{R}^n$ is the boundary of a convex polytope.

2 Geometry of Locally-Convex Immersions

The immersion $\tau : M \to \mathbb{R}^n$ induces a metric $d_\tau$ on $M$ by

$$d_\tau(p, q) = \inf_{\sigma(p, q) \subset M} |\tau(\sigma(p, q))|$$
where $|r(\text{arc}[p,q])| \in \mathbb{R} \cup \{ \infty \}$ stands for the length of the $r$-image of an arc $\text{arc}[p,q]$ joining these points on $\mathcal{M}$. This metric is called the $r$-metric. Suppose $\mathcal{M}$ is complete with respect to $d_r$. $(\mathcal{M}, r)$ has the following four useful properties.

**Lemma 1** Any two points of $\mathcal{M}$ can be connected by an arc of a finite length. Thus $\mathcal{M}$ is not only connected, but also arcwise connected.

**Lemma 2** The metric topology defined by the $r$-metric is equivalent to the original topology on $\mathcal{M}$.

van Heijenoort’s (1952) proofs of these lemmas, given for $\mathbb{R}^n = \mathbb{R}^n$, work for $\mathbb{X}^n = \mathbb{S}^n$ without changes. The following two propositions follow from standard results of general topology (see e.g. Dugundji, 1966).

**Proposition 3** Immersion $\tau$ is closed, i.e. $\tau(S)$ is closed in $\mathbb{X}^n$ for any closed $S \subset \mathcal{M}$.

**Proposition 4** If on a bounded (in $r$-metric) closed subset $S \subset \mathcal{M}$ mapping $\tau$ is one-to-one, then $\tau$ is a homeomorphism between $S$ and $\tau(S)$.

Recall that a set $C \subset \mathbb{R}^{n+1}$ is called a pointed cone if (1) $\lambda C \subset C$ for any $\lambda \in \mathbb{R}$, and (2) $0 \subset C$. A cone is called salient if it does not contain any linear subspace except for $0$. If $S \subset \mathbb{R}^{n+1}$, then we denote by $p \cdot S$ the cone with apex $p$ over $S$. By a circle on a sphere we always mean a great circle. We assume that $\mathbb{S}^n$ is embedded as the standard unit sphere into $\mathbb{R}^{n+1}$. For $x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ we denote by $c_x$ the central projection from $\mathbb{S}^n$ onto $T_x \subset \mathbb{R}^{n+1}$, its tangent plane at $x$.

**Theorem 5** Let $\tau : \mathcal{M} \to \mathbb{S}^n$ be a locally convex complete immersion of a connected $(n - 1)$-manifold $\mathcal{M}$. Then $(\mathcal{M}, r)$ is either strictly locally convex in at least one point, or $\tau(\mathcal{M}) = \partial(\mathbb{S}^n \cap C)$, where $C$ is a pointed convex cone in $\mathbb{R}^{n+1}$, in which case we say that $\tau$ is conical.

**Proof.** If there is $\tau(p)$ that is an extreme point for some convex witness $K_p$, then by a classical theorem of Straszewicz (see [13]) there are exposed points of $K_p$ arbitrarily close to $\tau(p)$. In this case there is a point of strict local convexity near $\tau(p)$. Otherwise, each $\tau(p)$ is an interior point of a segment on $\tau(\mathcal{M})$. From now on let us assume that this is the case and show that this assumption implies that $\tau$ is a homeomorphism onto $\partial(\mathbb{S}^n \cap C)$ for some pointed convex cone $C$.

We know that all $\tau(p)$’s are interior points of segments on $\tau(\mathcal{M})$. If a segment through $\tau(p)$ cannot be extended to a circle on $\tau(\mathcal{M})$, which is the $r$-image of a closed curve through $p$ on $\mathcal{M}$, then pick an end point of a maximal segment $I_p$ through $\tau(p)$ and call it $\tau(a)$. Since $\tau(a)$ is not extreme, it must lie in the interior of another segment. There is a support plane $H_a$ for $a$ such that $\text{int} I_p$ and $H_a$ do not intersect near $\tau(a)$. Then $r^{-1}(H_a)$ is either a closed submanifold of dimension at least 1 or a closed submanifold with boundary. If $\tau^{-1}(H_a)$ is a submanifold with boundary, then $c_x \circ \tau^{-1}(H_a)$ must have parallel lines on the boundary (which correspond to intersecting half-circles on $\mathbb{S}^n$), furthermore, through any two points on the boundary there are two parallel lines in the boundary (see [7], Propositions 1, p. 532, and Main Theorem, p. 531, for details). Let $R \subset T_x$ be a maximal subspace contained in $r \circ \tau^{-1}(H_a)$. The $r$-preimages of points at infinity of $R$ form a flat closed submanifold of $\mathcal{M}$. Thus $\mathcal{M}$ must have a flat closed submanifold of dimension at least 1.

Let $\mathcal{F} \subset \mathcal{M}$ be a closed flat submanifold of maximal dimension; denote by $F^\perp_{\mathcal{F}}$ its orthogonal complementary subspace of $\mathbb{S}^n$ at $\tau(y)$. If $\dim \mathcal{F} = n - 1$, then $\mathcal{F} = \mathcal{M}$ and we are done. Since $(\mathcal{M}, r)$ is locally convex and does not have points of strict convexity, there is an open set $U_{\mathcal{F}} \subset \mathcal{M}$ that contains $\mathcal{F}$ such that $\tau(U_{\mathcal{F}})$ is of the form $\partial(0 \cdot F \times C_F) \cap \mathbb{S}^n$, where $C_F$ is a salient $(n - 1 - \dim F)$-dimensional convex cone in $0 \cdot F^\perp_{\mathcal{F}}$. Note that $\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M}))_y$ is a locally convex hypersurface in $F^\perp_{\mathcal{F}}$, which is strictly convex at $y$. If $\dim F^\perp_{\mathcal{F}} = 2$, then obviously $\tau$ is conical, that is, radially convex. For the pointed cone $\mathcal{C}$ is the product of a linear $(n - 1)$-subspace in $\mathbb{R}^{n+1}$ and an angle in a complimentary linear 2-subspace.

If $\dim F^\perp_{\mathcal{F}} > 2$, then upon applying van Heijenoort’s theorem to $c_y \circ \tau : \tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})) \to c_y(F^\perp_{\mathcal{F}})$, we see that $c_y \circ \tau(\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})))$ is a complete convex hypersurface in $c_y(F^\perp_{\mathcal{F}}) \subset T_y$. But this is true for all $y \in \mathcal{F}$. Furthermore, if $y$ and $y'$ are sufficiently close, local convexity implies that $\tau(\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})))$ and $\tau(\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})))$ are isometric. Since $\mathcal{F}$ is compact, $\tau(\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})))$ are isometric for all values of $y \in \mathcal{F}$ and the isometry mapping $\tau(\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})))_y$ to $\tau(\tau^{-1}_y(F^\perp_{\mathcal{F}} \cap \tau(\mathcal{M})))_y$ preserves the distance of each point.
of \( r(\tau^{-1}(F^+ \cap \tau(M))) \) to \( F \). The isometry can be chosen as a minimal rotation \( \rho_{yy'} \) of \( S^n \) around the linear subspace of \( \mathbb{R}^{n+1} \), orthogonal to \( r(y) - 0 \) and \( r(y') - 0 \), that maps \( y \) to \( y' \). If \( r: \tau^{-1}(F^+ \cap \tau(M)) \to F^+ \) is strictly convex at some \( z \neq y \), then any geodesic segment \( I_z = r(I_z) \), where \( I_z \) is a flat open 1-submanifold through \( z \in M \), is transversal to \( F^+ \). So, any support plane \( H_z \) for \( (M, \tau) \) at \( I_z \) intersects with the support plane at \( F \) not over all of \( F \). But then for any open \( U_y \subset M \), such that \( y \in U_y \), there will be \( y' \in U_y \cap F \) such that \( \rho_{yy'} \) will move \( z \) to a point lying on the other side of \( H_z \), relative to the convex witness of \( z \). Thus, for \( y \in F \) hypersurface \( c_y \circ \tau(\tau^{-1}(F^+ \cap \tau(M))) \subset \tau(y)(F^+) \) cannot be strictly convex anywhere but at \( y \), which means it is a convex salient cone in \( \tau(y)(F^+) \). This implies that \( \tau \) is conical.  

The notion of convex part, introduced by Van Heijenoort (1952), happens to be very useful for working with local convexity. A convex part of \( (M, \tau) \), (centered) at a point of strict convexity \( \tau(o) \), is a connected subset \( C \) of \( \tau(M) \), whose \( \tau \)-preimage \( \mathcal{C} \) is open in \( M \), such that: (1) \( o \in \tau^{-1}(C) \), (2) \( \partial C = H \cap \tau(M) \), where \( H \) is some hyperplane in \( \mathbb{X}^n \), not passing through \( \tau(o) \), (3) \( C \) lies on the boundary of a convex body \( K_C \) bounded by \( C \) and \( H \). We call \( H \cap K_C \) the lid of the convex part \( C \). Let \( H_o \) be a supporting hyperplane at \( \tau(o) \). We call the open half-space defined by \( H_o \), where \( C \) lies, the positive half-space and denote it by \( H_o^+ \).

We will prove this lemma by perturbation argument, which reduces the spherical case to that of \( \mathbb{R}^n \). Since \( \partial S^o = \partial C, \ S \subset H_o \ (s > 0), \ \text{and dim} \ S = n - 1 \), we conclude that \( \text{conv} C \cap H_o = o \). Since, by Lemma 1, \( M \) is arcwise connected, all of \( \text{conv} C \), except for \( o \), lies in the positive halfspace \( H_o^+ \). Thus, there is a hyperplane \( H \) in \( S^n \), arbitrarily close to \( H_o \) and orthogonal to \( s \), such that \( [C] \) lies in an open halfspace \( H^+ \) defined by \( H \). Let \( p \) be the pole of \( S^n \) with respect to \( H \) that lies in \( H_o^+ \), and let \( \tau_p : H^+ \to T_p \) be the central projection on the tangent plane \( T_p \subset R^{n+1} \). \( \tau_p^{-1}(H^+ \cap M) \) is obviously a manifold. Map \( \tau_p \circ \tau \) is a locally-convex immersion of \( \tau^{-1}(H^+ \cap M) \) into \( T_p \) with a point of strict convexity, \( o \). Any Cauchy sequence on \( M \) under the \( \tau_p \circ \tau \)-metric is also a Cauchy sequence under the \( \tau \)-metric. Thus \( \tau_p \circ \tau : \tau^{-1}(H^+ \cap M) \to T_p \) is complete and satisfies the conditions of van Heijenoort’s theorem. \( \tau_p \circ \tau \) maps a (spherical) convex part at \( o \) onto an Euclidean convex part; it also maps the fiber bundle \( \{H_A \}_{(l,H_o)} \) to a fiber bundle in \( T_p \). In the case of \( \mathbb{X}^n = \mathbb{R}^n \) this lemma was proved in [8]. Thus, either \( M = \mathbb{C} \cup S, \) or \( C \) is a proper subset of a larger convex part centered at \( o \), and defined by the same bundle \( \{H_A \}_{(l,H_o)} \). 

van Heijenoort proved that a completely locally convex immersion \( G \) of a connected manifold \( M \) (dim \( M = n - 1 \)) into \( \mathbb{R}^n \) is the boundary of a convex body, if \( G \) has a point of strict convexity. For \( n = 3 \) this result, according to van Heijenoort, follows from four theorems in Alexandrov’s book (1948). Jonker and Norman (1973) proved that if \( f \) does not have a point of strict convexity, \( f(M) \) is the direct affine product of a plane locally convex curve and a subspace \( L \cong \mathbb{R}^{n-2} \) of \( \mathbb{R}^n \).
Theorem 8 Let \( \tau : M \to \mathbb{S}^n \) \((n > 2)\) be an immersion of a connected manifold \( M \) \((\dim M = n-1)\), satisfying the following conditions:
1) \( M \) is complete with respect to the metric induced on \( M \) by \( \tau \),
2) the immersion is locally convex at all points.

Then \( \tau(M) \) is the boundary of a convex body and \( \tau \) is an embedding.

Proof. By Theorem 5 we only need to prove the theorem when there is a point of strict convexity. If \( o \in M \) is such a point, then by Theorem 6 there is convex part centered at \( o \). Consider the union \( C \) of all convex parts, centered at \( o \). \( \partial C \subset H_\zeta \) is equal to \( H_\zeta \cap \tau(M) \). \( \partial C \) bounds a closed convex set \( D \) in \( H_\zeta \). Two mutually excluding cases are possible.

Case 1: \( \dim D < n-1 \). Then, following the argument of van Heljenoort (Part 2: pp. 239-230, Part 5: p. 241, and Part 3: II on p. 231), we conclude that \( C \cup D \) is the homeomorphic \( \tau \)-image of the \((n-1)\)-sphere \( \xi \cup r^{-1}_\xi(D) \subset M \). Since \( M \) is connected, \( \xi \cup r^{-1}_\xi(D) = M \), and \( \tau(M) \) is a convex embedding of \( M \).

Case 2: \( \dim D = n-1 \). By Lemma 7 \( C \) is either a subset of a bigger convex part, or \( C \), together with the lid \( D \), is the homeomorphic \( \tau \)-image of \( M \). Since the former alternative is excluded by definition of \( C \), \( \tau : M \to \mathbb{S}^n \) is an embedding onto \( \partial(C \cup D) \).

3 Locally convex PL-surfaces

Let \( P \) be a fixed cell-partition of \( M \). A \( k \)-face is the \( \tau \)-image of a \( k \)-cells of \( P \). Cells and faces are always relatively open.

Theorem 9 Let \( \tau : M \to \mathbb{R}^n \) \((n > 2)\) be a complete PL-realization of a connected manifold \( M \) \((\dim M = n-1)\) such that
1) the realization is locally convex in at least one (interior) point of each \((n-3)\)-cell.
2) \( \tau(M) \) is bounded or is strictly locally convex in at least one point.

Then, \( \tau : M \to \mathbb{R}^n \) is an embedding on the boundary of a convex body defined by (possibly infinitely many) affine inequalities.

Proof. By the definition of PL-realization and local convexity of \( \tau \), map \( \tau \) is an immersion. Because of (1), \( \tau : M \to \mathbb{R}^n \) is locally convex at all points of its \((n-3)\)-cells.

We know that for \( k = n-3 \) the immersion \( \tau \) is locally convex at all \( k \)-cells. We proceed by reverse induction in \( k \). Suppose we have shown that \((M, P; \tau) \) is locally convex at each \( k \)-face, \( 0 < k \leq n-3 \) (and therefore at all faces of higher dimensions). Let \( F \in P \) be a \((k-1)\)-face of \((M, P; \tau) \). Consider \( \text{Star} F \cap S_F \), where \( S_F \) is a sufficiently small \((n-k)\)-sphere centered at some point of \( F \) and such that \( \dim(F \cap S_F) = 0 \). \((M, P; \tau) \) is locally convex at \( F \) if and only if the hypersurface \( S \cap \text{Star} F \) on \( S_F \) is convex. Since \((M, P; \tau) \) is locally convex at each \( k \)-face, \( S_F \cap \text{Star} F \) is locally convex at each vertex, and thus locally convex everywhere. \( \tau^{-1}(S_F \cap \text{Star} F) \) is complete in \( \tau \)-metric and thus, by Theorem 8, \( S \cap \text{Star} F \) is an embedded convex hypersurface in \( S_F \). So, \((M, P; \tau) \) is locally convex at all points of \( F \).

This induction argument show that \( \tau \) is locally convex at all vertices, and, therefore, at all points. If \( \tau(M) \) is bounded, then by local convexity and Straszewicz’s theorem (see [13]) it has at least one strictly convex point. The metric induced by \( \tau \) is indeed complete. By van Heijenoort’s theorem and Theorem 8, \( M \) is the boundary of a convex body.

Corollary 10 Suppose \( \tau \) is bounded, complete, and is an immersion on the star of each \((n-3)\)-face. If \((M, P) \) is \((n-3)\)-simple, i.e. exactly three \((n-1)\)-cells make contact at each \((n-3)\)-cell, then \( \tau(M) \) is the boundary of a convex polytope.

4 New Convexity Checker for PL-surfaces

We present a polynomial-time algorithm for checking the convexity of PL-realizations in \( \mathbb{R}^n \) (in the sense outlined above) of a closed connected compact \((n-1)\)-manifold \( M \) \((n > 2)\) with a given cell-partition \( P \). We assume the algebraic (more precisely, ring-theoretic) computational model with the basis consisting of comparison, addition, subtraction, and multiplication. If any of the preliminary procedures return false, the main procedure halts and passes false as the final answer.
The idea of the algorithm is to check for local convexity at each \((n - 3)\)-face of the surface, which is, in turn, is reduced to checking the convexity of a PL-realization of a cone in \(\mathbb{R}^3\). The reduction from the star of an \((n - 3)\)-face to a cone in \(\mathbb{R}^3\) is done by the procedure Reduce-To-3D. Subroutine Produce-Polygon-In-2D reduces the convexity check of a cone in \(\mathbb{R}^3\) to checking the convexity of a plane polygon. The latter task is solved by the Boolean function \(\text{Check-Polygon’s-Convexity}\) that is, in turn, is used by the main procedure at most \(f_{n-3}\) times. Both Reduce-To-3D and \(\text{Check-Polygon’s-Convexity}\) can detect violations of the convexity or immersion properties, in which case they produce the final answer \(\text{false}\) and terminate the main procedure Convexity-Checker.

We omit the pseudocode for Reduce-To-3D, for this procedure is doing standard linear algebra. We only note that this and the other two subroutines can be implemented together, without affecting the algebraic complexity bounds given below, so that only polynomials of degree at most \(n\) are evaluated.

The input to Produce-Polygon-In-2D is a wheel graph \(W_k\) \((k > 1)\), each of whose vertices \(v\) is equipped with a point \(p(v)\) in \(\mathbb{R}^3\) – in other words, the input is a rectilinear realization \(p : W_k \rightarrow \mathbb{R}^3\) of \(W_k\) in \(\mathbb{R}^3\). For notational simplicity we assume that the vertex set \(V(W_k)\) of \(W_k\) is \([0, 1, \ldots, k]\), where \(k\) is the centre and \([0, \ldots, k - 1]\) is a cyclic order on the wheel. We also assume that \(k\) is \(\text{realized}\) at the origin 0. The output of Produce-Polygon-In-2D, which is also the input (unless non-convexity or non-immersion have been detected) to \(\text{Check-Polygon’s-Convexity}\), is a polygon in the plane. Different approaches, although with the same complexity, to the problem solved by \(\text{Check-Polygon’s-Convexity}\) have been suggested by Devillers et al (1998), and Mehlhorn et al (1999).

\[
\text{Algorithm 1 Produce-Polygon-In-2D}(W, p)
\]

1) Check if there is a support plane at the origin:
\[
c \leftarrow \sum_{i \in V(W)} p(i) \{c \text{ is the centroid of vectors } p(i)\}
\]
\[
k \leftarrow |V(W)| - 1
\]
for \(i = 0\) to \(i = k - 1\) do
\[
\text{if } p(i) \cdot c < 0 \text{ then return: } \text{false}; \text{ terminate}
\]
end if
end for
2) Check if all rays of the cone are distinct:
for \(i = 0\) to \(i = k - 1\) do
\[
\text{if } p(i) \times p(i + 1 \mod k) = 0 \text{ then return: } \text{false}; \text{ terminate } \{\text{not an immersion}\}
\]
end if
end for

\[
\text{Let } \mathcal{K} \subset \mathbb{R}, (\mathbb{Z} \subset \mathcal{K}), \text{ be the base ring of the computational model, i.e., all numerical data, such as vertex coordinates must come from } \mathcal{K}.
\]

\[
\text{Let } \mathbb{Z}[\text{Input}] \text{ be the ring polynomial with integer coefficients in the input parameters}
\]

3) Compute the polygon:
Find \(\min\{\alpha \in \mathcal{K}_{>0} \mid (\text{lin } p(i)) \cap (\text{lin } c)_{\alpha}^{\perp} \in (\mathbb{Z}[\text{Input}])^3\}\)
for \(i = 0\) to \(i = k - 1\) do
\[
q(i) \leftarrow (\text{lin } p(i)) \cap (\text{lin } c)_{\alpha}^{\perp}\}
\{q(i) \text{ denotes } \text{lin } p(i) \cap (\text{lin } c)_{\alpha}^{\perp}\}
end for

Input Conventions
Suppose the input is given as the subposet \(\text{SP} < \text{P}\) of cells of dimensions \(n - 3, n - 2, n - 3\) and the equations of \((n - 1)\)-faces. Assume that the dimension of each cell \(C \in \text{SP}\) is stored with it. There are mutual links between \((n - 1)\)-cells in \(\text{SP}\) and the records that have keep their equations. When we operate with any cell \(C \in \text{SP}\), we assume that all cell’s attributes, such as its dimension, links to incident cells in \(\text{SP}\), and the equation (if \(\dim C = n - 1\)) are accessible to \(C\) at any time at unit cost. In the beginning all \((n - 3)\)-cells of \(\langle M, P \rangle\) are put into a stack Ridges.

If the input is given by the subposet \(\text{SP}'\) of cells of dimensions \(n - 3, n - 2, n - 3, 0\), equipped with the coordinates of the vertices, extra time is needed to extract linear-algebraic information about the stars
Algorithm 2 Check-Polygon’s-Convexity($W$, $q$)

Check-Polygon’s-Convexity is now tracing the PL-circuit $[q(0), \ldots, q(k-1), q(0)]$ in the plane and terminates the procedure with answer $false$ whenever it encounters any of the following: (1) the curve made a turn in the direction opposite to the first non-zero turn $[q(i_1), q(i_1 + 1), q(i_1 + 2)]$; (2) edges $[q(i), q(i + 1)]$ and $[q(i + 1), q(i + 2)]$ are on one line and intersect; (3) $[q(0), \ldots, q(k-1), q(0)]$ has too many critical points in one of the linearly independent directions $q(i_1 + 1) - q(i_1)$ and $q(i_1 + 2) - q(i_1 + 1)$. For (3) we take two directed adjacent non-collinear edges and declare them axes $x$ and $y$. Any convex PL-circuit that passes through $x$ and $y$ must have two critical connected subsets in each direction. $Ver$ and $Hor$ are sequences of signs (+ and −) indicating increase and decrease. If the curve keeps increasing, or stays constant, in the direction, say $x$, we do not change $Ver$. It is easy to check if they are valid sequences for a convex curve and this is done by Boolean function Test. Update takes sequences $Ver$ and $Hor$, the next directed edge on the circuit $[q(0), \ldots, q(k-1), q(0)]$ and updates $Ver$ and $Hor$.

$k \leftarrow |V(W)| - 1$
$j \leftarrow 0$
while $q(j + 1 \mod k) - q(j) \times q(j + 2 \mod k) - q(j + 1 \mod k) = 0$ do
  $j \leftarrow j + 1$
end while

$n \leftarrow ((q(j + 1) - q(j - 1)) \times (q(j) - q(j - 1)))$ \{n is the positive rotation\}
$y \leftarrow q(j) - q(j - 1); x \leftarrow q(j + 1) - q(j)$ \{y, x are the “vertical” and “horizontal” axes\}
if $x \cdot y \geq 0$ then
  $Ver \leftarrow(+,+); Hor \leftarrow(+,+)$ \{initialization of sign sequences\}
else
  $Ver \leftarrow(-,+); Hor \leftarrow(-,+)$
end if
for $i$ from $j$ to $i = k + j - 1$ do
  Update($Ver, Hor, q(i + 1 \mod k) - q(i \mod k)$)
  if \{$Test(Ver, Hor) = false$ \ OR \ {$q(i + 2 \mod k) - q(i \mod k) \times (q(i+1 \mod k) - q(i \mod k)) \cdot n < 0$\} \ then
    return: $false$; terminate
  end if
end for
return: $true$
of \((n-3)\)-faces, which is at most \(O(f_{0,n-3})\). This extra term does not affect complexity if \(n = 3\), or if \((M, P)\) consists of cells with bounded \((n)\) number of vertices.

Algorithm 3 Convexity-Checker

\[
\text{while } \text{Ridges} \neq \emptyset \text{ do}
\begin{align*}
\text{Pop an } (n-3)\text{-face } F \text{ from Ridges} \\
(W, p) &= \text{Reduce-To-3D}(\text{Star } F) \\
(W, q) &= \text{Produce-Polygon-In-2D}(W, p) \\
\text{if } \text{Check-Polygon's-Convexity}(P, f) = false \text{ then} \\
\text{return: } false; \text{ terminate}
\end{align*}
\text{end if}
\text{end while}
\text{return: true}
\]

Complexity estimates

Denote by \(f_k\) the number of \(k\)-cells of \((M, P)\), and by \(f_{kl}\) – the number of incidences between \(k\)-cells and \(l\)-cells in \((M, P)\). The stack Ridges is addressed at most \(f_{n-3}\) times. The other steps take at most \(c \cdot f_{n-2,n-3}(\text{Star } F)\) arithmetic operations for each \(F\), where \(c\) does not depend on \(F\). Thus, the steps repeated for the stars of all \((n-3)\)-faces require \(O(f_{n-2,n-3})\) operations. Therefore, the total number of operations for this algorithm is \(O(f_{n-2,n-3})\).

4.1 Preprocessing for \(n = 3\)

Since the algorithm runs in linear time in the number of 0-cells \(f_0(\text{of } (M, P)\) when \(M\) is spherical, but a sequence of non-spherical PL-manifolds of dimension 2 can have the number of edges which grows quadratically in \(f_0\), it is desirable to check the topological type of the input by just counting edges in the input poset \(\text{SP} < P\). Once their number exceeds \(cf_0\), where \(c\) is some constant that is easy to calculate, we stop and declare the input non-convex. This preserves \(O(n)\) running time bound for PL-surfaces in \(\mathbb{R}^3\). One may wonder if this is necessary, since it seems very likely our algorithm will quickly encounter a non-convex vertex, if the input is a PL-sphere-with-handles. Surprisingly, Betke and Gritzmann (1984), proved that any orientable (non-spherical) connected closed 2-manifold can be PL-embedded into \(\mathbb{R}^3\) so that it has only 5 non-convex vertices, and this number cannot be made smaller! The same question for PL-immersions of non-orientable closed 2-manifolds is wide open.

4.2 Concluding Remarks

Remark 11 The algorithm processes the stars of all \((n-3)\)-faces independently. On a parallelized computer the stars of all \((n-3)\)-faces can be processed in parallel.

Remark 12 The algorithm does not use all of the poset \(P\) of \((M, P)\).

Remark 13 The algorithm requires computing polynomial predicates only. The highest degree of algebraic predicates that the algorithm uses is \(n\), which is optimal (see Devillers et al, 1998).

Remark 14 If the vertices of \((M, P)\) are realized by \(v\) in \(\mathbb{R}^n\) so that there is a 3-dimensional coordinate subspace \(L\) of \(\mathbb{R}^n\) such that all the subspaces spanned by \((n-3)\)-faces are (affinely) complementary to \(L\), the stars of \((n-3)\)-faces can be projected on \(L\) and all computations be done as in \(\mathbb{R}^3\). This reduces the degree of predicates from \(n\) to \(3\). Therefore, for sufficiently generic realizations the algorithm has the algebraic degree of at most 3 and the complexity that does not directly depend on \(n\).

1) The complexity of the algorithm is asymptotically equal to that of Devillers et al (1998) and Mehlhorn et al (1996-1999) in the case of 2D simplicial surfaces in \(\mathbb{R}^3\); for \(n > 3\) our algorithm is asymptotically faster than theirs. While Devillers et al. state running time \(O(n)\) for all dimensions, it is clearly a typo. First, this is clearly impossible by fundamental theorems on polyhedral combinatorics, even for PL-spheres. Second,
the pseudocode in Section 3 of [5] has time $d-2 \sum_{j=0}^{d-2} f_j + \sum_{\Gamma \in \mathcal{F}} \sum_{\text{dim } F = d-2}^{d} \sum_{j=3}^{d} f_{j-1,F}$. Mehlhorn et al do not give bound for $\mathbb{R}^n$, but mention a standard reduction to Linear Programming, which result in our bounds.

2) Devillers et al. and Mehlhorn et al. algorithms require the surfaces to be simplicial;

3) Devillers et al. and Mehlhorn et al. algorithms assume extra information, namely, the input surface is known to be oriented and comes together with coherently oriented facet normals. This is unnecessary.

4) What Devillers et al. and Mehlhorn et al. call ”local convexity at $(n-2)$-faces” is not in fact a local property, since it assumes the orientability of the surface.

Remark 15 Our algorithm can also be applied without changes to compact PL-hypersurfaces in $\mathbb{S}^n$ or $\mathbb{H}^n$.

References


