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THE LOSS OF TENSION IN AN INFINITE MEMBRANE WITH HOLES DISTRIBUTED ACCORDING TO A POISSON LAW

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Abstract

What is the effect of punching holes at random in an infinite tensed membrane? When will the membrane still support tension? This problem was introduced by Connelly in connection with applications of rigidity theory to natural sciences. The answer clearly depends on the shapes and the distribution of the holes. We briefly outline a mathematical theory of tension based on graph rigidity theory and introduce a probabilistic model for this problem. We show that if the centers of the holes are distributed in $\mathbb{R}^2$ according to a Poisson law with density $\lambda > 0$, and the shapes are i.i.d. and independent of the locations of their centers, the tension is lost on all of $\mathbb{R}^2$ for any λ. After introducing a certain step-by-step dynamic for the loss of tension, we establish the existence of a nonrandom $N = N(\lambda)$ such that N steps are almost surely enough for the loss of tension. Also, we prove that $N(\lambda) > 2$ almost surely for sufficiently small λ. The processes described in the paper are related to bootstrap and rigidity percolation.

Keywords: (Bootstrap) percolation; Poisson process; tension; rigidity

AMS 2000 Subject Classification: Primary 52A; 52C; 60K; 82B
Secondary 73B; 82C

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1. Introduction

Let M be a tensed membrane or film in space clamped on its boundary. If we make a small convex hole in this membrane, the tension redistributes over the rest of the membrane. However, the situation is more complicated when there are several holes. For example, if a nonconvex hole or a couple of convex overlapping holes are punched in the membrane, the tension is lost on at least the convex hull of this set (see Figure 1).

It is less intuitive but still true that the tension may be lost at some subset of the complement of a collection of convex nonoverlapping holes. For example, the convex hull of the three holes shown in Figure 2 cannot support tension. If the area where the tension ought to be lost is termed defective, all three polygons in Figure 2 must coalesce into one big defect. This fact follows from Theorem 4.1.10 of Rybnikov (1999) that relates the existence of tension to projections of convex polyhedral surfaces. Such a coalescence effect of nonoverlapping holes was first noticed by Connelly (1998).

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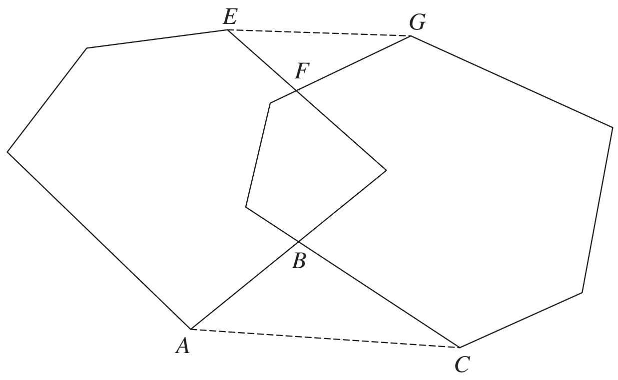

FIGURE 1: Two overlapping holes.

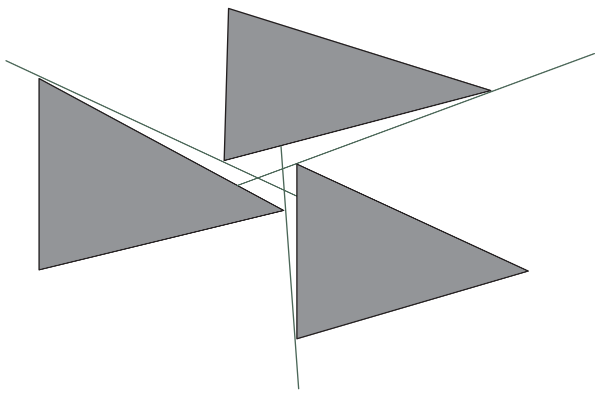

FIGURE 2: Three nonoverlapping holes.

In Section 2, we give basic definitions from rigidity theory and define the notion of a spider web which we use to describe a tensed medium. In Section 3, we give a formal treatment of tension in the language of spider webs. There we define the notion of a set which supports tension, and provide extra physical motivation for studying spider webs (Theorem 2.1). Our mathematical treatment of a tensed medium is discrete in nature and based on ideas from rigidity theory, as opposed to classical methods of mechanics of continuous media that use smooth functions, tensor fields, and differential equations for describing elastic properties of materials. In the planar case, holes with smooth boundary can be approximated by polygons. As a result, the theory based on graph rigidity seems to be better suited for computer simulations. This graph-theoretic treatment of tension was introduced by Connelly (1998). Moreover, the existence of a ‘discrete’ tension in the complement of the holes has some important physical implications for rigidity properties of large planar graphs realized in the complement of the holes (Theorem 3.1) and has an interesting interpretation in terms of projections of convex polyhedral surfaces on the plane (see Rybnikov (1999)).

The main goal of the paper is to investigate the existence of a subset supporting tension in the complement of an infinite set of holes in $\mathbb{R}^2$. We prove that if locations of the holes are distributed according to a Poisson law with $\lambda > 0$ and their shapes (properly defined) are i.i.d. and independent of the hole locations, then the tension is lost on all of the complement of the holes, which means that the subset supporting tension is empty almost surely. Surprisingly, this result holds for *any* $\lambda > 0$ and *any* distribution of the shapes of the holes. Thus, no phase transition is present. Our results remain valid if we replace $\mathbb{R}^2$ by $\mathbb{R}^d$, $d \geq 3$, though we do not have a nice physical interpretation of this model in higher dimensions.

Note that, in our model, after the hole creation the retained medium remains static. Should the medium be able to deform after the holes have been punched, it would possibly arrive at a new stressed configuration in equilibrium. Thus, for a medium with good elastic properties, our results on the loss of tension imply that, for any density of the defects, in order to preserve stability, the system has to rearrange itself. Using computer simulation, Tang and Thorpe (1987), (1988) studied rigidity of large finite networks of elastic springs of natural length zero where a tension is supplied by the boundary frame. In their model, as soon as bonds (edges) are removed the network (framework) deforms around the missing edges and the sites move to new equilibrium positions. Their simulation results suggest that there is a nontrivial threshold that lies between the critical thresholds for rigidity percolation (TRP) and connectivity percolation (TCP). The value of this threshold depends on the value of the external force. As the external force changes from zero to infinity, the value of the threshold changes from TRP to TCP. Of course, if the probability distribution of holes is such that the retained medium is not even connected, the material will not be able to rearrange itself to support tension even if the medium is elastic. This question is addressed in the connectivity percolation problems, which do have critical probabilities (for the distribution of holes) below which the medium remains almost surely connected, and above which it is almost surely disconnected; see e.g. Menshikov and Sidorenko (1987) and Meester and Roy (1996).

Additionally, we introduce certain dynamics in the model, to accommodate the fact that it may take time for a few overlapping holes to coalesce into a bigger one which is a convex hull of them. A rough description of the model is as follows. The positions of the holes are distributed according to a Poisson distribution on $\mathbb{R}^d$, $d \geq 2$, with density $\lambda > 0$, and the shapes of the holes are i.i.d. and independent of the Poisson process. The holes are termed *defects of the zeroth generation*. A defect of the kth generation ($k \geq 1$) is iteratively defined as the convex hull of a connected component of defects of the $(k-1)$th generation. For example, the defects of the first generation are the convex hulls of all connected holes. A more formal description of the defects can be found in Section 3. There we show that if for each point of $\mathbb{R}^2$ there is a k such that this point is covered by a defect of the kth generation, tension is lost on all of $\mathbb{R}^2$. The main theorem given in Section 4 says that there exists a nonrandom $N(\lambda)$ such that a defect of the Nth generation is all of $\mathbb{R}^d$ almost surely. In particular, on $\mathbb{R}^2$ this result implies that, for any $\lambda > 0$, there is no subset supporting tension in the complement of the holes.

In Section 4.3, we define N_{cr} as the smallest index generation for which the defect of that generation covers all $\mathbb{R}^d$. We show that N_{cr} is almost surely nonrandom and can take on values different from 1 and 2 (which are, in a sense, the trivial ones).

The loss-of-tension model we describe is similar to bootstrap percolation models on trees introduced by Chalupa *et al.* (1979) and later on d-dimensional lattices by Kogut and Leath (1981). In these models, points are independently occupied with a low density and the resulting configuration is taken as the initial state for dynamics based on some collection of local rules in which the occupation status of a point is updated according to the configuration of its neighbors. Van Enter (1987) conducted a rigorous analysis of these models (see also Aizenman and Lebowitz (1988)).

Since then, there have been many publications on this topic. For the latest results on bootstrap percolation see Dehghanpour and Schonmann (1997) and Gravner and McDonald (1997). For

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a review of bootstrap percolation models see Adler (1991). Gravner and Schonmann (personal communication) had some ideas on how to extend their results for the continuous case, but as far as the authors know, there have been no results on the model which we describe and analyze below.

Finally, we want to mention that, although the geometrical facts presented in Sections 2 and 3 are very important for the motivation of our paper, the probabilistic part of the paper is self-contained and is interesting per se. The reader who is interested in the probabilistic results only can now safely proceed directly to Section 4 after reading Definitions 3.2–3.4.

2. Rigidity and its applications

Over the past two decades two-dimensional and three-dimensional random central-force networks have been used by physicists for modeling the elastic behavior of glasses and proteins within the framework of effective medium theory (see Thorpe (1983), Jacobs and Thorpe (1995), (1996), Duxbury *et al.* (1998), and Thorpe and Duxbury (1999)).

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It turns out that real glasses are well represented by generic random networks (in mathematics of rigidity there is a tendency to use the term framework instead of the term network preferred by physicists). The success of these methods resulted in good characterization of elastic properties of glasses such as $\mathrm{Ge}_x\mathrm{As}_y\mathrm{Se}_{1-y}$ (Thorpe (1983)). The rigidity analysis of central-force networks has also been used for characterization of physical properties of other substances such as proteins and semiconductors (see Thorpe and Duxbury (1999)). While mathematical theory of generic (combinatorial) rigidity percolation can be found in Holroyd (1998), (2000) and Grimmett (1999), the results of this paper have applications to the geometric area of rigidity theory (see e.g. Connelly (1982)).

A realization of a graph in $\mathbb{R}^d$ is a mapping of its vertex set into $\mathbb{R}^d$. A bar-and-joint framework is a connected graph with at most countably many vertices, together with its realization in $\mathbb{R}^d$. We consider only *discrete frameworks*: any compact subset of $\mathbb{R}^d$ may contain only a finite number of vertices. Denote by $F(E, V, V_0; \boldsymbol{p})$ a framework in $\mathbb{R}^d$ with the edge set E, the vertex set $V = \{v_i\}$ with a pinned (fixed in $\mathbb{R}^d$) subset of vertices $V_0 \subset V$, and the realization $\boldsymbol{p}$ in $\mathbb{R}^d$. Here $\boldsymbol{p}$ can be regarded as a vector in $\mathbb{R}^{dV}$ whose components are the coordinates of all the vertices $\{v_i\}$ of F: $\boldsymbol{p} = \{v_{11}, \dots, v_{1d}; \dots; v_{|V|1}, \dots, v_{|V|d}\}$. For any subset of vertices $V' \subseteq V$, $\boldsymbol{p}(V')$ will stand for the realization of the vertices of V' in $\mathbb{R}^d$. We denote the graph of the framework by $G(E, V, V_0)$, where V_0 is the set of vertices that must be pinned in a realization. Vertices that are not pinned are called *free*. If $V_0 = \varnothing$, we will write simply $F(E, V; \boldsymbol{p})$. Two vertices v and w of $G(E, V, V_0)$ are called *adjacent* if there is an edge of E connecting them. The set of all the vertices adjacent to v is denoted by $A(v)$.

Definition 2.1. An equilibrium stress (or self-stress) is an assignment of real numbers $\omega_{vw} = \omega_{wv}$ to edges of E (a *tension* if the sign of ω is positive, or a *compression* if the sign is negative) so that the equilibrium conditions

$$\sum_{w \in A(v)} \omega_{vw}(\boldsymbol{p}(v) - \boldsymbol{p}(w)) = 0$$

hold at each vertex $v \in V \setminus V_0$.

Definition 2.2. A framework $F(E, V, V_0; \boldsymbol{p})$ that has an equilibrium stress, positive on all edges, is called a *spider web*.

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In other words, $F(E, V, V_0; \boldsymbol{p})$ is a *spider web* if it supports an equilibrium tension. Proposition 2.1 follows directly from the definition of spider webs.

Proposition 2.1. *If $F(E, V; \boldsymbol{p})$ is a spider web and $v \in V$, then $\boldsymbol{p}(v)$ lies in the relative interior of* conv $\boldsymbol{p}(A(v))$.

Proposition 2.2. *If $F(E, V; \boldsymbol{p})$ is a spider web in $\mathbb{R}^d$ and $\boldsymbol{p}(V)$ is not a point, then F has infinitely many edges and vertices and their convex hull is an affine subspace of $\mathbb{R}^d$.*

Proof of Proposition 2.2. Assume that $F(E, V; \boldsymbol{p})$ has only finitely many vertices. Then the convex hull of $\boldsymbol{p}(V)$ is a convex polytope whose vertex set is a subset of $\boldsymbol{p}(V)$. Let $\boldsymbol{p}(v)$ be a vertex of this polytope. By Proposition 2.1, $\boldsymbol{p}(v)$ lies in the interior of conv $\boldsymbol{p}(A(v))$. Therefore, it cannot be a vertex of conv $\boldsymbol{p}(V)$, unless $\boldsymbol{p}(V)$ is a point, a contradiction.

Fix a standard Euclidean metric on $\mathbb{R}^d$. Two frameworks in $\mathbb{R}^d$ are called *edge equivalent* if they have the same graph and the same lengths of all edges. Two edge-equivalent frameworks in $\mathbb{R}^d$ are called *congruent* if all distances between corresponding pairs of vertices are the same or, in other words, if they are equivalent with respect to the isometry group of $\mathbb{R}^d$.

Definition 2.3. A finite framework $F(E, V, V_0; \boldsymbol{p})$ is called *rigid* in $\mathbb{R}^d$ if there is an open set $N(\boldsymbol{p})$, $\boldsymbol{p} \in N(\boldsymbol{p}) \subset \mathbb{R}^{d|V|}$, such that any other realization $\boldsymbol{q}$ of the graph $G(E, V, V_0)$ satisfying the following conditions is congruent to $F(E, V, V_0; \boldsymbol{p})$:

(i) $F(E, V, V_0; \boldsymbol{q})$ is edge equivalent to $F(E, V, V_0; \boldsymbol{p})$,

(ii) $\boldsymbol{q} \in N_{\boldsymbol{p}}$,

(iii) the pinned vertices of $F(E, V, V_0; \boldsymbol{q})$ coincide with those of $F(E, V, V_0; \boldsymbol{p})$.

If a framework F satisfies the above definition with $N(\boldsymbol{p}) = \mathbb{R}^{d|V|}$, it is called *globally rigid*. Various notions of rigidity are used in graph rigidity theory and its applications. The next definition is one of the possible generalizations of the above for the case of infinite frameworks.

Definition 2.4. An infinite framework $F(E, V, V_0; \boldsymbol{p})$ in $\mathbb{R}^d$ is referred to as (globally) *pseudorigid* if, for any finite subset V' of V, the framework obtained from $F(E, V, V_0; \boldsymbol{p})$ by pinning all of the vertices in $V \setminus V'$ is (globally) rigid.

This concept is closely related to the notion of prestress stability (see Connelly and Whiteley (1996) for definitions), based on the energy approach to rigidity, which is typical for physics and engineering. For a mathematical treatment of this approach, see Connelly (1982). Consult Jacobs and Thorpe (1995), (1996) and Duxbury *et al.* (1998) for an exposition from the physicists' point of view. To avoid confusion, let us notice that the above definition is not the one used by Grimmett (1999) or Holroyd (1998), (2000) in their study of generic rigidity percolation. They call a framework $F(E, V; \boldsymbol{p})$ rigid if any finite subframework of F is contained in a rigid finite subframework of F. Rigidity in $\mathbb{R}^d$ obviously implies pseudorigidity in $\mathbb{R}^d$. However, rigidity in $\mathbb{R}^2$ does not imply pseudorigidity in $\mathbb{R}^3$. For example, an infinite graph triangulating $\mathbb{R}^2$ is always rigid in the plane; however, it may not be pseudorigid in space. Let us illustrate the differences between rigidity and pseudorigidity: the regular triangular lattice is rigid in $\mathbb{R}^2$ and pseudorigid in $\mathbb{R}^3$ but not rigid in $\mathbb{R}^3$, whereas the square lattice is pseudorigid in $\mathbb{R}^3$ but not rigid in $\mathbb{R}^d$ $(d \geq 2)$. These examples clearly demonstrate that a graph, rigid in $\mathbb{R}^2$, need not be rigid in $\mathbb{R}^3$. The pictures produced by Jacobs and Thorpe's (1995), (1996) computer program give many examples of $\mathbb{R}^2$-rigid, but not $\mathbb{R}^3$-rigid, random graphs. Meanwhile, the spider web property implies pseudorigidity in $\mathbb{R}^d$ for any $d > 1$.

Author: are these the references that you meant?

Theorem 2.1. *Let $F(E, V; \boldsymbol{p})$ be a (possibly infinite) spider web in $\mathbb{R}^2$. Then $F(E, V; \boldsymbol{p})$ is prestress stable and globally pseudorigid in $\mathbb{R}^d$ $(d > 1)$.*

This result can be derived from Connelly (1982) and Connelly and Whiteley (1996), where this is proved for finite frameworks. The proof directly applies to infinite frameworks, since the pseudorigidity is defined via finite subframeworks of F.

3. Tension

By a convex tiling of a closed planar set with piecewise-linear boundary or no boundary we mean a locally finite partition of this set into subsets of three types: open convex polygons, open segments called edges, and points called vertices. The one-skeleton of a tiling is a framework whose vertex set is the vertex set of the tiling, and whose edge set is the tiling's edge set. A tiling only with triangles such that any two triangles can only make contact either at a common vertex or at a full common edge is called a *triangulation* (see Figure 3).

In Figure 3, the triangulated set is the complement of four shaded polygons inside the bounding heptagon. A countable or finite set of open polygons $\mathcal{H}$ is called a discrete set of holes if the number of polygons intersecting any compact set of the plane is finite. We call the elements of $\mathcal{H}$ holes and denote by H the pointwise union of the holes.

Definition 3.1. Let M be a set with a polygonal boundary or no boundary in $\mathbb{R}^2$, and let $\mathcal{H}$ be a discrete collection of holes in M. We say that the membrane with holes $(\mathcal{M}, \mathcal{H})$ supports tension if $M \setminus H$ admits a tiling whose skeleton has edge set E and vertex set V such that the framework $(E, V, V \cap \partial M)$ is a spider web. Let S be a closed subset of $M \setminus H$. We say that tension is lost on S if there is no closed subset A of $M \setminus H$ such that A supports tension and contains S.

In Definition 3.1, a general convex tiling supporting tension can be replaced by a triangulation supporting tension. Indeed, let $\mathfrak{T}$ be a convex tiling whose one-skeleton is a spider web with an equilibrium stress τ. Each tile T of $\mathfrak{T}$ can be thought of as the base of a convex pyramid P_T in three-space with the apex projecting onto an *interior* point $\boldsymbol{v}$ of T. Denote by G_T the vertical projection of the one-skeleton of P_T on the plane of $\mathfrak{T}$ and by $E_{\text{int}}(G_T)$ the interior edges of the projection. By the Maxwell–Cremona correspondence (see e.g. Crapo and Whiteley

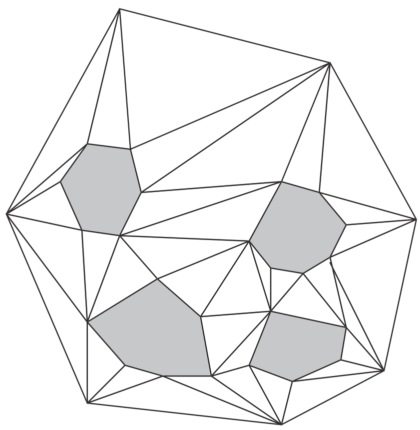

FIGURE 3: A triangulation.

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(1994)) the linear operator that projects P_T onto G_T induces an equilibrium stress on G_T, which is tension on the interior edges of the projection and compression on the boundary edges; this stress is defined up to scale by the geometry of P_T. Denote this stress by ω. Since equilibrium stresses form a linear space, without loss of generality we may assume that ω is small enough, i.e. for all edges e of G_T, $|\omega(e)| < \tau(e)$. Add the interior edges $E_{\text{int}}(G_T)$ and vertex $\boldsymbol{v}$ to the framework $(E, V, V \cup \partial M)$. Define a new stress τ_{new} on the resulting framework $(E \cup E_{\text{int}}(T), V \cup \boldsymbol{v}, V \cup \partial M)$ in the following way:

1. if e is not an edge of G_T, then $\tau_{\text{new}}(e) := \tau(e)$;
2. if e is an edge of T, then $\tau_{\text{new}}(e) := \tau(e) + \omega(e)$;
3. if e is an edge of $E_{\text{int}}(G_T)$, then $\tau_{\text{new}}(e) := \omega(e)$.

Then τ_{new} is an equilibrium tension on $(E \cup E_{\text{int}}(T), V \cup \boldsymbol{v}, V \cup \partial M)$. We can assume that all tiles of $\mathfrak{T}$ are assigned distinct natural numbers: $\mathfrak{T} = \{T_1, T_2, \ldots\}$. We triangulate inductively the tiles of $\mathfrak{T}$ according to the above triangulation procedure: the equilibrium tension is redefined on each step as explained above. The value of tension on each particular edge can only be affected twice by this procedure, since it is incident to only two tiles. Thus, $\mathfrak{T}$ can be triangulated so that the resulting triangulation supports tension.

The ability of a planar set to support tension has important implications for rigidity of graphs triangulating this set.

Theorem 3.1. *Let $\mathcal{H}$ be a discrete collection of convex open polygons in $\mathbb{R}^2$, possibly overlapping. If $(\mathbb{R}^2 \setminus H, \mathcal{H})$ supports tension, then the one-skeleton of* any *triangulation of $\mathbb{R}^2 \setminus H$ is prestress stable and globally pseudorigid in $\mathbb{R}^3$.*

Theorem 3.1 was proved by Connelly (1998, unpublished) with the use of Theorem 2.1. The above theorem suggests an interpretation of the area that supports tension as stable, and of the area that cannot support tension as amenable to deformations requiring little energy.

Author: is Connelly (1998, unpublished) the same reference as Connelly (1998) on p. 1?

If we wanted to study tension in a three-dimensional medium with voids, we would have to define first the notion of a set in $\mathbb{R}^3$ which supports tension. Note that not all spider webs in dimensions higher than two can be interpreted as one-skeletons of polyhedral tilings (see Connelly and Whiteley (1993), (1996)). That is why the notion of a set supporting tension for $\mathbb{R}^d$ $(d > 2)$ should be different. For example, we could require the existence of a three-dimensional spider web (in the complement of the holes) whose vertex set contains the vertex set of the holes.

Author: please give details of Connelly and Whiteley (1993).

Let us now make some observations about holes. First, if a hole is nonconvex, then there is no triangulation of the complement such that its one-skeleton supports an equilibrium tension. For instance, the equilibrium at vertex $\boldsymbol{v}$ in Figure 4 is impossible if all edges incident to this vertex are under tension (the arrows in the figure show the directions of tension forces at $\boldsymbol{v}$). Therefore, if two holes overlap, tension is lost on a convex hull of their union. Intuitively, the loss of tension is rather obvious—in Figure 1, the pieces ABC and EFG are ‘floppy’ in the space. The following statement summarizes this observation.

Proposition 3.1. *Let H be an open subset of $\mathbb{R}^2$ with piecewise-linear boundary. If $(\mathbb{R}^2, H)$ supports tension, then all connected components of H are convex polygons.*

As we mentioned before, the above proposition cannot be directly generalized for dimensions greater than two. Obviously, a saddle point of a three-dimensional void (‘three-dimensional hole’) can serve as a vertex of a spider web realized in the complement of the hole. Therefore,

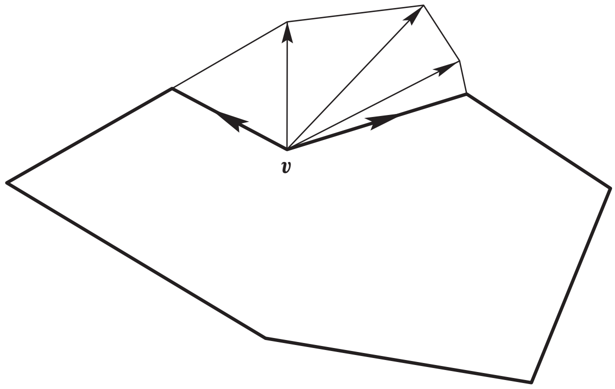

FIGURE 4: Nonconvex hole.

our result for iterated convex hulls in the independent Poisson model (Theorem 4.1) does not imply that for any $\lambda > 0$ tension is lost on all of the complement of the holes in $\mathbb{R}^d$ when $d \geq 3$.

Let us formally introduce the notion of defects. The initial defects are the holes themselves. Let $\mathcal{H}$ denote the set of the holes. Physically, they are the areas where the material of the membrane has been removed or damaged. Definitions 3.2–3.4 iteratively define the notion of defects of the kth generation. In these definitions it is no longer assumed that the holes are polygons.

Definition 3.2. Let $\mathcal{H}$ be a set of holes. Elements of $\mathcal{H}$ are called defects of the zeroth generation.

Definition 3.3. A connected component (understood topologically) of defects of the kth generation is referred to as a k-cluster.

Definition 3.4. A defect of the $(k+1)$th generation is the convex hull of a k-cluster.

Figure 5 provides an illustration to our definitions. Here we used a solid line for the boundary of holes (the defects of the zeroth generation), a dashed line for the boundary of a defect of the first generation, and a dotted line for the boundary of a defect of the second generation. By definition, no tension is possible in holes.

Lemma 3.1. *Let our membrane M be all of $\mathbb{R}^2$. Then tension is lost on a defect of generation k for any $k \geq 0$.*

Proof. Proceed by induction in generation of defects. For $k = 0$ the lemma is true by definition. Assume we have proved the lemma for defects of the kth generation.

Suppose tension is not lost on all of $\operatorname{conv}(C^k) \setminus C^k \neq \varnothing$. Thus, there is a supporting tension subset $S \subset \mathbb{R}^2 \setminus C^k$, such that $S \cap (\operatorname{conv}(C^k) \setminus C^k) \neq \varnothing$. Let $T(S)$ be a triangulation of S whose one-skeleton is a spider web. Since $\operatorname{conv}(C^k) \setminus C^k \neq \varnothing$, there is a vertex $\boldsymbol{v}$ of $T(S)$ such that any line passing through $\boldsymbol{v}$ has a nonempty intersection with $\operatorname{int} C^k$. Furthermore, we can assume that $\boldsymbol{v}$ is a boundary vertex of $T(S)$. Consider a small disk $D(\boldsymbol{v})$ centered at $\boldsymbol{v}$. The disk $D(\boldsymbol{v})$ is naturally partitioned into alternating sectors of S and $\mathbb{R}^2 \setminus S$ (these sectors look like 'pie slices'). Combining the fact that any line passing through $\boldsymbol{v}$ has a nonempty intersection with $\operatorname{int} C^k$ and Proposition 2.1, we conclude that all sectors of $\mathbb{R}^2 \setminus S$ have angle less than π at $\boldsymbol{v}$. For that reason and because there is no line passing through $\boldsymbol{v}$ such that C^k lies in one of the half-planes defined by this line, there must be at least two disconnected slices

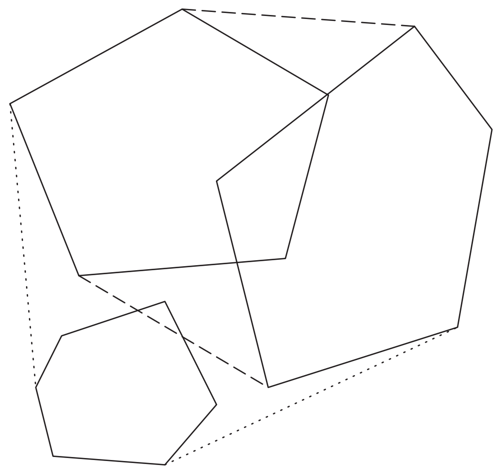

FIGURE 5: Defects of different generations.

of $\mathbb{R}^2 \setminus S$ in $D(\boldsymbol{v})$ that have nonempty intersection with C^k. These two slices of $\mathbb{R}^2 \setminus S$ span two disconnected infinite convex cones with vertex $\boldsymbol{v}$; both cones have common points with C^k. Since the boundary of $\mathbb{R}^2 \setminus S$ is polygonal and at any point of $\partial(\mathbb{R}^2 \setminus S)$ the inner angles must be at most π (see Proposition 3.1 and Figure 4), these two cones contain two disjoint connected components of C^k. This contradicts the assumption that C^k is connected.

Finally, note that the definition of defect does not cover all subsets that cannot support tension, since the area where tension is lost might be, in fact, the convex hull of a number of nonoverlapping holes (see the second paragraph of Section 1). However, our main result shows that even without taking into account the coalescence of some nonoverlapping holes, we can prove that in our model the tension is lost on all of $\mathbb{R}^2$ almost surely.

4. Percolation on iterative defects

In this section, we show that if the centers of holes are distributed in $\mathbb{R}^2$ according to Poisson law and their shapes are i.i.d., tension is lost on all of $\mathbb{R}^2$. Moreover, the following results on iterative convex hulls are true not only for dimension two, but also for $\mathbb{R}^d$ (see Section 4.2). While discussing iterative convex hulls, it is convenient to adopt a slightly more general definition of a hole and consider not only polygonal holes.

Definition 4.1. A hole (f-hole) centered at $p \in \mathbb{R}^d$ is a region

$$H(p, f) = \left\{ p + f\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)\boldsymbol{x} \;\middle|\; \|\boldsymbol{x}\| \leq 1 \right\},$$

where f is a continuous positive function defined on a unit $(d-1)$-sphere.

Definition 4.2. We say that a collection of holes $\mathcal{H} := \{H_1(p_1, f_1), H_2(p_2, f_2), \ldots\}$ is uniformly bounded in size from below if there exists an $r > 0$ such that $f_i(\boldsymbol{x}) \geq r$ for all f_i and $\|\boldsymbol{x}\| = 1$. Similarly, we say that $\mathcal{H}$ is uniformly bounded in size from above if there exists an $R > 0$ such that $f_i(\boldsymbol{x}) \leq R$ for all f_i and $\|\boldsymbol{x}\| = 1$.

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Consider a d-dimensional Poisson point process with density λ. Let $Y = Y(\omega)$ be the set of points of some realization ω of the process. Each point $y \in Y(\omega)$ is the center of a hole $H(y, f_y)$, where $f_y : C(\mathbb{S}^{d-1}, \mathbb{R}_+) \to \mathbb{R}_+$ is some positive and continuous function.

Fix a probability measure μ on $C(\mathbb{S}^{d-1}, \mathbb{R}_+)$. (Here $C(\mathbb{S}^{d-1}, \mathbb{R}_+)$ denotes the set of all continuous functions from a unit $(d-1)$-sphere to the set of positive numbers. The choice of a σ-algebra can be arbitrary, provided all the events mentioned further in the paper are measurable.) Suppose that for each y the function f_y is drawn from the distribution μ independently of the other functions and the configuration ω. Denote by $\mathcal{P}(\lambda, \mu)$ the constructed 'holey' process with density λ and the distribution of holes' shapes μ. Note that under certain conditions (e.g. $\mathcal{H}$ is bounded in size from above and below) there exists $\lambda_c = \lambda_c(\mathcal{H}, \mu)$ such that for $\lambda < \lambda_c$ there is no infinite cluster of holes (0-cluster in our notation), and for $\lambda > \lambda_c$ such cluster exists almost surely, see Menshikov and Sidorenko (1987). Definition 4.1 also was introduced there. Below is our main result (recall Definitions 3.2–3.4).

Theorem 4.1. *For any distribution μ on the space of positive continuous functions on the unit sphere and any $\lambda > 0$ there exists a nonnegative integer $N = N(\mu, \lambda)$ such that the N-cluster is $\mathbb{R}^d$ almost surely.*

By Lemma 3.1 this theorem implies that, in the independent Poisson model, tension is lost on all of $\mathbb{R}^2$ almost surely. To prove the above theorem, we reduce it to a simpler form, Lemma 4.2. First, observe the following.

Lemma 4.1. *It is sufficient to prove Theorem 4.1 under the additional assumption that the set of defects is bounded from below by some $r > 0$.*

Proof. By definition, the shape of a hole is defined by a continuous positive function $f(u)$ on the unit sphere $\mathbb{S}^{d-1}$. Since $\mathbb{S}^{d-1}$ is compact and $f(u)$ is continuous, $f(u)$ achieves its minimum on $\mathbb{S}^{d-1}$; this minimum is strictly positive, because $f(u) > 0$ for any $u \in \mathbb{S}^{d-1}$. Let $\rho(f) := \min_{u \in \mathbb{S}^{d-1}} f(u) > 0$. Then ρ is a real-valued positive random variable. Thus, there is an $r > 0$ such that $p_0 := \mathrm{P}(\rho \geq r) > 0$.

Let $\tilde{\mathcal{P}}(\lambda, \mu, r)$ be a process obtained from $\mathcal{P}(\lambda, \mu)$ by removing all the holes $H(y, f_y)$ with $\rho(f) < r$. This process is distributionally equivalent to $\mathcal{P}(\tilde{\lambda}, \tilde{\mu})$ where $\tilde{\lambda} = \lambda p_0$ and the distribution of the shapes of defects $\tilde{\mu}$ is such that the radius of the inscribed circle for any defect is at least r (the distribution $\tilde{\mu}$ is obtained by conditioning μ on the event $\rho(f) \geq r$, which has a positive probability). Consequently, the set of holes of $\tilde{\mathcal{P}}(\lambda, \mu, r)$ is now uniformly bounded in size from below. If the statement of Theorem 4.1 holds for any $\tilde{\mu}$ such that $\rho(f) > r > 0$ and any $\tilde{\lambda} > 0$, then Theorem 4.1 holds for an arbitrary μ, because the set of holes in the 'tilde' model (i.e. with the distribution $\tilde{\mu}$) is stochastically smaller than that of the original one.

Second, note that, by the result of the above lemma, we can assume that each hole $H(y, f_y)$ (defect of the zeroth generation) contains a ball of radius r centered at y. Since taking convex hulls is a monotonic operation (namely, if $A \subseteq B$ then $\operatorname{conv}(A) \subseteq \operatorname{conv}(B)$), it suffices to prove Theorem 4.1 for the case where all holes are balls of radius r. Moreover, without loss of generality we can assume that $r = 1$ because we can always rescale the space (see Zuev and Sidorenko (1985), also Meester and Roy (1996, Proposition 2.10)). Thus, Theorem 4.1 will follow from the statement below.

Author: please give details of Zuev and Sidorenko (1985).

Lemma 4.2. *Let $\mathcal{H}$ be a collection of balls of unit radius whose centers are distributed in $\mathbb{R}^d$ according to Poisson law with parameter $\lambda > 0$. Then there exists a nonnegative integer $N = N(\lambda)$ such that there is an infinite N-cluster whose convex hull is $\mathbb{R}^d$ almost surely.*

4.1. Proof of Lemma 4.2 for $d = 2$

We will present the proof for the planar case, i.e. $d = 2$. The proof for the case $d \geq 3$ is merely a corollary, and is given in Section 4.2.

Let us define an infinite increasing stochastic sequence of sets $G(k)$ starting from an arbitrary hole $H(y, f_y)$. Pick a hole (which is a unit disk) from $\mathcal{H}$; denote it by $G(0)$. Find all disks from $\mathcal{H}$ which have nonempty intersection with $G(0)$ and take the convex hull of these disks and $G(0)$; denote the resulting set by $G(1)$. Then consider all disks of $\mathcal{H}$ which intersect $G(1)$, and take the convex hull of these disks and $G(1)$; denote this set by $G(2)$. Iterating this procedure, we construct sets $G(3), G(4), \ldots$. If, for some k, no further element of $\mathcal{H}$ intersects with $G(k)$, we can conclude that $G(k + l) \equiv G(k)$ for all $l \geq 0$.

Proposition 4.1. *The set $G(k)$ is contained in some defect of the kth generation, which contains $G(0)$.*

This follows directly from the definitions of $G(k)$ and a defect of the kth generation.

Let $(\rho_{\boldsymbol{a}}, \varphi_{\boldsymbol{a}})$ denote the polar and $(x_{\boldsymbol{a}}, y_{\boldsymbol{a}})$ the Cartesian coordinates of a point $\boldsymbol{a} \in \mathbb{R}^2$. For simplicity, assume that $G(0)$ is centered at the origin. Denote by $\mathcal{C}(k)$ the circle of radius k centered at the origin.

Proposition 4.2. *With a positive probability depending only on λ,*

$$\mathcal{C}(k) \subseteq G(k) \quad \textit{for all } k = 1, 2, 3, \ldots. \tag{4.1}$$

This implies that $G(k)$ eventually covers all of $\mathbb{R}^2$ as $k \to \infty$.

Proof. Here we actually prove a stronger result: the inclusion (4.1) holds even if $G(k + 1)$ were constructed using $G(k)$ and *only* those circles of $\mathcal{H}$ whose centers lie inside the ring

$$R_{k+1} = \{\boldsymbol{a} \in \mathbb{R}^2 : \ k + \tfrac{1}{2} \leq \rho_{\boldsymbol{a}} \leq k + 1\}.$$

Denote by $\tilde{G}_k$ the process which uses only these circles to grow, with $\tilde{G}(0) := G(0)$. Clearly, $\tilde{G}(k) \subseteq G(k)$ for any k.

The process $\tilde{G}(\cdot)$ is an increasing process taking values in the subsets of $\mathbb{R}^2$. Observe that $\tilde{G}(k) \subseteq \mathcal{C}(k + 1)$. Let E_k be the event $\{\mathcal{C}(k) \subseteq \tilde{G}(k)\}$. Obviously,

$$\mathrm{P}(E_{k+1} \,|\, E_k, E_{k-1}, \ldots, E_1) \geq \mathrm{P}(E_{k+1} \,|\, \tilde{G}(k) = \mathcal{C}(k))$$

since we know that the probability $\mathrm{P}(E_{k+1} \,|\, \tilde{G}(k) = A)$ is increasing in $A \subset \mathbb{R}^2$.

Therefore, if we could find a sequence $\{\gamma_k\}$, $k = 1, 2, \ldots$, such that $\gamma_k \in (0, 1)$ for all k, $\sum_k \gamma_k < \infty$, and

$$\mathrm{P}(E_{k+1} \,|\, \tilde{G}(k) = \mathcal{C}(k)) \geq 1 - \gamma_k, \tag{4.2}$$

then $\mathrm{P}(E_{k+1} \,|\, E_k, E_{k-1}, \ldots, E_1) \geq 1 - \gamma_k$, and as a result

$$\mathrm{P}\left(\bigcap_{k=1}^{\infty} E_k\right) \geq \prod_{k=1}^{\infty}(1 - \gamma_k) > 0. \tag{4.3}$$

Obviously, $\mathrm{P}(E_{k+1} \,|\, \tilde{G}(k) = \mathcal{C}(k)) > 0$ for all k, and therefore we can always pick γ_k from (4.2) such that each $\gamma_k < 1$, and to check (4.3) it suffices to find and estimate γ_k only

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for large k. Suppose that $\tilde{G}(k) = \mathcal{C}(k)$. Break the ring $R_{k+1} = \mathcal{C}(k+1) \setminus \mathcal{C}(k+\frac{1}{2})$ into $M = M(k) := \lceil 2\pi(k+2)^{1/2} \rceil$ congruent pieces

$$R_{k+1}^{(i)} = \{\boldsymbol{a} \in R_{k+1} : \varphi_{\boldsymbol{a}} \in (\alpha i, \alpha(i+1)]\}, \qquad i = 0, 1, \ldots, M-1,$$

where

$$\alpha := \alpha(k) = \frac{2\pi}{M} \le \frac{1}{\sqrt{k+2}}. \tag{4.4}$$

Let

$$E_{k+1}^{(i)} = \{R_{k+1}^{(i)} \text{ contains at least one center of a circle from } \mathcal{H}\}.$$

The events $E_{k+1}^{(i)}$ are independent for different values of i, and, consequently, the probability of the event $\tilde{E}_{k+1} := \bigcap_{i=0}^{M-1} E_{k+1}^{(i)}$, that is, 'each $R_{k+1}^{(i)}$ contains such a center', is

$$\mathrm{P}(\tilde{E}_{k+1}) = (1 - \mathrm{e}^{-\lambda(k/2+3/8)\alpha})^M \ge 1 - M\,\mathrm{e}^{-\lambda(k/2+3/8)\alpha} = 1 - C_1 k^{1/2}\,\mathrm{e}^{-C_2 k^{1/2}}$$

for some positive constants C_1 and C_2 and large k.

Our next step is to show that $\tilde{E}_{k+1}$ implies E_{k+1} whenever $\mathcal{C}(k) \subseteq \tilde{G}(k)$. Once this is established, we can set

$$\gamma_k := C_1 k^{1/2}\,\mathrm{e}^{-C_2 k^{1/2}}, \tag{4.5}$$

and, as a result, $\sum \gamma_k < \infty$, which would yield (4.3), and thus prove Proposition 4.2.

Assume that $\tilde{E}_{k+1}$ occurs. Pick any point

$$\boldsymbol{a} \in \mathcal{C}(k+1). \tag{4.6}$$

Since each $R_{k+1}^{(i)}$ contains the center of a circle of $\mathcal{H}$, and every such circle obviously intersects $\mathcal{C}(k+1)$, there are points $\boldsymbol{b}$ and $\boldsymbol{c}$ in $\tilde{G}(k+1)$ such that

$$\begin{gathered}
\rho_{\boldsymbol{b}} \ge k + \tfrac{3}{2}, \qquad \rho_{\boldsymbol{c}} \ge k + \tfrac{3}{2}, \\
\varphi_{\boldsymbol{b}} \le \varphi_{\boldsymbol{a}} \le \varphi_{\boldsymbol{c}} \quad \mod 2\pi, \\
\varphi_{\boldsymbol{a}} - \varphi_{\boldsymbol{b}} \le \alpha \quad \mod 2\pi, \qquad \varphi_{\boldsymbol{c}} - \varphi_{\boldsymbol{a}} \le \alpha \quad \mod 2\pi,
\end{gathered} \tag{4.7}$$

Author: does (4.7) refer to all three lines, or just the last one?

where α is defined in (4.4). Since $\tilde{G}(k+1)$ is convex, it must contain all the points inside the triangle formed by the origin $\mathbf{0}$ and points $\boldsymbol{b}$ and $\boldsymbol{c}$. Let $\boldsymbol{h}$ be a point lying on the segment $[\boldsymbol{b}, \boldsymbol{c}]$ such that $[\mathbf{0}, \boldsymbol{h}]$ is orthogonal to $[\boldsymbol{b}, \boldsymbol{c}]$. Notice that this point has the smallest distance from the origin among all points of the segment $[\boldsymbol{b}, \boldsymbol{c}]$. Let us derive a lower bound on $\rho_{\boldsymbol{h}}$. From (4.7) we have that at least one of the angles formed by $[\mathbf{0}, \boldsymbol{h}]$ with $[\mathbf{0}, \boldsymbol{b}]$ and $[\mathbf{0}, \boldsymbol{c}]$ does not exceed α. Suppose it is the first one. Since $(k+\frac{3}{2})^2 > (k+1)(k+2)$, the length of $[\boldsymbol{h}, \boldsymbol{b}]$ is less than or equal to

$$\rho_{\boldsymbol{b}} \sin\alpha \le \rho_{\boldsymbol{b}}\alpha \le \frac{\rho_{\boldsymbol{b}}}{\sqrt{k+2}},$$

whence

$$\rho_{\boldsymbol{h}} \ge \sqrt{\rho_{\boldsymbol{b}}^2 - \frac{\rho_{\boldsymbol{b}}^2}{k+2}} \ge \sqrt{\frac{(k+\frac{3}{2})^2(k+1)}{k+2}} \ge k+1.$$

Consequently, (4.6) and (4.7) yield that the point $\boldsymbol{a}$ lies inside the triangle $\Delta(\boldsymbol{b}\mathbf{0}\boldsymbol{c})$ and, hence, belongs to $\tilde{G}(k+1)$.

The following statements follow from Proposition 4.2.

Corollary 4.1. *Suppose that while constructing $\tilde{G}(k+1)$ from $\tilde{G}(k)$, in the proof of Proposition 4.2 we are not allowed to use any circle whose center $\boldsymbol{a}$ is such that*

$$-k^{-2/3} \le \varphi_{\boldsymbol{a}} \le k^{-2/3} \mod 2\pi \quad \textit{or} \quad \pi - k^{-2/3} \le \varphi_{\boldsymbol{a}} \le \pi + k^{-2/3}.$$

Denote the resulting process by $G'(k)$. Let $p(k_0)$ be a conditional probability that the process $G'(k)$ ('avoiding' points close to the horizontal axis) can be continued indefinitely such that $\mathcal{C}(k) \subseteq G'(k)$ for all k, given that for some $k_0 > 1$ the circle $\mathcal{C}(k_0)$ ($k_0 > 1$) is already completely covered by some elements of $\mathcal{H}$ whose centers lie inside $\mathcal{C}(k_0)$. Then

$$\lim_{k_0 \to \infty} p(k_0) = 1.$$

Proof. First, the arguments of Proposition 4.2 for large k work in this case as well, since $k^{-2/3} = o(\alpha(k))$. Let the conditional probabilities corresponding to (4.2) with $\tilde{G}$ replaced by G' be $1 - \gamma'_k$. Since γ'_k are of the same form as γ_k given by (4.5), we again have $\sum \gamma'_k < \infty$. Next, observe that $p(k_0) = \prod_{k=k_0}^{\infty}(1-\gamma'_k) \ge 1 - \sum_{k \ge k_0} \gamma'_k$ and the sum on the right-hand side converges to zero as $k_0 \to \infty$.

Using Corollary 4.1 we can now finally prove Lemma 4.2. Fix some small $\varepsilon > 0$ and $k_0 > 1$ such that $p(k_0) \ge 1 - \varepsilon$. Let ν denote the probability that $\mathcal{C}(k_0)$ is completely covered by circles of $\mathcal{H}$. The value ν may be small; however, it is strictly positive and, therefore, there is a positive integer T such that

$$(1-\nu)^{T+1} < \varepsilon. \tag{4.8}$$

Consider $T+1$ circles of radius k_0 whose centers are located at the points

$$(0,0),\ (k_0^3, 0),\ (2k_0^3, 0), \ldots, (Tk_0^3, 0). \tag{4.9}$$

According to (4.8), the probability that at least one of them is completely covered by circles of $\mathcal{H}$ is larger than $1 - \varepsilon$. Pick any circle with a center from (4.9) (let it be, say, the ith one). The event {the process $G'(k)$ starting from $(ik_0^3, 0)$ can be continued indefinitely} is independent of the configuration inside the other circles, since when the process $G'(\cdot)$ reaches the jth circle $(j \ne i)$ $k \approx |i-j|k_0^3 =: mk_0^3$, $m \ge 1$, and therefore the jth circle lies entirely inside the cone

$$-\frac{1}{mk_0^2} < \varphi < \frac{1}{mk_0^2} \quad \text{or} \quad \pi - \frac{1}{mk_0^2} < \varphi < \pi + \frac{1}{mk_0^2},$$

and at the same time

$$\frac{1}{mk_0^2} \le \frac{1}{m^{2/3}k_0^2} \approx \frac{1}{k^{2/3}}.$$

Hence, with a probability exceeding $(1-(1-\nu)^{T+1}) \times p(k_0) > 1 - 2\varepsilon$, the circle of radius k_0 centered at one of the points of (4.9) is completely covered by elements of $\mathcal{H}$, and the process $G'(\cdot)$ starting from the ith circle can grow indefinitely.

Now, let us tile the plane with the boxes $L \times L$ where $L = 5Tk_0^3$, so that, for any $(i, j) \in \mathbb{Z}^2$, the box $B(i, j)$ consists of the points

$$\{(iL + x', jL + y') \text{ where } (x', y') \in (0, L]^2\}.$$

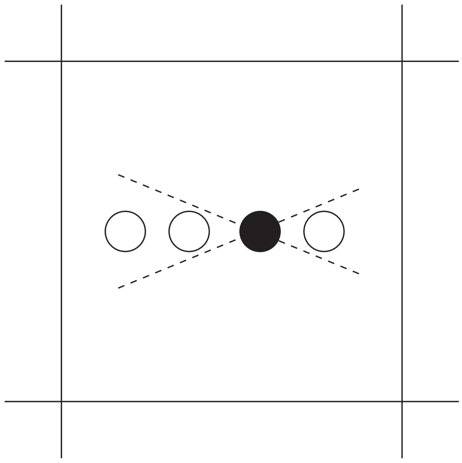

FIGURE 6: $T+1$ circles centered at the horizontal line coming through the middle of the square. At least one of them should be covered by elements of $\mathcal{H}$ for condition (i) to be satisfied.

We say that $B(i, j)$ is *open* if

(i) one of the $T+1$ circles of radius k_0 with the centers in the set

$$\{(i+0.6)L, Y_j\} \equiv \{(X_i, Y_j),\ (X_i + k_0^3, Y_j),\ (X_i + 2k_0^3, Y_j), \dots, (X_i + Tk_0^3, Y_j)\},$$

where $X_i = (i+0.4)L,\ Y_j = (j+0.5)L$, is completely covered by the elements of $\mathcal{H}$ (see Figure 6), and

(ii) the process $G'(\cdot)$ starting from a circle satisfying (i) reaches the circumference of radius $0.6L$ centered at a_{ij} (that is, $G'(k_1 - 1)$, where $k_1 = \lceil 0.6L \rceil$, contains the entire circle $a_{ij} + \mathcal{C}(k_1)$).

From the translation invariance and the previous arguments it follows that

$$\mathrm{P}(B(i, j) \text{ is open}) > 1 - 2\varepsilon. \tag{4.10}$$

We say that two boxes are connected if there exists a sequence of open boxes starting with one and ending with the other such that each pair of two consecutive boxes in this sequence share an edge. The 'box'-cluster containing the box $B(i, j)$ is the set of all boxes connected to it.

The events $E_{ij} = \{B(i, j) \text{ is open}\}$ in (4.10) are not independent, but fortunately are dependent only if two boxes share at least a point. This dependence is handled as follows. Accordingly to Liggett *et al.* (1997), if each of the E_{ij} has probability uniformly close to 1, conditioned on what happens outside of the neighborhood of $(i, j) \in \mathbb{Z}^2$ (more precisely, there is a number $M > 0$ such that, for any point of $(i, j) \in \mathbb{Z}^2$, the event E_{ij} is independent of $\{E_{i'j'},\ |i - i'| + |j - j'| > M\}$), then this random field dominates a product measure with a positive density which can be made arbitrarily close to 1, as soon as $\mathrm{P}(E_{ij})$ is close to 1. In our case, since ε was arbitrary, we could have chosen it such that this independent random field would have the density p exceeding 0.7.

Therefore, we can couple the boxes $\{B(i, j)\}$ with the site percolation process on $\mathbb{Z}^2$ where each site is open independently of the others with probability p.

However, as follows from Zuev (1987), the critical probability p_c for the site percolation on $\mathbb{Z}^2$ is smaller than 0.68 ($<p$) and therefore there exists almost surely an infinite open cluster

of boxes $B(i, j)$. As a result, Proposition 4.1 yields that there exists almost surely an infinite $(k_1 - 1)$-cluster.

To finish the proof of Lemma 4.2, we need the following statement, which is probably known, about the classical percolation model.

Proposition 4.3. *Consider a site percolation model on $\mathbb{Z}^2 \subset \mathbb{R}^2$ in the supercritical regime (i.e. $p > p_c$). Then the convex hull of the unique infinite cluster is $\mathbb{R}^2$.*

Proof of Proposition 4.3. First, we remark that the fact that the infinite cluster is unique whenever it exists follows from Aizenman *et al.* (1987). Let $C(\infty)$ be such an infinite cluster. On the plane with coordinates (x, y) consider the cone

$$A = \{(x, y) \in \mathbb{Z}^2 : \ x \geq 0, \ 0 \leq y \leq x\}.$$

Notice that the plane can be covered by exactly eight cones congruent to A, and they will have only common boundaries. Let E_A be the event $|A \cap C(\infty)| = \infty$, i.e. there are infinitely many points of the infinite cluster lying inside of A. Since this is a tail event, $\mathrm{P}(E_A)$ is either 0 or 1. If it were 0, it would imply that the probability that there are infinitely many points of $C(\infty)$ in each of the other seven cones congruent to A is also 0. This contradicts the fact that $|C(\infty)| = \infty$. Hence, $P(E_A) = 1$ and the same is true for the other cones. However, it is easy to see from geometrical observations that if each of these eight cones contains infinitely many points of $C(\infty)$ then $\mathrm{conv}(C(\infty)) = \mathbb{R}^2$.

Hence, the convex hull of the $(k_1 - 1)$-cluster in the continuous model is at least as large as that of an infinite open cluster for the site percolation on $\mathbb{Z}^2$ rescaled L times (see the description of coupling before Proposition 4.3), and thus Lemma 4.2 has been proven for $d = 2$.

4.2. Proof of Lemma 4.2 for $d \geq 3$

Let the coordinates of point $\boldsymbol{x} \in \mathbb{R}^d$ be $(x_1, x_2, \ldots, x_d)$. For $i, j \in \{1, 2, \ldots, d\}$ such that $i \neq j$, let

$$Q_{ij} = \{\boldsymbol{x} \in \mathbb{R}^d : \ x_k = 0 \text{ for all } k \in \{1, \ldots, d\} \text{ not equal to } i \text{ or } j\}$$

be a two-dimensional coordinate plane. Let us fix $i = 1$ and $j = 2$ for a moment and write $Q := Q_{12}$. For a defect $A \in \mathbb{R}^d$, let $A_Q := A \cap Q$ be the section of A by the plane Q. Since the set of holes consists only of unit balls, the process $\mathcal{P}$ generates a process $\mathcal{P}_Q$ on Q whose elements are circles of random radius (not exceeding one) with centers distributed according to a Poisson distribution.

Note that due to the nature of taking convex hulls, for any two defects $A, B \in \mathbb{R}^d$ we have

$$\mathrm{conv}(A_Q, B_Q) \subseteq (\mathrm{conv}(A, B))_Q.$$

Therefore, if we start the process of iterating holes (see Definitions 3.2–3.4) just on the plane Q starting from the realization of $\mathcal{P}_Q$, every defect of the kth generation in the planar process is contained in a section of some defect of the kth generation of the original d-dimensional process. However, for $d = 2$, Theorem 4.1 is already proven. Consequently, there exists a nonrandom number N such that for the two-dimensional process the defect of the Nth generation coincides with the plane Q almost surely. It means that almost surely there is a defect $A := A_{12}$ of the Nth generation for which $A_Q = Q$.

Now, by symmetry, it follows that for every possible pair (i, j) there exist a defect A_{ij} of (the same) Nth generation for which $A_{Q_{ij}} = Q_{ij}$ almost surely. At the same time, all planes Q_{ij} intersect, therefore so do A_{ij}'s. Hence we have

$$\operatorname{conv}\left(\bigcup_{1\le i<j\le d} A_{ij}\right) \supseteq \operatorname{conv}\left(\bigcup_{1\le i<j\le d} Q_{ij}\right) = \mathbb{R}^d,$$

and so the defect of the $(N+1)$th generation is $\mathbb{R}^d$.

4.3. Nontriviality and nonrandomness of N_{cr}

For every realization ω of the holey process $\mathcal{P}$, let $N_{\mathrm{cr}}(\omega)$ be the minimum number of times we take convex hulls of clusters required to fill up the whole $\mathbb{R}^d$, that is,

$$N_{\mathrm{cr}}(\omega) = \min\{k \ge 1 : \text{ the union of the defects of the } k\text{th generation is } \mathbb{R}^d\}.$$

By Theorem 4.1, $N_{\mathrm{cr}} \le N(\mu, \lambda) < \infty$ almost surely.

Since, for any positive integer m, the event $\{N_{\mathrm{cr}} = m\}$ is translationary invariant, from the ergodic theorem it follows that this event has probability either 0 or 1, and therefore the quantity N_{cr} is not random almost surely (see e.g. Meester and Roy (1996, Proposition 2.6)).

We say that a process is in the supercritical regime if there is an infinite connected component of holes (defects of the generation 0). Observe that if (i) the process is in the supercritical regime, (ii) the distribution of the shapes of the holes is invariant with respect to rotations, and (iii) the infinite cluster is unique, then $N_{\mathrm{cr}} = 1$ almost surely (the proof of Proposition 4.3 can easily be adapted for this case). If, on the other hand, there is no infinite cluster, then under certain conditions $N_{\mathrm{cr}} \ge 2$ almost surely. (It is conceivable *a priori* that the connected components of defects form an infinite sequence of concentrated nonoverlapping ring-shaped areas around the origin. Then, though the convex hull of each such ring is finite, the union of all of them is $\mathbb{R}^d$.)

A legitimate question about N_{cr} is whether this number can be greater than 2 with a nonzero probability. We answer this question in the following statement.

Theorem 4.2. *Let $d = 2$ and the set of holes consist of circles of radius $\frac{1}{2}$. If the rate λ of the Poisson process is sufficiently small, then $N_{\mathrm{cr}} > 2$ almost surely.*

The proof runs roughly as follows. First, we replace each defect of the first generation (convex hull of a 0-cluster) by a big circle containing it entirely inside. The radii of these circles are independent, and stochastically smaller than some exponential distribution. Later, we will be able to assume that their centers are also independent. Therefore, we have a Poisson Boolean percolation model, the percolation model in which the centers of the balls are points of a Poisson process and their radii are i.i.d. For this model (under some extra conditions on the radii distribution), it is known that if the density of the Poisson process is small enough, there is no infinite connected component of balls. Hence, all defects of the second generation are finite in size.

This proof also involves a number of couplings between continuous and discrete models. We start with auxiliary statements.

Consider a site percolation model on $\mathbb{Z}^2$ with two diagonals in every square, i.e. two distinct points $\boldsymbol{a}_1 = (x_1, y_1) \in \mathbb{Z}^2$ and $\boldsymbol{a}_2 = (x_2, y_2) \in \mathbb{Z}^2$ are connected by an edge if and only if

$$|x_1 - x_2| \le 1 \quad \text{and} \quad |y_1 - y_2| \le 1.$$

Denote this graph $\mathbb{Z}^2_{\otimes}$ and its vertex set $V(\mathbb{Z}^2_{\otimes}) \equiv \mathbb{Z}^2$.

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Let a vertex $\boldsymbol{a} \in V(\mathbb{Z}^2_\otimes)$ be occupied with probability $p > 0$ and vacant with probability $1 - p$ independent of the other vertices. An occupied path is a sequence of occupied vertices of $\mathbb{Z}^2_\otimes$ such that any two consecutive vertices are nearest neighbors in $\mathbb{Z}^2_\otimes$. A cluster containing point $\boldsymbol{a}$ is the set of all occupied vertices of $\mathbb{Z}^2_\otimes$ which can be connected to $\boldsymbol{a}$ by occupied paths.

Naturally embed $\mathbb{Z}^2_\otimes$ in $\mathbb{R}^2$ such that the points with integer coordinates coincide. Fix some vertex $\boldsymbol{a} \in V(\mathbb{Z}^2_\otimes)$, let $\rho = \rho(\boldsymbol{a})$ be the radius of the smallest circle centered at $\boldsymbol{a}$ which completely covers the cluster containing $\boldsymbol{a}$, and let $p_{\mathrm{cr}}(\mathbb{Z}^2_\otimes) \in (0, 1)$ be the critical probability for the site percolation on $\mathbb{Z}^2_\otimes$. The next statement follows immediately from Theorem 5.1 of Kesten (1982).

Author: please give details of Kesten (1982).

Proposition 4.4. *If $p < p_{\mathrm{cr}}(\mathbb{Z}^2_\otimes)$, then there exist $C, \mu > 0$ such that*

$$\mathrm{P}(\rho > t) \leq C \exp(-\mu t) \quad \text{for all } t > 0. \tag{4.11}$$

Next, consider a (continuous) Poisson Boolean model on $\mathbb{R}^2$ with rate λ' and a random radius of an occupied ball ρ satisfying (4.11) (see e.g. Meester and Roy (1996)). Let an occupied component be a collection of occupied circles such that they cannot be split into two nonoverlapping sets of $\mathbb{R}^2$ and which does not intersect any other occupied circle.

Proposition 4.5. *There exists $\lambda'_{\mathrm{cr}} > 0$ such that if $\lambda' < \lambda'_{\mathrm{cr}}$, the number of circles in any occupied component is finite almost surely.*

Proposition 4.5 follows immediately from Theorem 3.3 of Meester and Roy (1996). We need to adapt this statement to the following 'semicontinuous' model. Let $\boldsymbol{a} \in \mathbb{Z}^2$ be occupied with probability $p > 0$ and vacant with probability $1 - p$ independent of the other vertices.

With an occupied vertex $\boldsymbol{a}$ we associate a positive random variable $\rho_{\boldsymbol{a}}$. Suppose that all $\{\rho_{\boldsymbol{a}}\}$ are i.i.d. and the $\rho_{\boldsymbol{a}}$ satisfy (4.11) for some positive C and μ. Around every occupied vertex $\boldsymbol{a}$ we draw a circle of radius $\rho_{\boldsymbol{a}}$.

Corollary 4.2. *In the semicontinuous model, if p is smaller than some $p_{\mathrm{cr}} = p_{\mathrm{cr}}(C, \mu) > 0$, then the number of circles in any connected component of the union of occupied circles is finite almost surely.*

Proof. Let $\{\boldsymbol{a}_i\}$ be the set of all occupied vertices of $\mathbb{Z}^2_\otimes$. Embed $\mathbb{Z}^2_\otimes$ in $\mathbb{R}^2$ and for $\boldsymbol{a}_i = (x_i, y_i)$ consider a box

$$B_i = \{(x, y) \in \mathbb{R}^2 : x_i - \tfrac{1}{2} < x \leq x_i + \tfrac{1}{2},\ y_i - \tfrac{1}{2} < y \leq y_i + \tfrac{1}{2}\}.$$

Generate N_i points uniformly distributed on B_i, while N_i itself is drawn from a Poisson distribution with rate $\lambda' = -\log(1 - p)$ conditioned on $N_i \geq 1$. Enumerate these N_i points in an arbitrary way, and draw a circle of radius $\rho_i^{(1)} = \rho_{\boldsymbol{a}_i} + 1/\sqrt{2}$ around the first of these points. (This radius size will guarantee that this circle contains the circle of radius 1 from the other model entirely inside.) If $N_i > 1$, draw a circle of radius $\rho_i^{(j)}$ centered at the jth point for $j = 2, \dots, N_i$. Choose the $\rho_i^{(j)}$ such that they are i.i.d. for $i = 1, 2, \dots$ and $j = 1, \dots, N_i$.

Repeat this process independently at every occupied vertex of $\mathbb{Z}^2_\otimes$. Then it can be checked that the set of all such generated circles constitutes a Poisson Boolean model with rate λ'. Since the $\rho_i^{(j)}$ satisfy (4.11), possibly with a different constant C, we can apply Proposition 4.5.

On the other hand, every occupied component in a semicontinuous model lies entirely inside some occupied component of the Poisson Boolean model, which is finite.

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Now we return to a discrete model once again. As before, every site of $\mathbb{Z}^2_{\otimes}$ is occupied with probability p, and vacant with probability $1 - p$ independent of the other site. Every realization of this random field splits the set of occupied vertices into connected components, called clusters. From now on, suppose that $0 < p < p_{\mathrm{cr}}(\mathbb{Z}^2_{\otimes})$, so that there are no infinite clusters on $\mathbb{Z}^2_{\otimes}$. Embed $\mathbb{Z}^2_{\otimes}$ in $\mathbb{R}^2$ and for each cluster $C \subseteq V(\mathbb{Z}^2_{\otimes})$ of occupied sites call the smallest convex subset $H(C) \subseteq \mathbb{R}^2$ containing all its vertices the *convex hull* of this cluster. Though each of these convex hulls is finite almost surely, it is possible that there exists an infinite sequence $H_1, H_2, \dots$ of different convex hulls such that $H_i \cap H_{i+1} \neq \varnothing$ for $i = 1, 2, \dots$. However, if p is small enough, this does not happen.

Proposition 4.6. *There exists* $p_{\mathrm{conv}} \in (0, p_{\mathrm{cr}}(\mathbb{Z}^2_{\otimes}))$ *such that if* $p < p_{\mathrm{conv}}$, *then there exists no infinite sequence of overlapping convex hulls of open clusters of* $\mathbb{Z}^2_{\otimes}$ *as defined immediately above.*

Proof. Our goal is to reduce this statement to Corollary 4.2. Let $\{\eta_{j\boldsymbol{a}}\}$ be a collection of i.i.d. Bernoulli(p) random variables indexed by $j \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{Z}^2$. Fix a realization $\eta(\omega) \in \{0, 1\}^{\mathbb{N}\times\mathbb{Z}^2}$ and consider the following construction of the realization of a random field on $\mathbb{Z}^2_{\otimes}$.

Enumerate all vertices of $\mathbb{Z}^2_{\otimes}$ in an arbitrary order, $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_k, \dots$. Call the vertex $\boldsymbol{a}_1$ *marked*. If $\eta_{1\boldsymbol{a}_1} = 0$, let the vertex $\boldsymbol{a}_1 \in V(\mathbb{Z}^2_{\otimes})$ be vacant and skip to the next paragraph. Otherwise, set it to be occupied, and consider the set of eight vertices of $\mathbb{Z}^2_{\otimes}$ adjacent to it, denote them as $\{\boldsymbol{a}_{1:1}, \boldsymbol{a}_{1:2}, \boldsymbol{a}_{1:3}, \dots, \boldsymbol{a}_{1:8}\}$, and make them marked too. Next, with each marked site $\boldsymbol{a}_{1:j}$ repeat this procedure: if $\eta_{1\boldsymbol{a}_{1:j}} = 0$ then let $\boldsymbol{a}_{1:j}$ be vacant. If $\eta_{1\boldsymbol{a}_{1:j}} = 1$ then let $\boldsymbol{a}_{1:j}$ be occupied and consider all its neighbors in $\mathbb{Z}^2_{\otimes}$ which are *not marked* by this stage, make them marked and repeat the same procedure now with them. Iterate this process, using only random variables $\{\eta_{1\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{Z}^2}$, until the only newly marked vertices are vacant. This will happen in finite time almost surely since $p < p_{\mathrm{cr}}(\mathbb{Z}^2_{\otimes})$ and the process described above generates a cluster of occupied vertices containing point $\boldsymbol{a}_1$ in the site percolation model on $\mathbb{Z}^2_{\otimes}$. We remark that this process was introduced and studied by Menshikov (1985), (1986), and is referred to as the *method of generations*.

Author: please give details of Menshikov (1986).

Next, let $\boldsymbol{a}_k$ be the first point in the sequence $(\boldsymbol{a}_2, \boldsymbol{a}_3, \dots)$ which is not marked yet. Repeat the process described in the previous paragraph starting with $\boldsymbol{a}_k$, but this time use only random variables $\{\eta_{k\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{Z}^2}$ to determine whether a site being considered should be made occupied or vacant (exactly as in the previous paragraph). After this process also stops, find the first unmarked point in the set $(\boldsymbol{a}_{k+1}, \boldsymbol{a}_{k+2}, \dots)$ and repeat the procedure described in this paragraph again.

Repeating the algorithm of the previous paragraph infinitely many times, we will eventually exhaust the set of all vertices of $\mathbb{Z}^2_{\otimes}$. Each site is now occupied (let us call it $*$-occupied) with probability p independently of each other since the η are i.i.d. We write $\eta^*_{\boldsymbol{a}} = 1$ if the vertex $\boldsymbol{a} \in \mathbb{Z}^2$ is $*$-occupied and 0 if it is not. Then $\{\eta^*_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{Z}^2}$ is a collection of Bernoulli(p) i.i.d. random variables.

Let us refer to a realization of $\{\eta^*_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{Z}^2}$ as a $*$-*process* and write $\boldsymbol{a} \overset{*}{\sim} \boldsymbol{b}$ if there exists a path of $*$-occupied vertices connecting $\boldsymbol{a}$ and $\boldsymbol{b}$, namely if there exists a sequence of vertices $\boldsymbol{a} = \boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n = \boldsymbol{b}$ such that for each $i = 1, 2, \dots, n$ the vertices v_i and v_{i-1} are connected in the topology of $\mathbb{Z}^2_{\otimes}$, and all the $\boldsymbol{v}_i$ are $*$-occupied ($\eta^*_{\boldsymbol{v}_i} = 1$ for $i = 0, 1, \dots, n$).

Next, for a fixed integer J consider the family $\{\eta_{J\boldsymbol{a}},\ \boldsymbol{a} \in \mathbb{Z}^2\}$. This family generates an independent Bernoulli random field on the vertex set of $\mathbb{Z}^2_{\otimes}$. We will call this field the J-*process*. A vertex $\boldsymbol{v}$ is called J-occupied, if $\eta_{J\boldsymbol{a}} = 1$.

Author: do you mean '...if $\eta_{J_{\boldsymbol{v}}} = 1$'?

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Let C_J be the cluster of occupied vertices containing the point $\boldsymbol{a}_J$ on the Jth plane. More rigorously,

$$C_J = \{\boldsymbol{a} \in \mathbb{Z}^2 \setminus \boldsymbol{a}_J : \boldsymbol{a} \overset{J}{\sim} \boldsymbol{a}_J\} \subseteq \mathbb{Z}^2,$$

where $\boldsymbol{a} \overset{J}{\sim} \boldsymbol{b}$ if and only if there exists a path of J-occupied vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n$, connected in the topology of $\mathbb{Z}^2_\otimes$, with $\boldsymbol{v}_0 = \boldsymbol{a}$ and $\boldsymbol{v}_n = \boldsymbol{b}$. (This definition is a complete analogue of the definition for the $*$-process.) Also set $\boldsymbol{v} \overset{J}{\sim} \boldsymbol{v}$ by convention and note that C_J may be empty. Let the *radius* of cluster C_J be

$$\rho_J = \sup_{\boldsymbol{a} \in C_J} |\boldsymbol{a} - \boldsymbol{a}_J|$$

with the convention that $\rho_J = 0$ if $C_J = \varnothing$. Since the shapes of the C_J are i.i.d. (that is, the sets $\tilde{C}_J := \{\boldsymbol{a} - \boldsymbol{a}_J,\ \boldsymbol{a} \in C_J\}$ are i.i.d. for $J = 1, 2, \dots$) and $p < p_{\mathrm{cr}}(\mathbb{Z}^2_\otimes)$, the ρ_J are also i.i.d. finite random variables satisfying (4.11) (see e.g. Menshikov (1985), Menshikov *et al.* (1986)).

On the other hand, it is easy to see from the construction that every cluster of $*$-occupied sites (for the $*$-process) lies entirely inside C_J for some J. Indeed, since each such cluster of the $*$-process contains one of the $\boldsymbol{a}_J$ chosen by the iterative procedure above. At the same time, if $\boldsymbol{a}_J$ was chosen, and $\boldsymbol{a}$ is a vertex with $\boldsymbol{a} \overset{*}{\sim} \boldsymbol{a}_J$, then necessarily $\boldsymbol{a} \overset{J}{\sim} \boldsymbol{a}_J$.

Author: the sentence 'Indeed, … above' is incomplete – please correct.

Therefore, the convex hull of every such cluster lies inside the circle of radius ρ_J centered at $\boldsymbol{a}_J$, for some $J \geq 1$. Now the statement of Proposition 4.6 follows from Corollary 4.2 with $p_{\mathrm{conv}} = p_{\mathrm{cr}}(C, \mu)$.

Corollary 4.3. *For any $\beta > 0$, the statement of Proposition 4.6 with p_{conv} replaced by $p_{\mathrm{conv},\beta}$ holds even if every convex hull H of a nonempty cluster is replaced by a 'β-buffered convex hull'* $\mathrm{buff}_\beta(H)$, *where*

$$\mathrm{buff}_\beta(H) = \{\boldsymbol{a} \in \mathbb{R}^2 : \textit{there is } \boldsymbol{b} \in H \textit{ such that } |\boldsymbol{a} - \boldsymbol{b}| \leq \beta\}.$$

In other words, if $p < p_{\mathrm{conv},\beta}$, there exists no infinite sequence $H_0, H_1, H_2, \dots$ of convex hulls of open clusters in $\mathbb{Z}^2_\otimes$, such that $\mathrm{buff}_\beta(H_i) \cap \mathrm{buff}_\beta(H_{i+1}) \neq \varnothing$ *for all $i = 0, 1, 2, \dots$.*

Proof. This immediately follows from the proof of Proposition 4.6. The only difference is that we couple a convex hull of an open cluster not with the circle of radius ρ_J, but with the circle of radius $\rho_J + \beta$ centered at a_J. These new radii obviously also satisfy (4.11).

Proof of Theorem 4.2. Here we couple an independent Poisson model of holes with the semicontinuous percolation model on $\mathbb{Z}^2_\otimes$ described above. As usual, we embed $\mathbb{Z}^2_\otimes$ in $\mathbb{R}^2$ and consider an arbitrary vertex $\boldsymbol{a} = (x, y) \in V(\mathbb{Z}^2_\otimes)$. Set $\boldsymbol{a}$ to be occupied if there is at least one hole (that is, a circle of radius $\frac{1}{2}$ belonging to the set of holes) centered inside the box

$$B(x, y) = \{(x', y') \in \mathbb{R}^2 : x - \tfrac{1}{2} < x' \leq x + \tfrac{1}{2},\ y - \tfrac{1}{2} < y' \leq y + \tfrac{1}{2}\}$$

and vacant otherwise (see Figure 7). Then each of the vertices of $\mathbb{Z}^2_\otimes$ is occupied with probability $p = 1 - \mathrm{e}^{-\lambda}$ independently of the others. On the other hand, two holes have a nonempty intersection only if the sites on $\mathbb{Z}^2_\otimes$ corresponding to them are occupied and connected in $\mathbb{Z}^2_\otimes$. Consequently, every defect of the first generation (which is the convex hull of a connected component of holes) lies inside the $\frac{1}{2}(\sqrt{2} + 1)$-buffered convex hull of a cluster on $\mathbb{Z}^2_\otimes$, which is finite whenever $1 - \mathrm{e}^{-\lambda} < p_{\mathrm{cr}}(\mathbb{Z}^2_\otimes)$.

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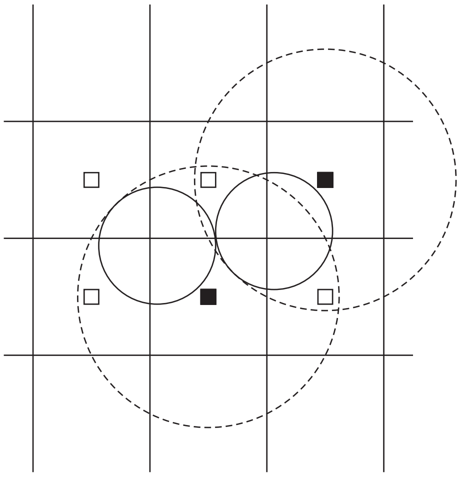

FIGURE 7: Embedding of $\mathbb{Z}^2_\otimes$ in $\mathbb{R}^2$.

By Corollary 4.3, as soon as λ is small enough, there is no infinite component of these $\frac{1}{2}(\sqrt{2}+1)$-buffered convex hulls. Therefore, all defects of the second generation must be finite, and hence $N_{\mathrm{cr}} > 2$.

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