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The Theory of Quality Translations with Applications to Tilings

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In this paper, we consider a broad class of simply connected complexes that includes, for example, the face-to-face tilings of E^n and S^n. Along with a complex (or tiling) we consider a set $\mathfrak{Q}$ of *qualities* that individually can be assigned to the various cells or tiles of the complex, and a group $\mathfrak{G}$ which permutes the elements of $\mathfrak{Q}$. These qualities might be colours, a set of positive real numbers or a set of affine functions. If each ordered pair of adjacent cells is associated with a group element $g \in \mathfrak{G}$, and a quality $q \in \mathfrak{Q}$ is assigned to any particular cell of the complex, qualities can be assigned to adjacent cells using the action of the group $\mathfrak{G}$ on $\mathfrak{Q}$. In fact, since the complex is assumed to be simply connected, a quality can be assigned to any cell of the complex by *translating* qualities along paths of adjacent cells using the group action. Our main theorem gives sufficient conditions that the quality assigned to a cell in this manner is independent of the particular path used for the *translation,* and is therefore uniquely determined. As applications of this theorem we establish an n-dimensional generalization of the converse of Maxwell's theorem on frameworks [10, 11], we obtain some theorems on colouring the cells of a complex, and we generalize the main part of Voronoi's famous theorem on primitive parallelohedra [16]. See [7] and the survey [1].

1. INTRODUCTION

In 1864, J. C. Maxwell, using geometrical methods, studied the forces on a planar framework that is in static equilibrium. He considered frameworks that are vertical projections of 1-dimensional skeletons of spatial polyhedral surfaces. In the case in which the polyhedral surface is infinite, the projections of infinite edges can be interpreted as external forces acting on the framework. Maxwell discovered that if the surface is convex and infinite, and the external forces are either all compressions or all tensions, then the edges of the framework are subjected to like forces and are respectively all under compression, or all under tension. He also discovered that if the polyhedral surface is convex and its boundary lies in a 2-plane, then either all the interior edges of the framework are under tension while the boundary edges are under compression, or vice versa. Maxwell formulated the converse theorems for each of these cases [1]. The first rigorous presentation of the converse theorems was given by W. Whiteley [18].

As an application of our *theorem on quality translation* (Theorem 3) we prove the converse of Maxwell's theorem on 2-dimensional frameworks, and also prove an n-dimensional generalization for the case in which the polyhedral surface is convex and infinite (see Theorem 4 of Section 6, and the comments immediately following the proof). In this application, the set of qualities $\mathfrak{Q}$ is taken to be the set of affine functions on E^n.

In 1908, G. F. Voronoi proved that a face-to-face tiling of E^n by translated copies of an arbitrary primitive parallelohedron is the affine image of a tiling by Dirichlet domains of an appropriate lattice [16]. Central to his proof is the construction of a *generatrissa* for the tiling. The *generatrissa* is an infinite convex polyhedral in E^{n+1} that projects onto the tiling. Theorems 4 and 5 of Section 6 follow from our theorem on quality translations, and give sufficient conditions for the existence of a generatrissa that generalizes the results of Voronoi to a broader class of tilings.

As an explicit example of an application of our main theorem, consider a

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face-to-face tiling of E^n, along with the colours black and white (the 2-point set of qualities $\mathfrak{Q}$). Under what conditions can a tiling be coloured as a chess-board, where adjacent tiles have different colours? If g permutes the two colours, then associate with each ordered pair of adjacent tiles the group element g of the permutation group on two objects. To attempt a chess-board colouring first assign a colour to one of the tiles, which for convenience we will call the origin. Then the other tiles are coloured by joining them to this origin with a chain of adjacent tiles, and alternatively colouring the elements of the chain. This method of colouring by *quality translation* will fail if the colours assigned depend on the particular chain used. It is easy to see that failures of this type occur when there is an $(n-2)$-face the star of which contains an odd number of tiles. Our theorem on quality translation establishes that a chess-board colouring can be achieved if the star of every $(n-2)$-face has an even number of tiles.

The prints of M. C. Escher provide many marvellous examples of colouring schemes for tilings (see Section 6).

It is natural to formulate our main results in terms of *Q-complexes,* which are defined in Section 1. Moreover, this level of generality is important in applications of the theory. Nevertheless, it is possible to skip the technicalities of Section 1 on a first reading, and we advise the reader to do this. If this advice is taken, the term 'Q-complex' should be mentally replaced in subsequent sections by the phrase 'a tiling of E^n', or 'a tiling of the surface S^n of a convex polyhedron', or the 'k-dimensional skeleton' ($k \geqslant 2$) of either of these tilings.

2. Q-COMPLEXES

The basic object that we study is a *Q-complex,* which is defined in the last paragraph of this section.

A polyhedral cell complex K^n is a complex in Euclidean space E^N in which the cells are convex polyhedra (the polyhedra may be unbounded, but each point of the complex belongs to at most a finite number of tiles). We shall consider a polyhedral cell complex as a topological space with a fixed decomposition into convex polyhedra.

The star $St(c^k)$ of a k-dimensional cell c^k is the union of all open cells the closures of which contain c^k. We shall say that an n-dimensional simply and strongly connected, dimensionally homogeneous complex $K^n \subseteq E^N$ is a *QRR-complex* if all of the stars $St(c^k)$, $k < n-1$, satisfy one or the other of the following two local conditions.

LOCAL CONDITION 1. *For* $0 \leqslant k \leqslant n-3$, $St(c^k)$ *is combinatorially equivalent to the product of* c^k *and a relatively open cone with a simply connected and strongly connected* $(n-k-1)$*-dimensional finite polyhedron as base.*

LOCAL CONDITION 2. *For* $k = n-2$, $St(c^k)$ *is combinatorially equivalent to the product of* c^k *and a relatively open cone with a connected* 1*-dimensional finite polyhedron as base.*

We shall refer to the base mentioned in Condition 1 as the *linked polyhedron* for $St(c^k)$, and to the base mentioned in Condition 2 as the *linked graph* (linked 1-dimensional polyhedron) for $St(c^{n-2})$. For additional details on these notions, see [8].

Additional generality can be added by considering CW-complex (see [17] for details on CW-complexes). Each k-dimensional cell ($k \leqslant n$) of a general n-dimensional CW-complex has a characteristic mapping from the unit ball B^k onto the closure of the cell, which is homeomorphic on the interior and continuous on the boundary of B^k. When all of the characteristic mappings are homeomorphisms on B^k, for all values of $k < n$, the cells can be triangulated as follows. If an interior point in each cell is selected as an apex, the cell can be triangulated as a cone, so that it becomes a simplicial star. The resulting simplicial cell-complex can be embedded into Euclidean space as a piecewise-linear complex, each flat piece corresponding to a simplicial part of a cell of the original cell-complex.

Notice that such a complex can be defined using Seifert and Threlfall's notion of cell complex (see [14], Section 67), where every cell is a cone with a homological sphere as base. By replacing homological sphere with combinatorial sphere in this definition, we obtain a CW-complex, where the characteristic mappings are homeomorphisms on entire balls. Such cell-complexes occupy an intermediate position between abstract CW-complexes and simplicial complexes and are of interest to us in our study of tilings of spheres with spherical polygons.

If a CW-complex with such cells satisfies both local conditions we will call it a *QCW-complex.* Since the results of this paper hold for both QRR- and QCW-complexes, it is convenient to introduce the class of Q-complexes which is the union of these two classes. All of our theorems will be stated in terms of Q-complexes.

3. PROPERTIES OF Q-COMPLEXES

PROPOSITION 1. *The star $St(c^k)$ of each k-cell of a Q-complex, and its closure, are simply connected when $n-2 \geqslant k \geqslant 0$.*

PROOF. This is a direct consequence of the local conditions. □

Every combinatorial manifold is an example of a Q-complex. The following lemma gives an additional large class of examples of Q-complexes. (In the statement of this lemma, we use the notation $Sk^k(K^n)$ to denote the k-dimensional skeleton of the complex K^n.)

LEMMA 1. *If K^n is a Q-complex and $n \geqslant k \geqslant 2$, then $Sk^k(K^n)$ is also a Q-complex.*

PROOF. It is known (and evident) that the 2-skeleton of a CW-complex has the same fundamental group as the complex itself. A QRR-complex can have infinite cells, so we briefly consider an arbitrary polyhedral cell complex with infinite cells. The infinite cells can be sequentially contracted onto their boundaries, starting with the n-dimensional infinite cells and ending up with the 1-dimensional ones. Since all cells of the resulting polyhedral cell complex are compact and convex, the polyhedral cell complex $Sk^k(K^n)$ is homologically equivalent to some CW-complex. Hence the 2-skeleton of an arbitrary polyhedral cell complex has the same fundamental group as the complex itself, because this statement holds for CW-complexes. It therefore follows that $Sk^k(K^n)$ is simply connected.

We now verify that $Sk^k(K^n)$ satisfies local conditions 1 and 2. To make our considerations more visible, we will treat linked polyhedra as bases of cones that are sections of stars by perpendicular disks in E^N. Consider the star in $Sk^k(K^n)$ of an arbitrary m-cell c^m belonging to $Sk^k(K^n)$ ($m \leqslant k-2$). We need to show that this start is combinatorially equivalent to the product of c^m and a relatively open cone with a connected polyhedron (in the case $m = k-2$) as base or a 1-connected polyhedron (in the case $m < k-2$) as base. The star of c^m in $Sk^k(K^n)$ includes all of the cells of K^n that make full contact with c^m, and have dimension at most k.

Let us consider the case $m < k-2$. The intersection of the star of c^m in $Sk^k(K^n)$ with an appropriately small orthogonal $(N-m)$-disk is a polyhedron of dimension at least 3. This polyhedron is a cone having a $(k-m+1)$-dimensional polyhedron P^{k-m-1} as base $(k-m-1 \geqslant 2)$. The intersection of the star of c^m in K^n with the same $(N-m)$-disk is a polyhedron of dimension at least 3. This polyhedron is a cone with a 1-connected polyhedron of dimension at least 2 as base. The fundamental group of the base coincides with the fundamental group of its 2-skeleton, i.e. with the fundamental group of P^{k-m-1}, since $k-m-1 \geqslant 2$. Thus the local conditions for a Q-complex hold when $m < k-2$.

Now consider the case $m = k-2$. The star of c^m in $Sk^k(K^n)$ includes all the cells of K^n that have full contact with c^m, and have dimension at most $m+2$. The intersection of this star with a small orthogonal $(N-k+2)$-disk is a 2-dimensional polyhedron, which is a cone with a 1-dimensional polyhedron as base, which we denote as P^1. The intersection of the star of c^m in K^n with the same $(N-m)$-disk is a polyhedron of dimension at least 2. According to local condition 2, this polyhedron is a cone with a connected polyhedron of dimension at least 1 as base. Since the 1-skeleton of this base coincides with P^1, we conclude that P^1 is connected, and that local condition 2 holds. □

LEMMA 2. *The intersection of the stars of two vertices of a Q-complex is either empty, or contains the star of the cell of minimal possible dimension that is contained in both.* □

PROOF. This statement is easy to verify. □

4. COMBINATORIAL PATHS ON Q-COMPLEXES

A *combinatorial path* in a Q-complex K^n is a finite ordered sequence $\mathfrak{p} = [c_1^n, \ldots, c_k^n]$ of n-cells, where consecutive n-cells share a common $(n-1)$-face. A trivial path is one consisting of a single element. A *combinatorial circuit* is a path with at least two different n-cells, and where the first and last cell coincide. A circuit is *k-primitive* $(0 \leqslant k \leqslant n-2)$ if all its elements belong to the star of a k-dimensional cell.

Two paths can be added if the last cell of the first path coincides with the first cell of the second path. For example, the sum of paths $[c_1^n, \ldots, c_i^n]$ and $[c_i^n, \ldots, c_k^n]$ is $[_1^n, \ldots, c_i^n, \ldots, c_k^n]$.

We shall say that two paths are equivalent if the first can be obtained from the second by either cancelling or adding a $(n-1)$-primitive circuits. For example, if $\mathfrak{p} = [c_1^n, \ldots, c_i^n, c_{i+1}^n, c_{i+2}^n, \ldots, c_k^n]$ and $c_i^n = c_{i+2}^n$, then $[c_i^n, c_{i+1}^n, c_{i+2}^n]$ is a $(n-1)$-primitive circuit which can be cancelled to give the equivalent path $\mathfrak{p}' = [c_1^n, \ldots, c_i^n, \ldots, c_k^n]$, so $\mathfrak{p}$ is equivalent to $\mathfrak{p}'$.

For each path $\mathfrak{p}$ there is an inverse path $-\mathfrak{p}$ obtained by reversing the order of the cells. A path can be added to its inverse, resulting in the trivial path, which is equal to the first cell of the path.

We shall say that a circuit is *k-reducible* if it can be reduced to the trivial path by cancelling k-primitive circuits.

5. REDUCTION OF COMBINATORIAL CIRCUITS

THEOREM 1. *Each combinatorial circuit is an n-dimensional Q-complex is $(n-2)$-reducible.*

LEMMA 3. *Each combinatorial circuit in an n-dimensional Q-complex is 0-reducible.*

PROOF. Consider an arbitrary combinatorial circuit which we will represent with a 1-dimensional circuit $\mathfrak{c}$ on which the n-cells of the combinatorial circuit are strung like the beads of a necklace. We require $\mathfrak{c}$ to be piecewise-linear with vertices interior to n-cells, and edges that avoid $Sk^{n-2}(K^n)$. Let $\mathfrak{c}$ be the image of the boundary of some square P^2 under a continuous mapping f. By the 1-connectivity of K^n, f can be extended to the square P^2 so that the image of P^2 lies in K^n. Let $C \subseteq R^N$ be a cube containing the image $f(P^2)$. The set of stars of vertices of K^n is an open covering for the compact set $C \cap K^n$. For this covering there is a Lebesgue number $\delta > 0$ such that when a subset of $C \cap K^n$ has diameter less than δ, this subset belongs entirely to at least one element of the covering.

By the uniform continuity of our mapping f, there is a quadrilliage of P^2 (see Figure 1) such that the image of each tile of the quadrilliage has diameter less than δ, and hence lies in the star of some vertex of K^n. In our argument below we will require that the images of the vertices of the quadrilliage all lie interior to n-cells. To achieve this, a small shift of some vertices of the quadrilliage may be required.

Consider an edge $[v_1, v_2]$ of a square of the quadrilliage. Let $St(c^0)$ be the star of a vertex c^0 that contains the images $f(v_1)$ and $f(v_2)$. If the images $f(v_1)$, $f(v_2)$ lie in a common cell c^n, they can be joined by the combinatorial path $[c^n, c^n]$; if they lie in adjacent cells c_1^n and c_2^n, then they can be joined by the combinatorial path $[c_1^n, c_2^n]$. If neither of these situations holds, we can join $f(v_1)$, $f(v_2)$ by a combinatorial path lying entirely in $St(c^0)$. Such a path exists by the strong connectivity of stars of cells that have dimension not exceeding $n-2$ (by the definition of a Q-complex). Thus in all cases there is a combinatorial path connecting $f(v_1)$ and $f(v_2)$ that lies in $St(c^0)$.

Representing the circuit $\mathfrak{c}$ as a 'sum of edges' of the small squares of the quadrilliage

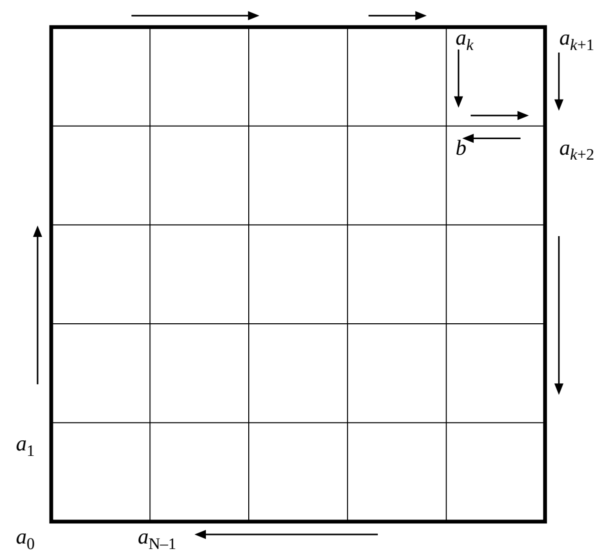

FIGURE 1.

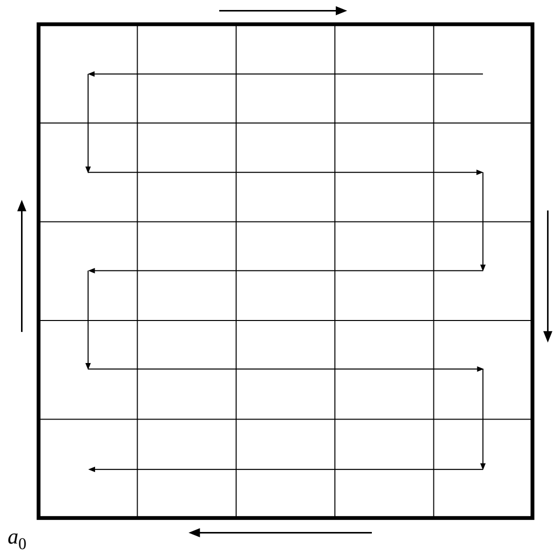

FIGURE 2.

(see Figure 2), it is easy to see how $\mathfrak{c}$ can be reduced to a trivial path. Adopting the notation introduced in Figure 2, we have the following path equivalences:

$$\begin{aligned}\mathfrak{c} &= [a_0, a_1, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_{N-1}, a_0]\\ &\sim [a_0, a_1, \ldots, a_k, a_{k+1}, a_{k+2}, b, a_k, b, a_{k+2}, \ldots, a_{N+1}, a_0]\\ &\sim [a_0, a_1, \ldots, a_k, b, a_{k+2}, \ldots, a_{N-1}, a_0].\end{aligned}$$

By cancelling the 0-primitive circuit $[a_k, a_{k+1}, a_{k+2}, b, a_k]$, we obtain the path

$$[a_0, a_1, \ldots, a_k, b, a_{k+2}, \ldots, a_{N-1}, a_0].$$

By repeating this process for each square of the quadrilliage, following the order indicated in *Figure 2*, the circuit $\mathfrak{c}$ is reduced to a trivial circuit. □

LEMMA 4. *Each combinatorial circuit which belongs to the star of a vertex of a n-dimensional Q-complex is $(n-2)$-reducible.*

PROOF. We shall prove Lemma 4 by induction on n. The statement of the lemma holds for the case $n = 2$ by Lemma 3. If $\mathfrak{p}$ is a combinatorial circuit belonging to the star of a vertex of an n-dimensional Q-complex K^n, there is a corresponding combinatorial circuit $\mathfrak{p}_1$ on the linked polyhedron of this star. By Lemma 2, $\mathfrak{p}_1$ is 0-reducible on the linked polyhedron LP^{n_1}, $n_1 = n - 1$, of the star. It follows from the induction hypothesis that $\mathfrak{p}_1$ is also $(n_1 - 2)$-reducible on the linked polyhedron. Passing from the linked complex to the star, we see that $\mathfrak{p}$ is $(n-2)$-reducible. □

PROOF OF THEOREM 1. By Lemma 3, circuits in a Q-complex are 0-reducible. It is a consequence of Lemma 4 that such circuits are also $(n-2)$-reducible. □

Let K^n be an arbitrary n-dimensional Q-complex, and let $\mathfrak{B}(K^n)$ be the space of equivalence classes of combinatorial paths on K^n (see Section 3). We shall say that a mapping f from $\mathfrak{B}(K^n)$ into a group $\mathfrak{G}$ is a morphism if;

(1) for any two paths $\mathfrak{p}_1$, $\mathfrak{p}_2 \in \mathfrak{P}(K^n)$ where the sum is defined, the quality $f(\mathfrak{p}_1 + \mathfrak{p}_2) = f(\mathfrak{p}_1)f(\mathfrak{p}_2)$ holds;

(2) f maps every trivial circuit into the identity element of $\mathfrak{G}$.

For any morphism f we easily infer that $f(-\mathfrak{p}_1) = (f(\mathfrak{p}_1))^{-1}$. The kernel of a morphism is the set of paths which are mapped into the unity element, and is a subset of the space of all paths. The kernel is closed under addition.

A morphism from $\mathfrak{P}(K^n)$ into a group $\mathfrak{G}$ can be constructed as follows. For each pair of adjacent n-cells c_i^n, c_j^n, a group element g is assigned to the path $[c_i^n, c_j^n]$, and the group element g^{-1} is assigned to the path $[c_j^n, c_i^n]$. Since an arbitrary path is the sum of paths of length 2, this mapping can be extended to $\mathfrak{P}(K^n)$ by using condition (1) for a morphism. It is clear that condition (2) is also respected by this construction.

THEOREM 2. *Let f be a morphism of $\mathfrak{P}(K^n)$ into $\mathfrak{G}$. If each simple $(n-2)$-primitive circuit lies in the kernel of f, then each circuit also lies in the kernel.*

PROOF. We first observe that any $(n-2)$-primitive circuit can be reduced to a trivial path by cancelling simple $(n-2)$-primitive circuits. If simple $(n-2)$-primitive circuits belong to the kernel of f, it therefore follows that $(n-2)$-primitive circuits also belong to the kernel of f. An application of Theorem 1 establishes that an arbitrary combinatorial circuit belongs to the kernel of f. □

6. QUALITY TRANSLATION

Let $\mathfrak{Q}$ be a set of 'qualities'. Here we consider the problem of assigning an element of $\mathfrak{Q}$, a quality, to each of the n-cells of K^n so that qualities assigned to adjacent cells are governed by rules associated with the common facet. Such a problem arises in constructing the generatrissa of a tiling of E^n. The generatrissa of a tiling is a convex polyhedral surface in E^{n+1} that projects vertically onto the tiling (see [1], [6], [12], [13] and [16]). In this case the qualities are affine functions defined on each of the cells, that support the corresponding facets of the polyhedral surface. The differences between functions on adjacent n-cells are completely determined by the edge in the dual graph that corresponds to the facet shared by the cells.

More formally, let K^n be a Q-complex, let $\mathfrak{q}$ be the set of qualities, and let $\mathfrak{G}$ be a permutation group that acts on $\mathfrak{Q}$. For each pair of adjacent n-cells c_i, c_j, a group element $g \in \mathfrak{G}$ is assigned to the two-element path $[c_i, c_j]$, and the inverse element g^{-1} is assigned to the inverse path $[c_j, c_i]$. This mapping can be extended to a morphism $f\colon \mathfrak{P} \to \mathfrak{G}$ on the set of equivalence classes of paths using the formula given in condition (1) for a morphism.

Suppose that a quality $q \in \mathfrak{Q}$ is assigned to one particular cell c_0, conveniently called the origin. Then a quality can be assigned to any other cell c_k by translating q along a path connecting c_0 to c_k, using the morphism f. If $\mathfrak{p} = [c_0, \ldots, c_k]$ is such a path, the quality assigned to c_k is given by the following formula:

$$qf(\mathfrak{p}) = qf([c_0, c_1]) \cdot f([c_1, c_2]) \cdot \ldots \cdot f([c_{k-1}, c_k])$$

PROPOSITION 2. *Equivalent paths assign the same quality*

PROOF. This proposition is a direct consequence of the formula

$$f([c_1, c_2, c_1]) = f([c_1, c_2])f([c_1, c_2]) = gg^{-1} = I$$ □

The condition that the qualities assigned to the tiles by translation are well defined and independent of path, is that every circuit lies in the kernel of f. A more manageable condition is given by the following theorem.

Theorem 3 (on quality translation). *A morphism f, from $\mathfrak{V}(K^n) \to \mathfrak{G}$, uniquely assigns qualities to the cells of a complex if and only if all simple $(n-2)$-primitive circuits lie in the kernel of f.*

Proof. This is a direct consequence of Theorems 1 and 2.

7. CONSTRUCTING THE GENERATRISSA

A convex polyhedral surface in E^{n+1} that projects onto a tiling $\mathfrak{T}$ of E^n was called a *generatrissa* by G. F. Voronoi. Such surfaces were first introduced J. K. Maxwell in his study of planar frameworks at equilibrium (Maxwell considered only surfaces with a finite number of faces). Later, Voronoi used the same construction in his study of infinite tilings with primitive parallelohedra. However, some important topological considerations were omitted in the proof of his important Theorem on the Generatrissa. In this section we give a new and complete proof of the main part of this theorem (Theorems 4 and 5) that is based on the theorem on quality (translation (Theorem 3).

If $\mathfrak{T}$ is a tiling of E^n with convex tiles, the *convex dual graph* of $\mathfrak{T}$ is a graph $G \subseteq E^n$, with the following properties:

(1) The vertices of G are in one-to-one correspondence with the tiles of $\mathfrak{T}$.

(2) The edges of G are in one-to-one correspondence with the facets of $\mathfrak{T}$, and are perpendicular to corresponding facets. If edge e corresponds to facet F^{n-1}, then the vertices of e correspond to the two tiles that belong to the star of F^{n-1}.

(3) If the vertices v_1 and v_2 of e correspond to the tiles T_1, T_2, then e is oriented so that a translation will place v_1 in the same half-space determined by F^{n-1} as T_1, and will place v_2 in the same half-space as T_2.

By those conditions the edges of G corresponding to facets that belong to the star of an $(n-2)$-dimensional face F^{n-2} form the boundary of a convex polygon in a 2-space perpendicular to F^{n-2}. This polygon is called the *dual convex polygon* of the star of the face F^{n-2}.

More generally, if *St* is the star of an $(n-k)$-dimensional face of $\mathfrak{T}$, then the *dual convex polytope* of *St* is a k-dimensional convex polytope $D^k(St)$ in E^n satisfying the following conditions:

(1) For $0 \leqslant m \leqslant k$, there is a one-to-one correspondence between the m-dimensional faces of $D^k(St)$ and the $(n-m)$-dimensional faces of *St.*

(2) If $d^s \subseteq d^t$ are faces of $D^k(St)$ corresponding to faces F^{n-s} and F^{n-t} of $\mathfrak{T}$, then $F^{n-t} \subseteq F^{n-s}$.

(3) For $0 \leqslant m \leqslant k$, each-dimensional face of $D^k(St)$ is perpendicular to the corresponding $(n-m)$-dimensional face of *St.*

(4) $Sk^1(D^k(St))$ is a convex dual graph for the star *St.*

The convexity of the convex dual polytope can be deduced from conditions (1)–(4), but we have chosen to avoid technicalities and include this statement in the definition.

The dual convex polytope and graph are defined up to affine translations and homotheties. In some cases there is additional flexibility and other affine transformations can be applied to these dual objects. This happens, for example, when the convex dual graph is a lattice.

Some authors [1], including Maxwell [10, 11], have used the term ‘reciprocal’ rather than ‘dual’ [12]. We have adopted the latter due to its more frequent use.

A tiling has a convex dual graph iff the tiling has a generatrissa. This equivalence for the genenral n-dimensional case was first noticed by F. Aurenhammer [2]. In this section we establish an n-dimensional generalization of the converse of Maxwell's theorem that guarantees the existence of a generatrissa under an hypothesis that is weaker than requiring that the tiling has a convex dual graph.

If $\mathfrak{p} = [T_1, \ldots, T_k]$ is a combinatorial path, let $E_{i,i+1}(\mathbf{x}) = 0$ be an equation for the hyperplane separating T_i and T_{i+1}, and be such that $E_{i,i+1}(\mathbf{x}) \leqslant 0$ for $\mathbf{x} \in T_i$, and $E_{i,i+1}(\mathbf{x}) \leqslant 0$ for $\mathbf{x} \in T_{i+1}$. We will say that a tiling is canonically defined if the linear equations determining the facets can be fixed up to sign, so that the equality $E_{1,2} + E_{2,3} + \cdots + E_{k-1,k} \equiv 0$ holds in all cases in which $\mathfrak{p}$ is a simple $(n-2)$-primitive circuit.

The converse of Maxwell's theorem on spider webs [1, 10, 11] can be formulated as follows: *If the* 1-*dimensional skeleton of a tiling of the plane is a spider web, then the tiling is the projection of a convex polyhedral surface.* In the 2-dimensional case spider webs and canonically defined tilings are equivalent, so the following theorem can be considered a generalization of the converse of the Maxwell's theorem on frameworks.

THEOREM 4 (Theorem 9 of [13]). *Assume that the tiling $\mathfrak{T}$ is canonically defined. Then $\mathfrak{T}$ has a convex dual graph and a generatrissa.*

PROOF. Using the theorem on quality translation we shall construct the convex dual graph and the generatrissa at the same time. The two sets of qualities that we will need are $\mathfrak{Q}_1$, on the elements of R^n, and $\mathfrak{Q}_2$, the set of affine, scalar valued functions on R^n. Both of these sets have a linear space structure and act on themselves by translations. We will denote the corresponding translation groups by $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

By hypothesis, the tiling is canonically defined and the equations determining the facets are fixed up to sign. For the two-element path $[T_1, T_2]$, the equation $E_{1,2} = a_1x^1 + \cdots + a_nx^n - c = \mathbf{ax} - c = 0$ separates T_1 and T_2, $E_{1,2}(\mathbf{x}) \leqslant 0$ on T_1, and $E_{1,2}(\mathbf{x}) \geqslant 0$ on T_2.

The mapping f_1 defined on two-element paths by the formula $f_1([T_1, T_2]) = \mathbf{a} \in \mathfrak{G}_1$ can be extended to a morphism $f_1 : \mathfrak{B}(\mathfrak{T}) \to \mathfrak{G}_1$ as above. This construction has the simple geometrical interpretation that if $v(T_1)$ is the vertex of the dual graph corresponding to T_1, then the vertex corresponding to T_2 is given by $v(T_2) = v(T_1) + f_1([T_1, T_2]) = v(T_1) + \mathbf{a}$. The second morphism $f_2 : \mathfrak{B}(\mathfrak{T}) \to \mathfrak{G}_1$ is defined by equations of the form $f([T_1, T_2]) = E_{1,2}(\mathbf{x}) = \mathbf{ax} - c$, so that if $l(T_1)$ is the affine function for the tile T_1, the affine function for T_2 is given by $l(T_1) + f([T_1, T_2]) = l(T_1) + E_{1,2} = l(T_1) + \mathbf{ax} - c$. Since $\mathfrak{T}$ is canonically defined, all simple $(n-2)$-primitive circuits lie in the kernel of both morphisms. It follows therefore by Theorem 3, that if a vertex of a convex dual graph is fixed for one tile (the 'origin'), the remaining vertices can be found by translating qualities (the elements of R^n) along paths by means of the action of the group $\mathfrak{G}_1$. Similarly, if an affine function is fixed for one tile, affine functions can be assigned to the others using the group $\mathfrak{G}_2$ and quality translation. The piecewise-linear function that results is continuous by construction (and by Theorem 3). That this function is convex follows from the fact that it is convex on the stars of facets by construction (for more details on the convexity of this function, see p. 281 of [6]). This function determines a convex polyhedral surface in E^{n+1} with facets projecting onto the tiles of $\mathfrak{T}$. It is a generatrissa for the tiling. □

Notice that the existence of a system of canonical equations for a tiling follows directly from the existence of a convex dual graph for the tiling, or the existence of a generatrissa.

In the case $n = 2$ it is easy to see that the existence of canonical equations for a tiling is equivalent to the existence of stresses for the corresponding framework, so that all edges are under tension, or under compression. Such a framework is called a *spider web* [1]. If a cell decomposition has a spider web stress, the vector sum of forces along the edges attached to each vertex equals zero. Taking these forces in clockwise order they can therefore be drawn as a closed polygon. By rotating this polygon clockwise through 90 degrees each of its edges is perpendicular to the corresponding edge of the spider web, and the polygon has become a circuit. Each rotated edge determines a linear equation for the corresponding edge of the tiling. By construction (and Theorem 5) all such equations are canonical. Since these arguments can be reversed they establish the equivalence of the existence of canonical equations for a tiling and the existence of a spider web stress.

Theorem 5 (Theorem 10 from [13]). *If the star of each* $(n-3)$*-face* $(n>2)$ *of a tiling* $\mathfrak{T}$ *has a convex dual polytope with all facets triangular, then* $\mathfrak{T}$ *has a generatrissa.*

Proof. The strategy of proof is to construct the convex dual graph for $\mathfrak{T}$, from which the existence of a generatrissa immediately follows (see the comments prior to Theorem 4). We will consider the $(n-1)$-skeleton $Sk^{n-1}(\mathfrak{T})$ as a Q-complex. For the set of qualities $\mathfrak{Q}$ and the group $\mathfrak{G}$ we take the positive real numbers R_+. The group $\mathfrak{G}$ acts on $\mathfrak{Q}$ by multiplication. The elements of $\mathfrak{Q}$ will be interpreted as the lengths of the edges of the convex dual graph G and are associated with the $(n-1)$-faces of $\mathfrak{T}$.

By hypothesis, the star of each $(n-2)$-face has only three $(n-1)$-faces, so the length of an edge of G corresponding to one of these faces uniquely determines the other two. (Obviously, if the star includes more than three $(n-1)$-faces, the lengths of the edges of the dual polygon would not be uniquely determined by the length of one of them.) As a consequence, for each path $[F_1^{n-1}, F_2^{n-1}]$ of length 2 in Sk^{n-1}, it is possible to fix a mapping $x \rightarrow \alpha x$, where α is the ratio of lengths of edges in G that correspond to the two faces F_1^{n-1} and F_2^{n-1}. The positive number α is interpreted as an element of the group $\mathfrak{G}$. This establishes a correspondence between all two-element paths and elements of $\mathfrak{G}$, which can be extended to a morphism on $\mathfrak{B}(Sk^{n-1})$ in the usual way.

Before applying Theorem 3, we need to obtain some information on the $((n-1)-2)$-primitive circuits in $\mathfrak{B}(Sk^{n-1})$. If F^{n-3} is an arbitrary $(n-3)$-face in $\mathfrak{T}$, let D^3 be the convex dual polytope for $St(F^{n-3})$. By hypothesis, the facets of D^3 are triangles, so if a single edge of D^3 is fixed all others are uniquely determined. Since $((n-1)-2)$-primitive circuits on $St(F^{n-3})$ correspond to circuits on the edges of D^3, this last statement has the interpretation that each $((n-1)-2)$-primitive circuit of $St(F^{n-3})$ belongs to the kernel of the morphism constructed above. Since the choice of the face F^{n-3} was arbitrary, it follows from Theorem 3 that the edges of G are uniquely determined by the length of one particular edge, and the existence of the convex dual graph G is established. □

The following theorem is a reformulation of Theorem 5.

Theorem 6. *If the star of each* $(n-3)$*-dimensional face* $(n>2)$ *of an* $(n-2)$*-primitive tiling* $\mathfrak{T}$ *can be canonically defined, then* $\mathfrak{T}$ *has a generatrissa.*

Proof. Let E^3 be a 3-plane intersecting some $(n-3)$-face F^{n-3} perpendicularly, and through some point v in the relative interior of F^{n-3}. Then, by hypothesis,

$St(F^{n-3}) \cap E^3$ is a $(n-2)$-primitive star, which can be extended to a star decomposition of E^3 by convex cones. By Theorem 4, this decomposition has a generatrissa $G \in E^4$. Let $p \in E^4$ be a point lying above this generatrissa, and let $C(p)$ be the cone dual to G that has the vertex p. Then the polytope $P^3 = C(p) \cap E^3$ is dual to $St(v)$, and therefore to $St(F^{n-3})$. P^3 has triangular facets since $St(v)$ is $(n-2)$-primitive. It follows from Theorem 5 that $\mathfrak{T}$ has a generatrissa.

8. Applications to the Colouring of Tilings

The colouring of tilings and tessellations is interesting from both the mathematical and artistic points of view. In crystallography coloured tilings have appeared in the study of coloured crystallographic groups (see [15]). In the work of M. C. Escher, there are many exquisite examples of coloured tesselations (See [9] for a comprehensive review of his work, complemented with a text by G. S. M. Coxeter; in referring to the catalogue of prints contained in there we will use the abbreviation *cat.*) Some additional examples are the wonderful coloured tessellations on the vaults and walls of the Mauritanian palaces in Alhambra, which have been admired from the time they were created in the fourteenth century.

It is natural to formulate theorems on the colouring of tilings and tesselations using the language of combinatorial manifolds, because manifolds have the same local structure as Euclidean space. More importantly, the proofs of Theorem 8 and 8* below make use of duality theory for manifolds. In particular, use is made of the fact that primitive cell decompositions, where each point of a manfold makes contact with at most $n+1$ n-cells, have dual cell decompositions that are triangulations (see [4], [8], [14].)

A *simply connected combinatorial manifold* can be defined as a QCW-complex M^n where the star of each cell is homeomorphic to the standard n-ball, and the boundary of the star is homeomorphic to the standard $(n-1)$-sphere. Such cell-decomposition of a manifold is called *combinatorial* (for more information on decompositions of manifolds, see [4]). The linked complexes of faces of all dimensions of combinatorial manifolds are standard spheres.

Suppose that $\mathfrak{T}$ is a combinatorial decomposition of a manifold. A 2-colouring, or black and white chess-board colouring of $\mathfrak{T}$, is one in which adjacent cells have different colours.

Theorem 7. *A cell-decomposition of a simply connected combinatorial manifold M^n has a chess-board colouring if the star of each $(n-2)$-cell has an even number of n-cells.*

Proof. The necessity of this condition is obvious, and the sufficiency is obtained by applying Theorem 3. For $\mathfrak{Q}$ we take the two-point set of colours black and white, and for $\mathfrak{G}$ the two-element group that permutes these colours. The morphism f is determined by requiring that f maps each path of length 2 onto the non-trivial element of $\mathfrak{G}$ that permutes the colours. Since the conditions of Theorem 3 are clearly satisfied, the manifold can be chess-board coloured. □

Tessellations with two colours occupy a prominent position in the graphic work of M. C. Escher. He created many woodcuts and engravings of chess-board coloured tilings and tesselations of the Euclidean plane, of Lobachevski's plane and of the sphere. For additional examples that illustrate our Theorem 7, we refer the reader to the well-known works of Escher [9].

The following pieces contain fragments of 2-coloured tilings of the Euclidean plane: Metamorphis II, III (*cat. #320, #446*), Verbum (*cat. #326*), Sun and Moon (*cat. #357*), Regular Division of the Plane with Birds (*cat. #361, #361A*), Plane Filling I, II (*cat. #373, 422*), Division (*cat. #411*) and Regular Division of the Plane I–VI (*cat. #416–421*).

The following pieces illustrate how tesselations and tilings of the Lobachevski plane (the Klein model) can be coloured: Cycle Limit I (*cat. #429*) and Heaven and Hell (*cat. #436*).

Generalizing to n-colours, we can ask under what condition a manifold can be coloured so, that adjacent n-cells have distinct colours. Of particular interest is the problem of finding the minimal number of colours required to colour a manifold. Escher has investigated this more general problem in some of his work; see, for example, Metamorphosis I, II, III (*cat. #298, #320, #426*), Cycle Limit II, II (*cat. #432, #434*), Development I (*cat. #300*), Cycle (*cat. #305*) and Reptiles (*cat. #327*).

Symmetry has certainly played an important role in the work of Escher. In our discussion we are concerned with only the local aspect of a colouring, our global condition being the simple connectivity of the combinatorial manifold.

Theorem 8. *Let* $\mathfrak{T}$ *be a primitive combinatorial decomposition of a simply connected* n*-manifold. Then* $\mathfrak{T}$ *can be coloured with* $n+1$ *colours iff all* 2*-dimensional cells have an even number of vertices.*

It is the dual statement that is easier to prove, and is formulated as follows.

Theorem 8*. *Let* $\mathfrak{T}$ *be a combinatorial triangulation of a simply connected* n*-manifold. Then the vertices of* $\mathfrak{T}$ *can be coloured with* $n+1$ *colours, the vertices of each simplex having distinct colours, iff the star of each* $(n-2)$*-simplex has an even number of* n*-simplexes.*

Proof of Theorem 8*. The necessity of this condition is easy to see, but for sufficiency we will need to apply Theorem 3.

There are $(n+1)!$ ways to colour the vertices of a reference n-simplex s_0^n that are consistent with the colouring scheme that the vertices have distinct colours. If the vertices of all the simplexes of $\mathfrak{T}$ are put into correspondence with these of s_0^n, then any colouring of an arbitrary simplex $s_k^n \in \mathfrak{T}$ that is consistent with the colouring scheme corresponds to one of the $(n+1)!$ possible colourings of the vertices of s_0^n. The colouring of a second simplex s_l^n can be compared with that of s_k^n and is related by a permutation of the vertices of s_0^n. We therefore take, for $\mathfrak{Q}$, the $(n+1)!$ possible colourings of the reference simplex s_0^n, and for $\mathfrak{G}$ the group of all permutations of the vertices of s_0^n.

Once the colours of the vertices of a symplex s_k^n are fixed, the colours of the vertices of an adjacent simplex s_l^n are uniquely determined by the colouring scheme. For any path of length 2 therefore, there is a unique element of $\mathfrak{G}$ which relates the colouring of the second symplex to that of the first. This mapping on paths of length 2 into $\mathfrak{G}$ can be extended to a morphism f on $\mathfrak{B}(\mathfrak{T})$ in the usual way.

Any circuit in the star of an $(n-2)$-simplex must cross a certain number of facets, and if the vertices of these facets are coloured by translation of qualities they must alternate between two types; $n-2$ of the colours are fixed for the facets, and the remaining colours must alternate between the two remaining possibilities, say ‘black and white’. In order that the colouring of the first and the last n-simplex in the circuit agree, an even number of facets must be crossed, which can happen if the star has even number of facets or, equivalently, an even number of n-simplexes. In other words, the circuit belongs to the kernel of the morphism iff the star has an even number of n-simplexes. An application of the theorem on quality translation completes the proof. □

The results quoted in this section are concerned with local conditions for coloring simply connected manifolds. However, we believe similar results can be obtained for much more general manifolds. It is interesting to note that M. C. Escher has investigated the colouring of non-simply connected surfaces: see, for example, Development II. (*cat. #310, #310A*), Predestination (the colouring of a Mobius strip; *cat. #372*), Swans (the colouring of cylinder surface; *cat.#408*), Smaller and Smaller (*cat. #413*), Fish (*cat. #414*), Whirlpools (*cat. #423*), Part of Life I (*cat. #424*), Sphere Surface with Fish (*cat. #427*) and Path of Life III (*cat. #445*).

COROLLARY OF THEOREMS 7 AND 8. *The tiling of E^n by primitive parallelohedra of the first type* (*see* [16]) *can be coloured with $n+1$ colours, but cannot be coloured with* 2 *colours* (*when $n>1$*).

PROOF. Such a tiling can be $(n+1)$-coloured because the 2-dimensional faces of a primitive parallelohedron of the first type are either parallelograms or hexagons. Such a tiling cannot be chess-board coloured because it is $(n-2)$-primitive (there are only three n-dimensional tiles in the star). □

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After our paper was accepted for publication, the authors received a preprint from W. Whiteley [19] which contained theorems that are very close to our Theorems 4 and 5 when $n=3$ (Whiteley established conditions for the existence of a polyhedral surface that projects onto a piecewise-linear realization of a primitive decomposition of a homological sphere in E^3).

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