Supertopes

Robert Erdahl, Konstantin Rybnikov

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Abstract

This document reflects the state of the subject as of the end of 2002. It is an updated and *corrected* version of a preprint initially produced in the end of the summer of 2001 by Rybnikov at the Department of Mathematics of Cornell University.

1 Introduction

A lattice Delaunay cell is *perfect* (a.k.a. extreme) if its Delaunay sphere is the only ellipsoid that circumscribes this cell. Such Delaunay cells correspond, roughly in one-to-one fashion, to extreme hypermetrics, initially studied in analysis and combinatorics (Deza and Laurent 1997). When the lattice is affinely mapped to $\mathbb{Z}^n$, the perfect Delaunay cell is mapped to a $\mathbb{Z}^n$-polytope, which is circumscribed with an empty *perfect ellipsoid* $f(\mathbf{x}) = c$: the defining property of such ellipsoid is that the *inhomogeneous* quadratic form $f(\mathbf{x})$ can be reconstructed in a unique way from its minimum on $\mathbb{Z}^n$ and all representations of this minimum. A perfect ellipsoid is an exact inhomogeneous analog of the notion of *perfect form,* introduced by Korkin and Zolotareff (1873) and later studied by Voronoi (1908-1909), Coxeter (1951), Conway, Sloane (1988), Martinet (1996) etc. It might not be so well known, but Korkin & Zolotareff (1873) came to the discovery of important lattices (and forms) E_n, A_n, D_n by trying to construct infinite series of perfect forms! Point lattices are very important to many areas of algebra, number theory, geometry, combinatorics, cryptography, communication theory, and the theory of approximations: e.g., see Conway and Sloane (1999). To understand properties of lattices we often need to understand certain polytopes associated with these lattices. Delaunay polytopes (also called *holes*) form one of the most important classes of such polytopes.

Positive definite quadratic forms (PDQFs) in n variables make an *open* cone $\mathfrak{P}_n$ of dimension $N = \frac{n(n+1)}{2}$ in $Sym_n(\mathbb{R}) \cong \mathbb{R}^N$, the space of quadratic forms, or symmetric matrices. The boundary of $\mathfrak{P}_n$ consists of positive semi-definite quadratic forms (referred to as PQFs). PDQFs serve as algebraic representations of *point lattices.* There is a natural one-to-one correspondence between isometry classes of n-dimensional lattices and integral equivalence classes (i.e. with respect to substitutions $\mathbf{x}'{=}A\mathbf{x}$, $A \in GL_n(\mathbb{Z})$) of PDQFs in n variables.

Conjugation by a fixed matrix from $GL_n(\mathbb{Z})$ is an invertible linear operator on $Sym_n(\mathbb{R})$. Therefore, conjugation defines a homomorphism $\mathcal{V}$ from $GL_n(\mathbb{Z})$ to $GL_N(\mathbb{Z})$, and $GL_n(\mathbb{Z})$ acts pointwise on $Sym_n(\mathbb{R})$. Two subsets of $Sym_n(\mathbb{R})$ are called arithmetically equivalent if they are equivalent with respect to the action of $\mathcal{V}(GL_n(\mathbb{Z}))$.

Definition 1 A partition $\mathfrak{R}$ *of* $\mathfrak{P}_n$ into *relatively open convex polyhedral* cones with apex at 0 is called a reduction partition if: (1) it is invariant with respect to $GL_n(\mathbb{Z})$; (2) there are only finitely many arithmetically non-equivalent cones in this partition; (3) for each cone C of R and any PQF φ in n indeterminates, φ can be $GL_n(\mathbb{Z})$-equivalent to at most finitely many forms from C.

Voronoi defined two polyhedral reduction partitions of $\mathfrak{P}_n$: these are the tilings by *perfect domains* and *domains for lattice types,* also called *L-domains.*

Definition 2 A convex polyhedron P in $\mathbb{R}^n$ is called a Delaunay cell of a lattice L with respect to a positive quadratic form $\varphi(x)$ if: (1) for each face F of P we have $conv(L \cap F) = F$; (2) there is a quadric $Q_{(P,\varphi)}(x) = 0$, circumscribed about P , whose quadratic part is $\varphi(x)$; (3) no points of $L \backslash P$ satisfy $Q_{(P,\varphi)}(x) \leq 0$.

When $\varphi(\mathbf{x}) = \sum x_i^2$, this definition gives the classical concept of Delaunay cell in $\mathbb{E}^n$ (1924, 1937). Delaunay tilings can be defined not only for lattices, but for any reasonable discrete point sets. These tilings have enormous applications in computational geometry, numerical methods, CAD, the theory of lattices, mathematical crystallography, etc.

Two n-forms φ_1 and φ_2 belong to the same L-type if the Delaunay tilings of $\mathbb{Z}^n$ with respect to these forms are affinely equivalent (the notion of L-type is, in fact, due to Voronoi (1908, vol. 133)). Each L-type domain is, of course, the union of infinitely many convex cones that are equivalent with respect to $GL_n(\mathbb{Z})$, acting pointwise on $Sym_n(\mathbb{R})$. L-type domains form a *reduction partition* of $\mathfrak{P}_n$.

The notions of Delaunay tiling and L-type are extremely important in the study of extremal and group-theoretic properties of lattices. For example, the analysis of Delaunay cells in the famous Leech lattice conducted by Conway, Sloane and Borcherds showed that 23 "deep holes" (Delaunay cells of radius equal to the covering radius of the lattice) in the Leech lattice are in one-to-one correspondence with even unimodular 24-dimensional lattices, classified by Niemeier (1973) that, in turn, give rise to 23 "gluing" constructions of the Leech lattice from root lattices. Barnes and Dickson (1967) and Dickson (1968) and, later, Delaunay et al. (1969, 1970) proved that the closure of each N-dimensional L-type domain has at most one local minimum of the ball covering density, and if such a minimum exists, the group of $GL_n(\mathbb{Z})-$automorphisms of the domain maps this form to itself. Using this approach Delaunay et al (1963, 1970) found the best lattice coverings in $\mathbb{E}^4$ and $\mathbb{E}^5$. The theory of L-types has numerous connections to combinatorics, in particular, to cuts, hypermetrics, and regular graphs (see Deza, Laurent 1997), and algebraic geometry. For example, Valery Alexeev (2001, 2002) recently discovered an interesting connection between L-types of g-dimensional PQFs and certain functorial compactification $\overline{AP}_g$ of the moduli space of abelian varieties A_g.

The L-type partition of the cone of PQFs is closely related to the theory of *perfect forms*. The *arithmetic minimum* of a form $\varphi(\mathbf{x})$ is its minimum on $\mathbb{Z}^n$. The integral vectors on which this minimum is attained are called the *minimal vectors* of φ: these vectors have the minimal length among all vectors of $\mathbb{Z}^n \backslash Ker(\varphi)$, when φ is used as the metrical form. For each PDF φ with the set of minimal vectors $\mathsf{P} \subset \mathbb{Z}^n$ there is a P-domain Π_{P} defined by $\Pi_{\mathsf{P}} = \{\sum_{\mathbf{p}\in\mathsf{P}} \omega_{\mathbf{p}}(\mathbf{p}\cdot\mathbf{x})^2 \mid \omega_{\mathbf{p}} > 0\}$. The P-domains form a *reduction partition* of $\mathfrak{P}_n$ which is called the *perfect* partition. Form $\varphi(\mathbf{x})$ is called *perfect* if it can be reconstructed up to scale from all representations of its arithmetic minimum. Indeed, uniqueness requires the existence of at least $n(n+1)$ minimal vectors. The perfect domains are open polyhedral N-dimensional cones, which fit together facet-to-facet to tile $\mathfrak{P}_n$. The P-domains with lesser dimension are relatively open faces of this tiling. In fact, P-domains from a partition of the convex hull of all rational rank one forms:

$$\bigsqcup_{\mathsf{P}} \Pi_{\mathsf{P}} = conv\{(\mathbf{p}\cdot\mathbf{x})^2 \mid \mathbf{p} \in \mathbb{Q}^n\} \subset \overline{\mathfrak{P}}_n, \tag{1}$$

where the union is taken over all sets of minimal vectors for PDQFs in n variables. Each extreme ray of this tiling lies on $\partial\mathfrak{P}_n$.

Intuitively, perfect lattices are those that have a large supply of minimal vectors, although a perfect n-lattice for $n > 8$ is not always spanned by its minimal vectors. A perfect form $\varphi(\mathbf{x})$ can obviously be described as a hyperplane in $Sym_n(\mathbb{R})$ that contains $N+1$ integer points whose coordinates are the images of the minimal vectors $\{\mathbf{v}_k | \, k = 1, ..., 2s\}$ under the Veronese-Voronoi mapping $V : \mathbf{v}_k \rightarrow \{v_k^i v_k^j | \, 1 \le i \le j \le n\}$. In fact, the intersection of the half-spaces (not containing $\mathbf{0}$) defined by the hyperplanes corresponding to perfect forms is a "polyhedron" with infinitely many faces, called the Voronoi polyhedron. Voronoi showed that this polyhedron has only finitely many faces which are not $GL_n(\mathbb{Z})$-equivalent, and therefore in each dimension there are only finitely many perfect forms up to $GL_n(\mathbb{Z})$-equivalence. Perfect forms play an important role in lattice sphere packings. Voronoi's theorem (1908, vol. 133) says that a form is extreme—i.e., a maximum of the packing density—if and only if it is perfect and eutactic (see Coxeter, 1951 for details).

A cone in $Sym_n(\mathbb{R})$ spanned by the images, under V, of the minimal vectors of a perfect form is called a perfect cone. The union of all perfect cones corresponding to forms integrally equivalent to φ is called the perfect domain of φ. For each perfect cone there are infinitely many $GL_n(\mathbb{Z})$-equivalent ones, so the perfect domain of φ consists of infinitely many equivalent perfect cones, just like an L-type domain consists of infinitely many convex cones. A fundamental theorem of Voronoi (1908, vol. 133) in the interpretation of Ryshkov (in Delaunay and Ryshkov 1971) says that the cone of PQFs is tiled face-to-face by perfect cones. Since there are only finitely many non-equivalent perfect forms, there is a finite set of perfect cones in $Sym_n(\mathbb{R})$ such that each form in n variables is equivalent to a form from one of these cones. Voronoi gave an algorithm finding all perfect domains for given n. This algorithm is known as Voronoi's reduction with perfect forms. For the computational analysis of his algorithm and its improvements see Martinet (1996). Perfect forms have been completely classified in dimensions $n \leq 7$, but already for $n = 9$ there are billions of them. The theory of perfect forms was used for finding the best lattice packings in low dimensions and for classifying maximal finite subgroups of $GL_n(\mathbb{Z})$ for small values of n (Ryshkov 1972). Among important recent developments in the theory of perfect forms and L-types is our result (R.E. and K.R. 2002) showing that, to the contrary of the Voronoi (1909) and Dickson (1972) conjecture, the L-partition of the cone of PQFs is *not* a refinement of the perfect partition for $n \geq 6$. We are now trying to find connections between L and perfect partitions. This work is based on studying the pattern of scalar products between the Voronoi vectors of a PDQF φ and the minimal vectors of the perfect form to the domain of which φ belongs. One of distant goals in this direction is to understand the connections between the packing and covering problems for lattices.

The notion of perfect Delaunay cell is an inhomogeneous analog of the notion of perfect form.

Definition 3 Let $P \subset R^n$ be a Delaunay polytope in Z^n for a PDQF $\varphi(x)$. P is called perfect if the only ellipsoid circumscribed about P is the one defined by the form $\varphi(x)$. In this case the inhomogeneous quadratic form defining this ellipsoid is referred to as a *perfect ellipsoid.*

The vertices of a perfect Delaunay polytope are analogs of the minimal vectors for a perfect form: the minimal possible number of vertices of a perfect Delaunay cell is $\frac{n(n+1)}{2} + n$, while the minimal number of minimal vectors of a perfect forms is $\frac{n(n+1)}{2}$. Perfect Delaunay polytopes are important not only to the theory of lattices, but also to the theory of cuts and hypermetrics started in analysis by I. J. Schoenberg in 1935-37 (for references see Deza & Laurent, 1997). In our research we are primarily interested in geometric aspects of perfect Delaunay cells and ellipsoids, although we keep in mind possible connections with distance regular graphs: e.g. the 1-skeletons of the Delaunay cells of E_6 and E_7 are well-known strongly-regular graphs—the Schlafli and Gosset graphs.

1-dimensional L-types that are interior to the cone of PQFs are very rare in low dimensions. They first occur in dimension 4: D_4 has an extreme L-type. Any new perfect Delaunay cell would give a new example of an extreme L-type. *Prior to our work only finitely many examples of extreme L-types and perfect ellipsoids have been known.* The significance of extreme L-types is much due to their relation to the structure of Delaunay and Dirichlet-Voronoi tilings. Dirichlet-Voronoi polytopes of higher-dimensional lattices are important to the theory of quasicrystalls, coding theory, information quantization, etc (e.g. see Conway & Sloane (1999), Senechal (1995)). The significance of extreme L-types for geometry of lattices is illustrated by the following theorems.

Theorem 1 *(Ryshkov 1998; Erdahl 2000) The Voronoi polytope for any PDQF φ is the Minkowski sum of Voronoi polytopes for quadratic forms lying on the extreme rays of the L-cone of φ. Their arrangement in space is determined by φ. The Delaunay tiling of $\mathbb{Z}^n$ for a PQF φ is the intersection of the Delaunay tilings of forms lying on the extreme rays of the L-cone of φ.*

These propositions are dual formulations of the same fact. For example, the Delaunay tiling of the plane with respect to the form $f = x^2 + y^2 + (x - y)^2$ is just the intersection of the Delaunay tilings for forms x^2, y^2 and $(x - y)^2$: by vertical strips $k < x < k + 1$, $k \in \mathbb{Z}$, by horizontal strips $k < y < k + 1$, $k \in \mathbb{Z}$, and by slanted strips $k < x - y < k + 1$, $k \in \mathbb{Z}$. The Voronoi cell for f is the sum of three segments orthogonal with respect to f to these three families of strips. Therefore, Voronoi polytopes for extreme L-types are "building blocks" for Voronoi polytopes of arbitrary forms, while Delaunay tilings for extreme rays of the L-cone of f are coarsenings of the Delaunay tiling of f.

Very few perfect Delaunay polytopes in low dimensions are known. It is known that there are no perfect Delaunay polytopes in dimensions less than 6 (Erdahl 1975). The canonical examples of perfect Delaunay polytopes are the Gosset polytopes in $\mathbb{E}^6$ and $\mathbb{E}^7$, e.g. see Coxeter (1934, 1973), Erdahl (1992), Deza et al. (1995). Other examples include two 16-dimensional polytopes in BW_{16} and its sublattice, three 15-dimensional polytopes in sections of BW_{16}, two polytopes in 22 and 23 dimensional sections of the Leech lattice (Deza et al. 1995; Deza et al. 1992). Most of these examples are manifestations of such phenomena as extreme sets of equiangular lines and extreme spherical two-distance sets that have been intensively studied in algebraic combinatorics (see Lemmens and Seidel, 1973).

2 Constructing Perfect Delaunay Cells

In 2001 we constructed an infinite series of big Delaunay cells Υ_n not coming from equiangular lines or two-distance sets. This series starts from (the affine image in $\mathbb{Z}^n$ of) Gosset polytope in E_6. The construction of this series is based on the *infinite series of Delaunay simplexes of relative volume* $n-3$ found by Erdahl and Rybnikov (2002b). This is the best known infinite series of big Delaunay simplexes; it improves upon Ryshkov's (1973) series of Delaunay simplexes of volume r in dimension $n = 2r+1$ (it is interesting that Ryshkov's series of big simplexes is also related to his 1973 series of perfect lattices not generated by its perfect vectors). Polytopes Υ_n are constructed by supplementing the vertices of the simplex of volume $n-3$ in some very special way (see below). This construction generalizes the embedding of the simplex of volume $n-3$ into the Gosset polytope in $\mathbb{E}^6$. We hope to generalize the lattice E_8 to an infinite series of lattices with interesting geometric and arithmetic properties.

Lattice Delaunay polytopes of large relative volume and/or many vertices are of special interest to the study of lattice L-types and perfect forms. Polytopes with many vertices normally occur in highly symmetric lattices, such as E_n $(n = 6, 7)$, Barnes-Wall lattice, BW_{16}, sections of Leech lattice, Λ_{24}, etc. Simplexes of large volume are found in lattices with special symmetries (e.g. Λ_{24}) and their perturbations (e.g. some L-type domains with extreme ray of type E_n $(n = 6, 7)$. Delaunay simplexes are very special cases of empty lattice simplexes that have been attracting interest of mathematicians due to their importance in integer programming (e.g. see Haase and Zielger, 2000). While there are pretty sharp results on empty lattice simplexes, not much is known about Delaunay empty lattice simplexes. The volume of an empty lattice simplex can be arbitrary for $n \geq 4$, but the volume of a Delaunay n-simplex is, according to Lovász (unpublished), bounded from above by $n!(2^n/\binom{2n}{n})$ (see Deza & Laurent (1997) for a proof). It is not even known if the maximal volume of a Delaunay simplex grows linearly, polynomially, or exponentially in the dimension. The biggest Delaunay simplex, we know of, lives in the Leech lattice and has volume $85n$ $(n = 24)$, but the best infinite series known is linear: $n-3$. It would be interesting to improve Lovász bound, since it would improve an upper bound on the number of lattice points one has to check to verify if a given lattice polytope is Delaunay (Deza & Laurent, 1997). Rybnikov is also interested in *algorithmic approaches to determining whether an empty lattice polytope is Delaunay.* This problem may be important to cryptography, as it is an inhomogeneous counterpart of the *shortest vector problem.* He is interested in both, algorithms for computers, and "algorithms for humans", i.e. analytical techniques, involving arithmetics and geometry, that allow proving the Delaunay property in the cases when the form is very symmetric, similar to the method of projective inequalities (Dieter, 1975; Anzin, 1991) for the problem of determining the shortest vectors.

2.0.1 Supertopes

We constructed the following infinite series of *supertopes* Υ_n in $\mathbb{Z}^n$ for all $n \geq 6$. Below are the vertices of Υ_n:

$[0^n] \times 1$	$[1, 0^{n-1}; 0] \times (n-1)$	$[1, 0^{n-2}; -1] \times (n-1)$
$[1^2, 0^{n-3}; -1] \times \frac{(n-1)(n-2)}{2}$	$[0, 1^{n-2}; -(n-4)] \times (n-1)$	$[1^{n-1}; -(n-3)] \times 1$

Here we use a short-hand notation for families of vectors obtained from some n-vector by all circular permutations in strings of symbols that are separated by commas and bordered on the sides by semicolons and/or brackets. We realized the equation of the ellipsoid circumscribed about Υ_n should have the following form:

$$d\sum_{\mathbf{i=1}}^{\mathbf{n-1}} x_{\mathbf{i}}^2 + 2m \sum_{\mathbf{1\leq i<j\leq n}}^{\mathbf{n-1}} x_{\mathbf{i}}x_{\mathbf{j}} + 2e\sum_{\mathbf{i=1}}^{\mathbf{n-1}} x_{\mathbf{i}}x_{\mathbf{n}} + bx_{\mathbf{n}}^2 - d\sum_{\mathbf{i=1}}^{\mathbf{n-1}} x_{\mathbf{i}} + l_{\mathbf{n}}x_{\mathbf{n}} = 0 \qquad (2)$$

Theorem 2 *(Erdahl, Rybnikov, Kemp, Saliola 2001) Equation* $\mathbf{x^tQ_nx}+\mathbf{Lx}=\mathbf{0}$ *defines an ellipsoid circumscribed about polytope* Υ_n *for* $n \geq 6$. *Here* $\mathbf{Q_n}=(q_{ij})$ *is a symmetric positive matrix of the above form where*

$$q_{ii} = 2 - 5n + n^2 \text{ for } i < n,$$

$$q_{nn} = -2 - 3n + n^2,$$

$$q_{ij} = 12 - 7n + n^2 \text{ for } i < j < n,$$

$$q_{in} = +4 - 5n + n^2 \textit{ for } i \neq n.$$

$\mathbf{L}$ *is a linear functional defined by*

$$\mathbf{l} \cdot \mathbf{x} = \sum_{\mathbf{i=1}}^{\mathbf{i=n-1}} (-q_{\mathbf{ii}})x_{\mathbf{i}} + l_{\mathbf{n}}x_{\mathbf{n}} = (-2+5n-n^2) \sum_{\mathbf{i=1}}^{\mathbf{i=n-1}} x_{\mathbf{i}} + (6+5n-n^2)x_{\mathbf{n}}. \qquad (3)$$

This ellipsoid is unique.

Michael Greene, a Harvard undergraduate who worked in the summer of 2001 with Rybnikov in the **R**esearch **E**xperience for **U**ndergraduates Project at Cornell proved that $\mathbf{x^tQ_7x}+L\mathbf{x}=\mathbf{0}$ is Delaunay by checking it directly with a computer program that he wrote in MATHEMATICA system (this program is available by request from Rybnikov: krybniko@cs.uml.edu). Recently, we proved the Delaunay property for all of the series.

In 2002 Bob Erdahl has found an important realization of Υ_n as a section of a centrally-symmetric polytope C_n in $\mathbb{Z}^{n+1}$. For $n=6$, this realization is, in fact, 2_{21} (Gosset 6D polytope) as a section of 3_{21} (Gosset 7D polytope).

In general, consider $\mathbb{Z}^n$ as a subspace of $\mathbb{Z}^{n+1}$ defined by $x_{n+1}=1$. Then take two copies of the original Υ_n : a shift of Υ_n by vector $[0^n;1]$ and an inverted copy of Υ_n, i.e. $-\Upsilon_n$. It turns out that one can complement these two polytopes with two points, so that the convex hull of the resulting set is a perfect $(n+1)$-dimensional Delaunay polytope.

The two points that need to be added to the vertices of $\Upsilon_n+[0^n;1]$ and $-\Upsilon_n$ are $[1^n;(n-3)]$ and $[-1^n;-(n-4)]$. Denote the resulting point set by $vert\, C_{n+1}$. Points of $vert\, C_{n+1}$ can be partitioned into pairs corresponding to the segments passing through $[0^n;\frac{1}{2}]$. These segments are the diagonals of the $(n+1)$-dimensional polytope C_{n+1}, mentioned above. Let us now consider lattice Λ_{n+1} generated by the diagonals of C_{n+1}. $2\mathbb{Z}^{n+1}$ is a sublattice of Λ_{n+1} : if one can prove that the diagonals are the minimal vectors of Λ_{n+1} with respect to some quadratic form φ, then one has a proof that they are minimal vectors in their parity class in $\mathbb{Z}^{n+1}$ relative to this form φ. Vectors $B=\{[0^n;1], [2,0^{n-1};1], [2^n;2n-7]\}$ are linearly independent and, therefore, form a rational basis of Λ_{n+1}. The Gramm matrix of B, with respect to $Q_n(\mathbf{x}',\mathbf{x}')\bigoplus x_{n+1}^2$ where $\mathbf{x}=[\mathbf{x}';x_{n+1}]$, has an astonishingly simple form: it is $I+\alpha J$, where $\alpha=\frac{1+\binom{n-4}{2}}{8(n-5)}$ and J is the $n+1$ by $n+1$ matrix of all 1's.

The philosophy behind the construction of supertopes is in generalizing the affine structure of 2_{21} to a general dimension. While most important affine properties are preserved in this generalization, the group-theoretic ones are not. In our research on supertopes we have been using software POLYMAKE written by Gawrilow and Joswig in TU Berlin.

We hope that one can simplify the analysis of supertopes by using the symmetries of the supertopes, in particular, the so-called "long triangles" (e.g $[1^2;0^{n-3};-1]$, $[0^2;1^2;0^{n-5};-1]$, $[0^4;1^2;0^{n-7};-1]$ in Υ_n). These triangles are similar to 45 long equatorial triangles (of norm 2) on the boundary of 2_{21}, the Delaunay cell of E_6 and an affine image of our 6-supertope. Right now we are checking if these triangles are playing in an n-supertope a role, similar to that of the long triangles of 2_{21}: the projection of 2_{21} on the 4-plane perpendicular to the long triangle is a 24-cell (Ivič Weiss, 1986). In the summer of 2002, within the framework of Cornell REU, Rybnikov worked with Cornell undergraduate student Joseph Palin on connections between "long triangles" and arithmetic properties of inhomogeneous quadratic forms defined by polytopes Υ_n.

Long triangles (of norm 2) in E_n are interesting for another reason. Arrangements of equilateral triangles of side length $\sqrt{2}$ are crucial in Blichfeldt's (1935) argument (published without proof) that the densest lattices for $n = 6, 7, 8$ must have D_5 as a sublattice of the same minimum (see Vetchinkin (1982) for a proof of Blichfeldt's result).

Voronoi vectors of a positive definite form $\varphi(\mathbf{x})$ are, by definition, the minimal integral vectors, relative to $\varphi(\mathbf{x})$, in their parity classes, i.e. mod 2. Roughly speaking, each Delaunay face with inner diagonals (such faces are called *primitive elements* of the tiling) is the convex hull of Voronoi vectors from the same parity class with common midpoint; these diagonals are the only minimal vectors in their parity class. The converse is also true. Although these facts have been first observed and used by Voronoi (1909), Baranovskii (1991) was the first to formulate and prove them as a theorem. The (partial for $n = 5$) classification of L-types for $n < 6$ was obtained via the theory of *primitive elements.* We are trying to generalize the Voronoi-Baranovskii theory of parity classes to *triangles* of Voronoi vectors. Voronoi triangles seem to be very important for the problem of densest lattice packings (methods of Blichfeldt and Vetchinkin) and the theory of perfect forms in general. The problem here is that not all Voronoi triangles from can play a role similar to that of Voronoi vectors. E_6 and the series of forms from the above theorem, basically, serves as a laboratory for this project.

We hope that our work on positive inhomogeneous quadratic forms may lead to a better understanding of the structure of lattices and the relationship between the theories of L-types and perfect forms. This could shed new light on how the packing and covering problems are related to each other.

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