# On Convexity of Hypersurfaces in the Hyperbolic Space

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**Abstract**

In the Hyperbolic space $\mathbb{H}^n$ ($n \geq 3$) there are uncountably many topological types of convex hypersurfaces. When is a locally convex hypersurface in $\mathbb{H}^n$ globally convex, that is, when does it bound a convex set? We prove that any locally convex proper embedding of an $(n-1)$-dimensional connected manifold is the boundary of a convex set whenever the complement of $(n-1)$-flats of the resulting hypersurface is connected.

## 1 Introduction

In this paper we study convex geometry of unbounded hypersurfaces in the hyperbolic space $\mathbb{H}^n$ ($n \geq 2$). The focus of our study is on local geometric properties that guarantee the global convexity of a hypersurface. The main result of the paper states that a proper locally-convex embedding of a connected $(n-1)$-manifold $\mathscr{M}$ into $\mathbb{H}^n$ ($n \geq 2$), where the complement of the union of flat $(n-1)$-dimensional submanifolds is connected, is the boundary of a convex body. In general, a hypersurface in $\mathbb{H}^n$ or $\mathbb{R}^n$ is called convex if it is the boundary of a *convex body, possibly unbounded.* Local convexity of a hypersurface $M$ at a point $p$ is understood in the sense of existence of an $M$-neighborhood of $p$ that lies on the boundary of a convex body. A point of local convexity $p \in M$ is called a point of strict convexity if there is a hyperplane $H$ through $p$ such that a punctured $M$-neighborhood of $p$ lies in one of the *open* halfspaces defined by $H$.

While studying the convexity in the hyperbolic space it is instructive to think of $\mathbb{H}^n$ in terms of the Beltrami-Klein model, where the space is represented by the interior of the unit ball in $\mathbb{R}^n$ and the geodesics are straight line segments. One immediately sees that the convexity properties of bounded objects in $\mathbb{H}^n$ are no different from those of their Euclidean counterparts. The convexity theory for unbounded objects in $\mathbb{H}^n$ is quite different from that for unbounded objects in $\mathbb{R}^n$: for example, there are uncountably many topological types of unbounded convex surfaces in $\mathbb{H}^n$ (Kuzminykh, 2005).

In all examples we will think of $\mathbb{H}^3$ (and $\mathbb{H}^n$) as of the interior of the unit ball in $\mathbb{R}^3$ (in $\mathbb{R}^n$), where the geodesic segments are the straight line segments under the hyperbolic metric given by the natural logarithm of the corresponding double ratio (Beltrami-Klein model).

A *$k$-flat* is a subset of $M$, which also belongs to a $k$-dimensional subspace of $\mathbb{H}^n$ and is open and connected in this subspace. A $k$-flat which is not a proper subset of another flat of any dimension is called a *$k$-face.* Obviously, each $k$-flat is contained in some unique $m$-face for some $m \geq k$ (note that for a general surface $M$ faces may have non-empty intersections).

The main results are stated in the following theorems.

**Theorem 1** *Let $f : \mathscr{M} \to \mathbb{H}^n$ be a proper embedding of a connected $(n-1)$-manifold into the hyperbolic space $\mathbb{H}^n$. Suppose also that the complement of the union of all $(n-1)$-faces of $f(\mathscr{M})$ is connected. If $f$ is locally-convex, then $f(M)$ is the boundary of a convex body.*

For $n \geq 3$ and bounded $f$ the above theorem is derived in our paper from van Heijenoort's (1952) local-to-global convexity theorem, which asserts that a complete locally-convex immersion of a connected $(n-1)$-manifold ($n \geq 3$) into $\mathbb{R}^n$ with at least one point of strict convexity is a homeomorphism onto the boundary of a convex body. Subsequently, Jonker and Norman (1973) streamlined van Hejenoort's argument and proved that a complete locally convex immersion of a connected $(n-1)$-manifold ($n \geq 3$) into $\mathbb{R}^n$ fails to be the boundary of a convex body only when the surface is the direct affine product of a non-convex plane curve and a subspace of dimension $n-2$. Unfortunately, van Heijenoort's theorem cannot be extended to unbounded surfaces in the hyperbolic space without strengthening of the assumptions. For example, the immersed surface described in the following example (see Figure 1) is strictly locally-convex at all points and yet fails to serve as the boundary of a convex body.

**Example 1** Consider a surface defined parametrically by

$$(x = \sqrt{\frac{2}{3}(u-\frac{1}{3})^2(u+\frac{1}{3})\cos v}, y = v, z = u\cos v)$$

This defines an immersion of an open subset of the $(u, v)$-plane into $\mathbb{H}^3$ (this is not an immersion as a map from $\mathbb{R}^2$ to $\mathbb{R}^3$). However, the resulting surface does not bound any convex body in $\mathbb{H}^3$.

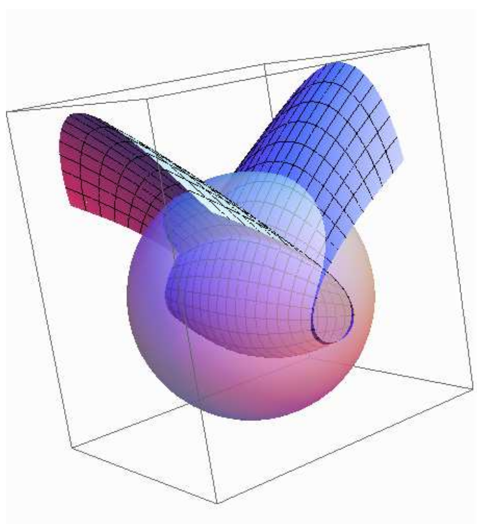

Figure 1: Strictly locally convex immersed surface in $\mathbb{H}^3$

The immersion in Example 1 is proper and complete. On the other hand, the embedding constructed in Example 2 is not proper, but complete. Both examples satisfy the completeness assumption (see next Section for definitions) made in van Heijenoort's (1952) and Jonker and Norman (1973).

**Example 2** Consider a surface defined parametrically by

$$\left\{ \frac{(\cos 1.8t)\,(1-y^2)}{e^{\frac{t}{10}}}, y, \frac{(\sin 1.8t)\,(1-y^2)}{e^{\frac{t}{10}}} \right\},$$

where

$$\frac{3\pi}{16} \leq t \leq \frac{3\pi}{2}; \quad -1 \leq y \leq 1.$$

This defines an embedding of an open subset of the $(t, y)$-plane into $\mathbb{H}^3$. However, the resulting surface does not bound any convex body in $\mathbb{H}^3$

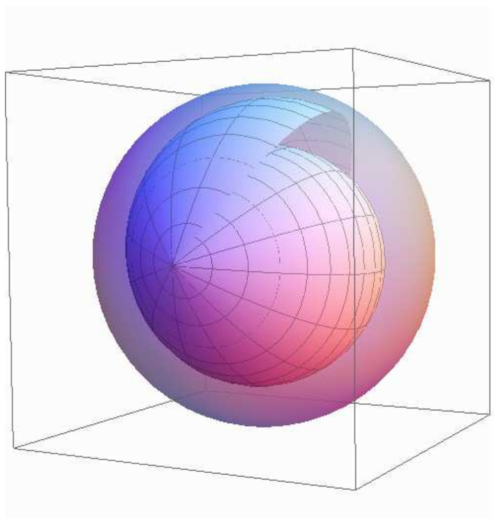

Figure 2: Strictly convex embedded surface shown in Klein's model of the Hyperbolic space. The surface has shell-like form. The shell "wraps around itself" infinitely many times.

It is only natural to ask at this point if replacing the immersion assumption by the embedding assumption, together with properness, would make van Heijenoort's local-to-global convexity criterion applicable to the Hyperbolic space. As illustrated by the following example the answer is again no.

**Example 3** Consider a surface in $\mathbb{R}^3$ defined in the following way. Let $T$ be a regular triangle of circumradius 1 on the $xy$-plane, centered at the origin. Consider a PL-function on the unit disk, which is (1) 1/2 at the origin, (2) zero on the sides and outside of the triangle $T$, (3) continued by linearity inside of $T$. The graph of the resulting function looks like a hat. Consider this graph as an embedded surface in the hyperbolic space (Beltrami-Klein model). It is strictly locally convex at the "tip of the hat", but does bound any convex body.

Theorem 1 shows that the embedding assumption together with a global condition that is stronger than the local convexity, but somewhat weaker than the strict local convexity do guarantee the global convexity. The following section introduces necessary terminology and definitions. The proofs are given in the last section.

## 2 Definitions and notation

From now on $\mathbb{X}$ (or $\mathbb{X}^n$) denotes $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, where $n \in \mathbb{N} = \{0, 1, \dots\}$. All maps are continuous.

**Definition 2** *A* surface in $\mathbb{X}$ *is a pair* $(\mathscr{M}, r)$ *where* $\mathscr{M}$ *is a manifold, with or without boundary, and* $r : \mathscr{M} \to \mathbb{X}$ *is a continuous map, hereafter referred to as a* realization *map.*

To avoid a common confusion caused by (at least three) different usages of `closed` in English texts on the geometry-in-the-large, we use this word for closed subsets of topological spaces only. We will not use the term "closed surface" at all; a closed submanifold stands for a submanifold which happens to be a closed subset in the ambient manifold. Whenever we want to include manifolds with boundary into our considerations we explicitly say so.

A map $i : \mathscr{M} \to \mathbb{X}$ is called an *immersion* if $i$ is a local homeomorphism; in such a case we also refer to $(\mathscr{M}, i)$ as a surface immersed into $\mathbb{X}$. This is a common definition of immersion in the context of non-smooth geometry in the large (e.g. see van Heijenoort, 1952); a more restrictive definition is used in differential geometry and topology, furthermore, some authors define an immersion as a continuous local bijection. Although the latter definition is not, in general, equivalent to the common one, it is equivalent to the common one in the context of the theorems stated in this paper. A map $e : \mathscr{M} \to \mathbb{X}$ is called an *embedding* if $e$ is a homeomorphism onto $e(\mathscr{M})$. Obviously, an embedding is an immersion, but not vice versa.

A set $K \subset \mathbb{X}$ is called *convex* if for any $x, y \in K$ there is a geodesic segment of minimal length with end-points $x$ and $y$ that lies in $K$. Right away we conclude that the empty set and all one point sets are convex. A *convex body* in $\mathbb{X}$ is a closed convex set of full dimension; *a convex body may be unbounded.* A map $r : \mathscr{M} \to \mathbb{X}$ is called *locally convex at* $p \in \mathscr{M}$ if we can find a neighborhood $\mathscr{N}_p \subset \mathscr{M}$ and a convex body $K_p \subset \mathbb{X}$ for $p$ such that $r|_{\mathscr{N}_p} : \mathscr{N}_p \to r(\mathscr{N}_p)$ is a homeomorphism and $r(\mathscr{N}_p) \subset K_p$. In such a case we refer to $K_p$ as a *convex witness for* $p$. (Here, as everywhere else, the subscript indicates that $K_p$ depends on $p$ in some way but is not necessarily determined by $p$ uniquely.) Thus, the local convexity at $\boldsymbol{p} = r(p)$ may fail because $r$ is not a local homeomorphism at $p$ or because no neighborhood $\mathscr{N}_p$ is mapped by $r$ onto the boundary of a convex body, or for both of these reasons. When it is clear from the context that we are discussing the properties of $r$ near $\boldsymbol{p} = r(p)$, we say that $r$ is convex at $\boldsymbol{p}$. If $K_p$ can be chosen so that $K_p \setminus r(p)$ lies in an open half-space defined by some hyperplane passing through $r(p)$, the realization $r$ is called *strictly convex* at $p$. We will also sometimes refer to $(\mathscr{M}, r)$ as strictly convex at $r(p)$.

Let us recall (see e.g. Rockafellar, 1997) that a point $\boldsymbol{p}$ on the boundary of a convex set $C$ is called *exposed* if $C$ has a support hyperplane that intersects $\overline{C}$, the closure of $C$, only at $\boldsymbol{p}$. Thus, an *exposed* point on a convex body $K$ is a *point of strict convexity* on the hypersurface $\partial K$. Conversely, for a point of strict convexity $p \in \mathscr{M}$ for $(\mathscr{M}, r)$ the image $i(p)$ is an exposed point of any convex witness for $p$. Local convexity can be defined in other, non-equivalent, ways (e.g., see van Heijenoort).

A hypersurface $(\mathscr{M}, r)$ is (globally) *convex* if there exists a convex body $K \subset \mathbb{X}^n$ such that $r$ is a homeomorphism onto $\partial K$. Hence, we exclude the cases where $r(\mathscr{M})$ is the boundary of a convex body, but $r$ fails to be injective. Of course, the algorithmic and topological aspects of such a case may be interesting to certain areas of geometry, such as origami.

# 3 Geometry of locally convex immersions

Recall that a path joining points $x$ and $y$ in a topological space $\mathcal{T}$ is a map $\alpha : [0,1] \to \mathcal{T}$, where $\alpha(0) = x$ and $\alpha(1) = y$. Denote by $\mathrm{Paths}_{\mathscr{M}}(x, y)$ the set of all paths joining $x, y \in \mathscr{M}$.

Any realization $r : \mathscr{M} \to \mathbb{X}^n$ induces a distance $d_r$ on $\mathscr{M}$ by

$$d_r(x, y) = \inf_{\alpha \in \mathrm{Paths}_{\mathscr{M}}(x,y)} |r(\alpha)|,$$

where $|r(\alpha)| \in \mathbb{R}_+ \cup \infty$ stands for the length of the $r$-image of the path $\alpha$ joining $x$ and $y$ on $\mathscr{M}$ (we call it the *$r$-distance*, because it is not always a metric).

Of course, for a general realization $r$ it is not clear *a priori* that there is a path of finite length on $r(\mathscr{M})$ joining $r(x)$ and $r(y)$ (where $x$ and $y$ are in the same connected component). The notion of *complete* realization is essential to the correctness of van Heijenoort's theorem. A realization $r : \mathscr{M} \to \mathbb{X}$ is called *complete* if every Cauchy sequence on $\mathscr{M}$ (with respect to the distance induced by $r$ on $\mathscr{M}$) converges. Completeness is a rather subtle notion: a space may be complete under a metric $d$ and not complete under another metric $d_1$, which is topologically equivalent to $d$ (i.e. $x_n \xrightarrow{d} a$ iff $x_n \xrightarrow{d_1} a$).

A realization is called *proper* if the preimage of every compact set is compact. A proper realization is always closed. For any given natural class of realizations (e.g. PL-surfaces, semialgebraic surfaces, etc) it is usually much easier to check for properness than for completeness. Furthermore, the notion of properness is topological, while that of completeness is metrical. Note that in some sources, such as the paper by Burago and Shefel (1992), completeness with respect to the $r$-metric is called *intrinsic completeness,* while properness is referred to as *extrinsic completeness.* The following is well-known for immersions (see e.g. Burago and Shefel, p. 50), but is also true for arbitrary proper realizations. The proof given here was suggested by Frank Morgan.

**Lemma 3** *A proper realization $r$ of any manifold $\mathscr{M}$ in $\mathbb{X}$ is complete.*

**Proof.** Let $\{x_n\} \subset \mathscr{M}$ be Cauchy. Then $\{r(x_n)\}$ is also Cauchy in the $r$-distance and, therefore, in the intrinsic distance of $\mathbb{X}$ as well. Since $\mathbb{X}$ is complete, $\{r(x_n)\}$ converges to some point $y$ of $\mathbb{X}$. Since $r(\mathscr{M})$ is closed, $y \in r(\mathscr{M})$.

For any $k \in \mathbb{N}_{>0}$ there is $j(k)$ such that for any $i \geq j(k)$ we have $d_r(x_i, x_{i+1}) < \frac{1}{2^k}$. Note that in this case $\sum_{k>0} d_r(x_{j(k)}, x_{j(k+1)})$ converges. As $\{r(x_n)\}$ is convergent, it lies in some compact set $S \subset \mathbb{X}$. Since $r$ is proper, $r^{-1}S$ is compact. Thus, $x_n$ have an accumulation point $x$. As $r$ is continuous and $r(x_n) \to y$ in $\mathbb{X}$, $r(x) = y$.

Let us show that $x_{j(k)}$ converges to $x$ in the $r$-distance. For each $k$ there is a path $p_k$ of length less than $\frac{1}{2^k}$ (in the $r$-distance) from $x_{j(k)}$ to $x_{j(k+1)}$. For each $k$ we can form a path $\alpha_k$ with source $x_{j(k)}$ by concatenating $p_i$, $p_{i+1}$,...,etc, for all $i \geq k$. Since $\{x_{j(k)}\}$ converges to $x$, $\alpha_k$ is a path from $x_{j(k)}$ to $x$. Since $\sum_{k>0} d_r(x_{j(k)}, x_{j(k+1)})$ converges, it is a path of finite length. Thus, $\{x_{j(k)}\}$ converges to $x$ in the $r$-distance. Since a subsequence of $\{x_n\}$ converges to $x$ in the $r$-distance, $\{x_n\}$ also converges to $x$ in the $r$-distance. ∎

The reverse implication is true for locally-convex immersions, but not, for example, for saddle surfaces (e.g. see Burago and Shefel, p. 50):

**Lemma 4** *(van Heijenoort) A complete locally-convex immersion of a connected* $(n-1)$*-manifold into* $\mathbb{X}^n$ *is proper.*

If $f : \mathscr{M} \to \mathbb{X}^n$ is a continuous map, then a priori there are three topologies on $\mathscr{M}$: the original (intrinsic) topology of $\mathscr{M}$, the topology induced by the metric of $\mathbb{X}^n$, and the $f$-distance topology. It turns out that for locally convex immersions all three topologies coincide.

**Lemma 5** *(van Heijenoort, 1952; pp. 227-228) Let* $f : \mathscr{M} \to \mathbb{X}^n$ *be a complete locally-convex immersion of an* $(n-1)$*-manifold* $\mathscr{M}$*. Then any two points in the same connected component of* $\mathscr{M}$ *can be connected by an arc of finite length. The topology on* $\mathscr{M}$ *defined by the* $f$*-distance is equivalent to the intrinsic (original) topology on* $\mathscr{M}$*.*

van Heijenoort's proofs of Lemmas 5 and 4 given in the original for $\mathbb{R}^n$ are valid, word by word, for $\mathbb{S}^n$ and $\mathbb{H}^n$, since these lemmas are entirely of local nature. If $f$ is a locally-convex immersion, then for a "sufficiently small" subset $\mathbb{S}$ of $\mathscr{M}$ the map $f|_{\mathbb{S}}$ is a homeomorphism and, therefore, the topology on $\mathbb{S}$ that is induced by the metric topology of $\mathbb{X}^n$ is equivalent to the intrinsic topology of $\mathbb{S}$ and, thanks to Lemma 5, to the $f$-distance topology. Thus, for sufficiently small subsets of $\mathscr{M}$ (but not $i(\mathscr{M})$ !) the three topologies considered in this section are equivalent – the fact that will be used throughout the text without an explicit reference to the above lemmas.

The following is our starting point.

**Theorem 6** (van Heijenoort, 1952) *If a complete locally convex immersion* $f$ *of a connected* $(n-1)$*-manifold* $\mathscr{M}$ *into* $\mathbb{R}^n$ *(*$n \geq 3$*) has a point of strict convexity, then* $f$ *is a homeomorphism onto the boundary of a convex body.*

This theorem was also proved by A. D. Alexandrov (1948) for $n = 3$. Note that there is no need to check the existence of a point of strictly convexity in the compact case:

**Lemma 7** *If* $f : \mathscr{M} \to \mathbb{R}^n$ *is a locally-convex immersion of a compact connected* $(n-1)$*-manifold* $\mathscr{M}$*, then* $f$ *has a point of strict convexity.*

**Proof.** As $\mathscr{M}$ is compact and $f$ is an immersion, $\mathsf{conv}\, f(\mathscr{M})$ is a compact subset of $\mathbb{R}^n$. Since $\mathsf{conv}\, f(\mathscr{M})$ is compact, it is also bounded and, in particular, does not contain lines. Any non-empty convex set, which is free of lines, has a non-empty set of extreme points (a point on the boundary of a convex set is extreme if it is not interior to any line segment contained in set's boundary). Thus $\partial\, \mathsf{conv}\, f(\mathscr{M})$ contains an *extreme* point. Straszewicz's theorem (e.g. Rockafellar, 1997, p. 167) states that the *exposed* points of a closed convex set form a dense subset of *extreme* points of this set. Thus, $\mathsf{conv}\, f(\mathscr{M})$ has an exposed point. Since an exposed point $y$ cannot be written as a *strict convex combination* of other points of the set, $y$ must lie in $f(\mathscr{M})$. Let $x$ be a point from $f^{-1}(y)$. Since $f$ is locally-convex at $x$ and there exists a hyperplane $H$ through $y$ that has empty intersection with $f(\mathscr{M}) \setminus y$, we conclude that the map $f$ is strictly locally-convex at $x$. ■

### 3.1 Local Convexity after Ehrhart Schmidt

An alternative vision of local convexity was studied by Ehrhart Schmidt (e.g. van Heijenoort, 1952): a point $p$ on the boundary of an open *set* $S$ is a point of local convexity in the sense of Schmidt if there exists a hyperplane through $p$ such that the intersection of a sufficiently small metric ball centered at $p$ with $S$ lies in one of the open halfspaces defined by the hyperplane. We will use this notion of local convexity in the proof of our main theorem.

**Theorem 8** *(E. Schmidt) Let $S$ be an open subset of $\mathbb{R}^n$. Suppose $S$ is locally convex at all points of its boundary in the sense of Schmidt. Then $S$ is convex.*

The proof can be found, e.g., in van Heijenoort (1952).

# 4 Topology

We will be using Alexander-Pontrjagin duality to establish that under our assumptions on the manifold $\mathscr{M}$ and its realization $f$ the complement $\mathbb{H}^n \setminus f(\mathscr{M})$ has exactly two connected components.

**Theorem 9** *(Pontrjagin, 1927, 1934) Let $K$ be a closed compact subset of $\mathbb{X}$, where $\mathbb{X}$ is $\mathbb{R}^n$ or $\mathbb{S}^n$. Let $A$ be an abelian group of coefficients and let $A^*$ denote its Pontrjagin dual. Then $\mathcal{H}_p(K, A)$ is dual to $\mathcal{H}_{n-p-1}(\mathbb{X} \setminus K, A^*)$, where $\mathcal{H}_p(-, G)$ stands for the $p$-th singular homology group of the augmented complex.*

This theorem was proven by Pontrjagin for $\mathbb{X} = \mathbb{R}^n$, but the proof goes word by word for $\mathbb{S}^n$. Pontrjagin himself remarked: "I limit myself to the case where the manifold is a Euclidean space, since, with existing methods, the generalization to the case of an arbitrary manifold does not present any great difficulty." It is also remarked in Alexandroff (1943, p. 327) that Pontrjagin's proof holds for any simply-connected manifold.

Alternatively, we can use Alexandroff's generalization of Alexander-Pontrjagin which is formulated for arbitrary closed subsets of manifolds and Alexandroff-Čech homology theory:

**Theorem 10** *(Alexandroff, 1943) Let $K$ be a closed subset of a manifold $\mathbb{Y}$ ($\dim \mathbb{Y} = n$). Let $A$ be an abelian group of coefficients and let $A^*$ denote its Pontrjagin dual. Then $H_p(K, A)$ is dual to $H_{n-p-1}(\mathbb{Y} \setminus K, A^*)$, where $H_k(-, G)$ stands for the $k$-th Alexandroff-Čech homology group (defined in terms of coverings) of the augmented complex.*

*For the rest of the paper $f : \mathscr{M} \to \mathbb{H}^n$ stands for a proper locally-convex embedding of a connected manifold $\mathscr{M}$ of dimension $n-1$, where $n \geq 2$. We denote $f(\mathscr{M})$ by $M$.*

We are now going to prove that $\mathbb{H}^n \setminus M$ has exactly two connected components.

**Theorem 11** *Let $f : \mathscr{M} \to \mathbb{H}^n$ be a proper embedding of a connected manifold $\mathscr{M}$ of dimension $n-1$, where $n \geq 2$. Then $\mathbb{H}^n \setminus f(\mathscr{M})$ has two connected components.*

**Proof.** Since $f$ is complete, we know that either $M = f(\mathscr{M})$ is a compact closed hypersurface or $M$ is unbounded. In the latter case let us consider the Alexandroff (one point) compactification of $\mathbb{H}^n$, denoted by $\overline{\mathbb{H}^n}$, which turns it into a topological $n$-sphere. Let us also consider the Alexandroff compactification $\overline{M}$ of $M$.

We first prove that $f$, extended by sending the point at infinity of $\mathscr{M}$ to the point at infinity of $\mathbb{H}^n$, is continuous at this point. Let $\{x_n\}$ be a sequence converging to the point of infinity of $\mathscr{M}$. If $\{f(x_n)\}$ does not converge to the point of infinity of $\mathbb{H}^n$, then there is an infinite subsequence of $\{x_n\}$ that is mapped into a compact subset $B$ of $\mathbb{H}^n$. Since $f$ is proper, $f^{-1}B$ is compact in $\mathscr{M}$. Thus, this subsequence has an accumulation point in $\mathscr{M}$. Since $\dim \mathscr{M} \geq 1$, the sequence $\{x_n\}$ cannot contain such a subsequence, which contradicts to our initial assumption.

By the result of the last paragraph we can rephrase our problem as follows: *Prove that $\overline{\mathbb{H}^n} \setminus \overline{M}$ has two connected components.* By Alexander-Pontrjagin-Alexandroff duality theorem $H_{n-1}(\overline{M}, \mathbb{Z})$ is isomorphic to $H_0(\overline{\mathbb{H}^n} \setminus \overline{M}, \mathbb{Z}^*)$, where the latter is understood in the sense of augmented complexes. Since $H_{n-1}(\overline{M}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ (the only cycle is represented by $\overline{M}$ itself), $H_0(\overline{\mathbb{H}^n} \setminus \overline{M}, \mathbb{Z}^*)$ is also isomorphic to $\mathbb{Z}$. The number of connected components is equal to the rank of the 0-th homology group (of the augmented chain complex) plus one. Thus, $\overline{\mathbb{H}^n} \setminus \overline{M}$ has two connected components. ■

# 5 Main Result

Denote by $Sk_{n-2}(M)$ the complement in $M$ of all $(n-1)$-dimensional faces of $M$. As before, $M = f(\mathscr{M})$.

**Theorem 12** *Let $f : \mathscr{M} \to \mathbb{H}^n$ be a proper locally-convex embedding of a connected manifold $\mathscr{M}$ of dimension $n-1$, where $n \geq 2$. Suppose $Sk_{n-2}(M)$ is connected. Then $f$ is a homeomorphism on the boundary of a convex body.*

**Proof.** Since $f$ is proper, it is also complete (Lemma 3). If $\mathscr{M}$ is compact, then by Theorem 6 we conclude that $f$ is a homeomorphism on the boundary of a convex body. By the results of the previous section $\mathbb{H}^n \setminus M$ consists of two connected components. Denote one of the components by $\underline{0}$ and the other by $\underline{1}$. If $Sk_{n-2}(M)$ is empty, then $M$ is a hyperplane and the theorem is proven. Otherwise, let $C$ be a function on that assigns to each $p \in Sk_{n-2}(M)$ the connected component to which convex witnesses of $p$ belongs. Since $p \in Sk_{n-2}(M)$ is not contained in any $(n-1)$-flat of $M$, it cannot have convex witnesses whose interiors lie in distinct connected components of $\mathbb{H}^n \setminus M$. Thus, we have a well-defined map $C$ from $Sk_{n-2}(M)$ to $\{\underline{0}, \underline{1}\}$ that assigns to each point $p$ of $M = f(\mathscr{M})$ the connected component containing its convex witness. Let us equip $\{\underline{0}, \underline{1}\}$ with discrete topology.

Let $K$ be a convex witness for $p$. If $p'$ is sufficiently close to $p$, then $K$ is also a convex witness for $p'$. Thus, $C(p) = C(p')$ and $C$ is continuous at $p$. The space $\{\underline{0}, \underline{1}\}$ is disconnected, while the space $Sk_{n-2}(M)$ is connected. Since $C : M \to \{\underline{0}, \underline{1}\}$ is continuous we conclude that either $C(M) = \underline{0}$ or $C(M) = \underline{1}$.

Let $F$ be an $(n-1)$-face of $M$. Unless $M = F$, the relative boundary of $F$ is non-empty. Each point of $F$ can be connected by an path contained in $F$ to a point of $\mathrm{rel}\partial F$. Since all points of rel$\partial F$ have their convex witnesses in the same component of $\mathbb{H}^n \setminus M$, we can unambiguously assign the same component to all points of $F$. In other words, the map $C$ can be continuously extended to all of $\mathscr{M}$. Thus, $M$ can be regarded as the boundary of an open set $S$ in $\mathbb{H}^n$ which contains convex witnesses for all points of $M$. The set $S$, considered as a subset of $\mathbb{R}^n$ satisfies the conditions of Ehrhart Schmidt's theorem and therefore is convex. Thus, $M$ is the boundary of convex body $S$ in $\mathbb{H}^n$. ■

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