A Reduction of Conway's Thrackle Conjecture

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Abstract. A *thrackle* is a drawing of a simple graph on the plane, where each edge is drawn as a smooth arc with distinct end-points, and every two arcs have exactly one common point, at which they have distinct tangents. Conway, who coined the term thrackle, conjectured that there is no thrackle with more edges than vertices – a question which is still unsolved. A full thrackle is one with n vertices and n edges, and it is called non-extensible if it cannot be a subthrackle of a counterexample to Conway's conjecture on n vertices. We define the notion of incidence type for a thrackle, which is the sequence of degrees of all vertices in increasing order. We introduce three reduction operations that can be applied to full subthrackles of thrackles. These reductions enable us to rule out the extensibility of many infinite series of incidence types of full thrackles. After defining the 1-2-3 set, we reduce Conway's conjecture to the problem of proving that thrackles from the 1-2-3 set are not extensible. Our result proves the hypothesis of Wehner, who predicted that a potential counterexample to Conway's conjecture would have certain graph-theoretic properties.

1 Introduction

In the late 1960's, John H. Conway conjectured that the number of lines of a thrackle on n vertices cannot exceed n, which is known as the **Thrackle Conjecture**. A *thrackle* is a plane drawing of an undirected simple graph (no loops or multiple edges) on n vertices by edges which are smooth arcs (called lines) between vertices, with the condition that every two lines intersect at exactly one point, and have distinct tangents there [1]. More formally, we study the property of a graph to have a thrackle drawing. For the sake of brevity we do not make a notational distinction between a graph with the thrackle property and a thrackle drawing of this graph, referring to both of them as thrackle.

For clarity, a thrackle has the following thrackle properties [4]:

1. the endpoints of each line are two vertices;
2. no line crosses itself;
3. each two lines intersect exactly once;
4. if two lines intersect, then they intersect in a single point.

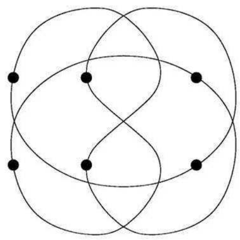

Fig. 1. A thrackle with six vertices and six lines [4]

For example, Figure 1 shows a thrackle with six vertices and six lines. The thrackle conjecture asks if it is possible to add a line between two vertices which intersects all the existing six lines just once.

In a graph, a path $v_0, v_1, ..., v_k$ forms a *simple cycle* if $v_0 = v_k$, all other v_i's are distinct, and the path contains at least one edge. If a simple cycle has n vertices, then it is called an *n-cycle*. Wehner [4] demonstrated that all the simple cycles except the 4-cycle can be thrackles, as shown in Figure 2 for several examples. The 4-cycle cannot be a thrackle, nor a subgraph of a thrackle, since any drawing of the 4-cycle has two lines that do not intersect or intersect more than once, which violates thrackle property 3. Using induction, Wehner proved that if an n-cycle can be a thrackle, so can the $(n + 2)$-cycle. Since apparently the 3-cycle and the 6-cycle can be thrackles, so can all the simple cycles (except the 4-cycle).

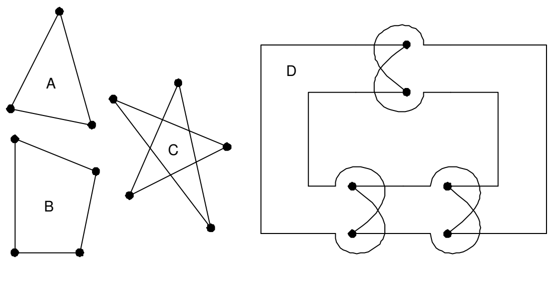

Fig. 2. Thrackles with cycles (except B): A. 3-cycle; B. 4-cycle; C. 5-cycle; D. 6-cycle [4]

After nearly forty years, despite many efforts by researchers, Conway's conjecture remains open. In [2], Lovasz, Pach, and Szegedy showed that every thrackle on n vertices has at most $2n - 3$ lines, instead of $O(n^{3/2})$. The $O(n^{3/2})$ bound follows from an upper bound on the number of edges in a simple graph without 4-cycles, which is a well-known theorem of graph theory. Five years later, Cairns and Nikolayevsky proved that the upper bound can be further lowered to $\frac{3}{2}(n-1)$ [3]. They also showed that the upper bound for generalized thrackles on n vertices (which are the thrackles that allow lines to intersect an odd number of times) is $2n-2$. Wehner predicted that a minimal counterexample to Conway's thrackle conjecture, if it exists, would contain two simple cycles of one of the following types: *Figure-8* (two simple cycles share a vertex), *Theta* (two simple cycles share a path) or *Dumb-bell* (two simple cycles are connected by a path) [4].

Can we lower the upper bound for a thrackle on n vertices further to n? Here we investigate the relationship between the vertex degrees of a graph and the thrackle properties. By defining *full* thrackle and thrackle *incidence type*, we demonstrate that full thrackles are related to each other, by means of reductions. We also show that full thrackles can be divided into equivalence classes in terms of their incidence types, and these full thrackle incidence types form a diagram with a natural hierarchy. We define and study a subset of thrackles called the 1-2-3 set, and prove Wehner's hypothesis on potential counterexamples to Conway's conjecture. From this, we reduce Conway's thrackle conjecture to the problem of proving the non-extensibility of the 1-2-3 set. An earlier version of this work appeared in [5].

2 Reduction Theory for Thrackles

2.1 Full thrackle and its incidence type

Lemma 1. If a line is removed from a thrackle, the result is still a thrackle.
Proof: Suppose T is a thrackle. Then all four thrackle properties must hold for T. Removing one line from T does not affect the rest of the lines and vertices in T, i.e., all four thrackle properties still hold for all the lines left. Therefore, the result is also a thrackle. □

Definition 1. *n-Cycle Thrackle:* A thrackle that is an n-cycle.

With this definition, we call the graphs A, C and D in Figure 2 a 3-cycle thrackle, a 5-cycle thrackle and a 6-cycle thrackle, respectively. It is easy to see that for an n-cycle thrackle, every vertex has a degree of 2.

Definition 2. *Full Thrackle:* A thrackle with n vertices and n lines.

With this definition, it immediately follows that all the n-cycle thrackles are full thrackles since they have an equal number of vertices and lines.

Lemma 2. For a full thrackle on n vertices, the sum of the degrees of all vertices $\sum deg(v_i) = 2n$, where $1 \leq i \leq n$.
Proof: For a full thrackle on n vertices, since it has n lines and each line starts and ends at a vertex, the sum of the degrees of all vertices $\sum deg(v_i)$ is $2n$. □

Lemma 3. If a full thrackle is not an n-cycle thrackle, then it has at least one vertex of degree 1.

Proof: By contradiction. For a full thrackle T on n vertices $v_1, v_2, \ldots, v_n$, suppose there is no vertex of degree 1. Then $deg(v_i) \geq 2$ for all $1 \leq i \leq n$. It follows that $\sum deg(v_i) \geq 2n$. By Lemma 2, we know that '>' cannot be satisfied. This forces $deg(v_i) = 2$ for all $1 \leq i \leq n$, which means T is an n-cycle thrackle, contradicting the assumption that T is not an n-cycle thrackle. □

Definition 3. *Incidence Type of a Thrackle:* A list of n integers sorted in increasing order, where each integer is the degree of a vertex of the thrackle.

For example, as shown in Figure 3, the incidence type of thrackle A is (2, 2, 2), and that of B is (1, 1, 2, 2).

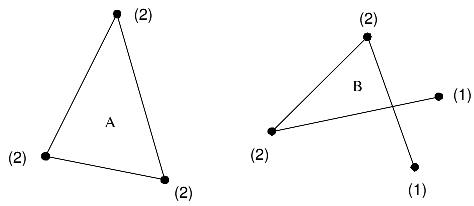

Fig. 3. The incidence types of thrackles: A. (2, 2, 2); B. (1, 1, 2, 2)

Lemma 4. For a full thrackle on n vertices, its incidence type cannot be of the form of $(\underbrace{1, ..., 1}_{n-2}, j, k)$, where j, k are integers and $2 \leq j \leq k$.

Proof: By contradiction. Let $deg(\mathbf{p}) = j$ and $deg(\mathbf{q}) = k$. By Lemma 2, $(n-2) + j + k = 2n$, so $j + k = n + 2$. By the thrackle properties $\mathbf{p}$ and $\mathbf{q}$ share an edge. Therefore, the total number of vertices of degree 1 that $\mathbf{p}$ and $\mathbf{q}$ are adjacent to is $(j-1) + (k-1) = (j+k) - 2 = (n+2) - 2 = n$. However, there are only $(n-2)$ number of vertices of degree 1, which is a contradiction. □

2.2 Reductions of thrackles

For a thrackle, we can remove a vertex of degree 1 along with its adjacent line. It is easy to see that the result is also a thrackle.

Definition 4. *α-Reduction:* The removal of a vertex of degree 1, together with the line adjacent to it from a thrackle. As a consequence, the degree of the vertex which was adjacent to the removed one is decreased by 1.

Lemma 5. The α-reduction of a full thrackle on n vertices is a full thrackle on $n-1$ vertices.
Proof: Removing one vertex of degree 1 and one line from a full thrackle on n vertices will leave the graph with $n-1$ vertices and $n-1$ lines. Thus, the result is a full thrackle on $n-1$ vertices. □

Figure 4 illustrates the α-reduction of the full thrackle (1, 2, 2, 3) to (2, 2, 2). By Lemma 3, we know that if a full thrackle is not an n-cycle thrackle, then it can be α-reduced to a thrackle with fewer vertices.

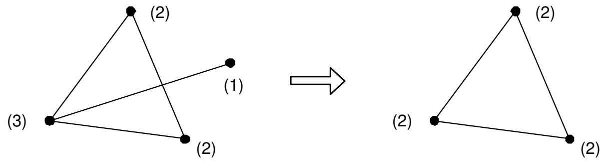

Fig. 4. The thrackle (1, 2, 2, 3) is α-reduced to the thrackle (2, 2, 2)

Can the n-cycle thrackles be reduced at all? Wehner showed a replacement of a 3-path by a 5-path method [4], which is used to prove that if an n-cycle can be a thrackle, so can the $(n+2)$-cycle. A 3-path or a 5-path refers to a graph fragment in an n-cycle, e.g., any three consecutive lines in an n-cycle will form a 3-path. We use Wehner's method reversely to remove two vertices in an n-cycle thrackle at a time. It is easy to see that a graph fragment as in Figure 5 can always be found in an n-cycle thrackle on five or more vertices. This operation consists of the removal of three lines and the addition of a new line. The result is an n-cycle thrackle with two fewer vertices [1].

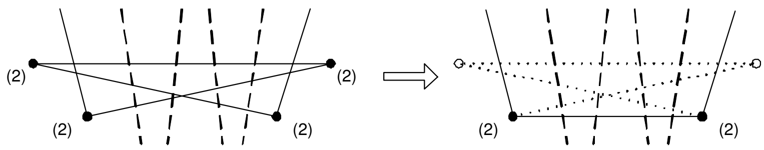

Fig. 5. A simple cycle thrackle $(..., 2, 2, 2, 2, ...)$ is β-reduced to a simple cycle thrackle $(..., 2, 2, ...)$ [4]. The solid lines represent the five consecutive edges involved in the reduction, and the dashed lines are the rest of the edges in the simple cycle thrackle.

Definition 5. *β-Reduction:* The removal of two adjacent vertices and three lines adjacent to them from a five-consecutive-edge subgraph in an n-cycle thrackle (except the 6-cycle thrackle). In addition, the two "dangling" vertices are connected with a new line (Figure 5).

As an example, the β-reduction of an 8-cycle thrackle to the 6-cycle thrackle is shown in Figure 6.

[1] This reduction method cannot be applied to the 6-cycle thrackle even though a graph fragment with five consecutive lines can be found in the 6-cycle thrackle.

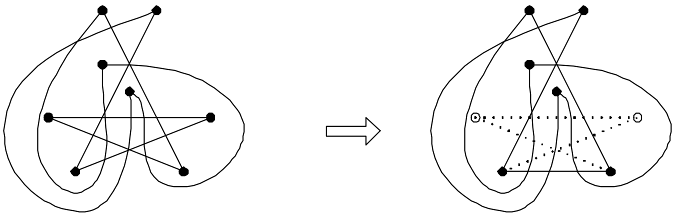

Fig. 6. The β-reduction of an 8-cycle thrackle to the 6-cycle thrackle

Lemma 6. A β-reduction of an n-cycle thrackle ($n \geq 5$ and $n \neq 6$) is an $(n-2)$-cycle thrackle.
Proof: The resulting thrackle has $n-2$ vertices and $n-3+1 = n-2$ lines. Since all thrackle properties still hold and every vertex has a degree of 2, it is an $(n-2)$-cycle thrackle. □

Note that for an n-cycle thrackle on $n \geq 5$ vertices, there are n ways to perform β-reduction. Also, β-reduction cannot be applied to the 6-cycle thrackle because the 4-cycle is not a thrackle. To reduce the 6-cycle thrackle, we need a different reduction method. Before we show it, let us characterize the 6-cycle thrackles first. Wehner introduced notions called zero-configuration, plus-configuration and minus-configuration [4], which are used to specify how the lines in a directed 4-path ($1 \to 2 \to 3 \to 4 \to 5$) are intersecting each other according to the order and the orientation with which the fourth line crosses the first and second line [4]. By using a computer program, he showed that the 6-cycle thrackles have no zero-configurations, but only plus-configurations [2]. This means that there exists only one type of the 6-cycle thrackle, i.e., all the 6-cycle thrackles are identical in terms of the order and the orientation with which the six lines intersect each other. Thus, the 6-cycle thrackles in Figure 6, Figure 1 and Figure 2 (D) can be considered to be equivalent, even though they look quite different.

Wehner demonstrated how to transform the 3-cycle thrackle into the 6-cycle thrackle [4]. Again, this can be used reversely to transform the 6-cycle thrackle back to the 3-cycle thrackle, as shown in Figure 7. This reduction proceeds as described below in Definition 6.

Definition 6. *γ-Reduction:* The operation of reduction of the 6-cycle thrackle to the 3-cycle thrackle, which involves the merge of three pairs of vertices and three pairs of lines. The operation proceeds as follows. First, pick a vertex in the 6-cycle thrackle as the starting vertex, and travel along the lines continuously in one direction to visit each of the other vertices exactly once. Eventually, we will return to the vertex where we started. While visiting, number each vertex in increasing order. Second, for each vertex, identify its *in*-line and *out*-line, where

[2] Wehner called it "++++++" [4]. Note that plus-configuration and minus-configuration are interchangeable, i.e., if we reverse the orientation of all lines in a plus-configuration, the result is a minus-configuration, and vice versa.

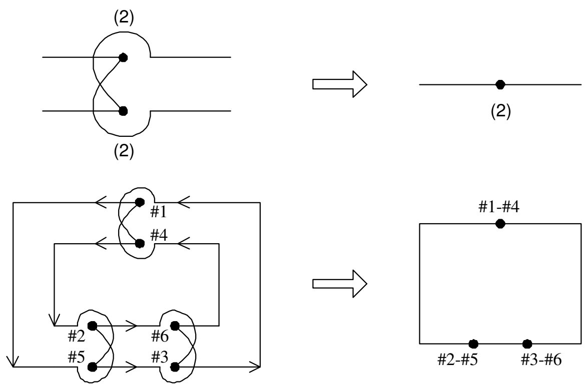

Fig. 7. The 6-cycle thrackle (2, 2, 2, 2, 2, 2) is γ-reduced to the 3-cycle thrackle (2, 2, 2)

the *in*-line is the line we use to "enter" the vertex and the *out*-line is the one we use to "leave" the vertex. Third, merge vertices #1 with #4, #2 with #5, and #3 with #6. In addition, merge the two *in*-lines and the two *out*-lines of each pair of vertices.

Lemma 7. The γ-reduction of the 6-cycle thrackle is the 3-cycle thrackle.
Proof: As previously mentioned, all 6-cycle thrackles are equivalent in terms of the order and the orientation with which the six lines intersect each other. Therefore, the reduction can be applied to all 6-cycle thrackles. Since three pairs of vertices and three pairs of lines are merged, it leaves the graph with three vertices and three lines, which constitute the 3-cycle thrackle. □

Next, we analyze the relationship between full thrackle incidence types, and show that all full thrackle incidence types form a poset.

2.3 Equivalence classes of full thrackles

Given a full thrackle on n vertices, there are only $2n$ degrees (Lemma 2) which have to be distributed over n vertices with $1 \leq deg(v_i) \leq n-1$ for each vertex v_i; this limits the possible incidence types. Among those incidence types, some can be full thrackle incidence types and some cannot. For example, for a full thrackle on 4 vertices, (1, 1, 3, 3) is a possible way to distribute the 8 degrees. However, by Lemma 4, this incidence type is not a thrackle.

We divide full thrackles into equivalence classes based solely on the number of vertices. To illustrate this concept, for each class of at most 7 vertices, we exhaustively enumerate the incidence types that satisfy the two constraints (the total number of degrees and the degree range for each vertex). These classes ($3 \leq n \leq 7$) are listed as follows (the field marked by † is the result of Lemma 4):

3-Vertex Class

Total number of degrees: 6
Degree range for each vertex: 1–2

	Incidence type	*Thrackle*	*Reduction*
1	(2, 2, 2)	Yes	N/A

4-Vertex Class

Total number of degrees: 8
Degree range for each vertex: 1–3

	Incidence type	*Thrackle*	*Reduction*
1	(1, 1, 3, 3)	No†	N/A
2	(1, 2, 2, 3)	Yes	$\alpha \to (2, 2, 2)$
3	(2, 2, 2, 2)	No	N/A

5-Vertex Class

Total number of degrees: 10
Degree range for each vertex: 1–4

	Incidence type	*Thrackle*	*Reduction*
1	(1, 1, 1, 3, 4)	No†	N/A
2	(1, 1, 2, 2, 4)	Yes	$\alpha \to (1, 2, 2, 3)$
3	(1, 1, 2, 3, 3)	Yes	$\alpha \to (1, 2, 2, 3)$
4	(1, 2, 2, 2, 3)	Yes	$\alpha \to (1, 2, 2, 3)$
5	(2, 2, 2, 2, 2)	Yes	$\beta \to (2, 2, 2)$

6-Vertex Class

Total number of degrees: 12
Degree range for each vertex: 1–5

	Incidence type	*Thrackle*	*Reduction*
1	(1, 1, 1, 1, 3, 5)	No†	N/A
2	(1, 1, 1, 1, 4, 4)	No†	N/A
3	(1, 1, 1, 2, 2, 5)	Yes	$\alpha \to (1, 1, 2, 2, 4)$
4	(1, 1, 1, 2, 3, 4)	Yes	$\alpha \to (1, 1, 2, 2, 4)$ $\alpha \to (1, 1, 2, 3, 3)$
5	(1, 1, 1, 3, 3, 3)	Yes	$\alpha \to (1, 1, 2, 3, 3)$
6	(1, 1, 2, 2, 2, 4)	Yes	$\alpha \to (1, 1, 2, 2, 4)$ $\alpha \to (1, 2, 2, 2, 3)$
7	(1, 1, 2, 2, 3, 3)	Yes	$\alpha \to (1, 1, 2, 3, 3)$ $\alpha \to (1, 2, 2, 2, 3)$
8	(1, 2, 2, 2, 2, 3)	Yes	$\alpha \to (1, 2, 2, 2, 3)$ $\alpha \to (2, 2, 2, 2, 2)$
9	(2, 2, 2, 2, 2, 2)	Yes	$\gamma \to (2, 2, 2)$

7-Vertex Class

Total number of degrees: 14
Degree range for each vertex: 1–6

	Incidence type	*Thrackle*	*Reduction*
1	(1, 1, 1, 1, 1, 3, 6)	No†	N/A
2	(1, 1, 1, 1, 1, 4, 5)	No†	N/A
3	(1, 1, 1, 1, 2, 2, 6)	Yes	$\alpha \to$ (1, 1, 1, 2, 2, 5)
4	(1, 1, 1, 1, 2, 3, 5)	Yes	$\alpha \to$ (1, 1, 1, 2, 2, 5) $\alpha \to$ (1, 1, 1, 2, 3, 4)
5	(1, 1, 1, 1, 2, 4, 4)	Yes	$\alpha \to$ (1, 1, 1, 2, 3, 4)
6	(1, 1, 1, 1, 3, 3, 4)	Yes	$\alpha \to$ (1, 1, 1, 2, 3, 4) $\alpha \to$ (1, 1, 1, 3, 3, 3)
7	(1, 1, 1, 2, 2, 2, 5)	Yes	$\alpha \to$ (1, 1, 1, 2, 2, 5) $\alpha \to$ (1, 1, 2, 2, 2, 4)
8	(1, 1, 1, 2, 2, 3, 4)	Yes	$\alpha \to$ (1, 1, 1, 2, 3, 4) $\alpha \to$ (1, 1, 2, 2, 2, 4) $\alpha \to$ (1, 1, 2, 2, 3, 3)
9	(1, 1, 1, 2, 3, 3, 3)	Yes	$\alpha \to$ (1, 1, 1, 3, 3, 3) $\alpha \to$ (1, 1, 2, 2, 3, 3)
10	(1, 1, 2, 2, 2, 2, 4)	Yes	$\alpha \to$ (1, 1, 2, 2, 2, 4) $\alpha \to$ (1, 2, 2, 2, 2, 3)
11	(1, 1, 2, 2, 2, 3, 3)	Yes	$\alpha \to$ (1, 1, 2, 2, 3, 3) $\alpha \to$ (1, 2, 2, 2, 2, 3)
12	(1, 2, 2, 2, 2, 2, 3)	Yes	$\alpha \to$ (1, 2, 2, 2, 2, 3) $\alpha \to$ (2, 2, 2, 2, 2, 2)
13	(2, 2, 2, 2, 2, 2, 2)	Yes	$\beta \to$ (2, 2, 2, 2, 2)

Sample full thrackles are also shown in Figure 8.

Observe that for each class listed (3, 4, 5, 6, 7-Vertex classes), if an incidence type can be the incidence type of a full thrackle, then it can be reduced to one or more thrackle incidence types in a lower class (i.e., a class with fewer vertices), except for the incidence type (2, 2, 2). Next, we will use induction to show that an incidence type of a full thrackle can be reduced to at least one full thrackle incidence type in a lower class.

Theorem 1. A full thrackle incidence type can be reduced to at least one full thrackle incidence type in a lower class, except for the incidence type (2, 2, 2).
Proof: Using induction on the number of vertices, we show that if an n-Vertex incidence type can be the incidence type of a full thrackle, then it can be reduced to at least one thrackle incidence type in a lower class.

Induction basis: The 3-Vertex class. This class only contains one full thrackle incidence type (2, 2, 2).

Induction hypothesis: Suppose that all full thrackle incidence types that have no more than $n-1$ vertices can be reduced (via α, β, or γ reduction) to one or more full thrackle incidence types in a lower class.

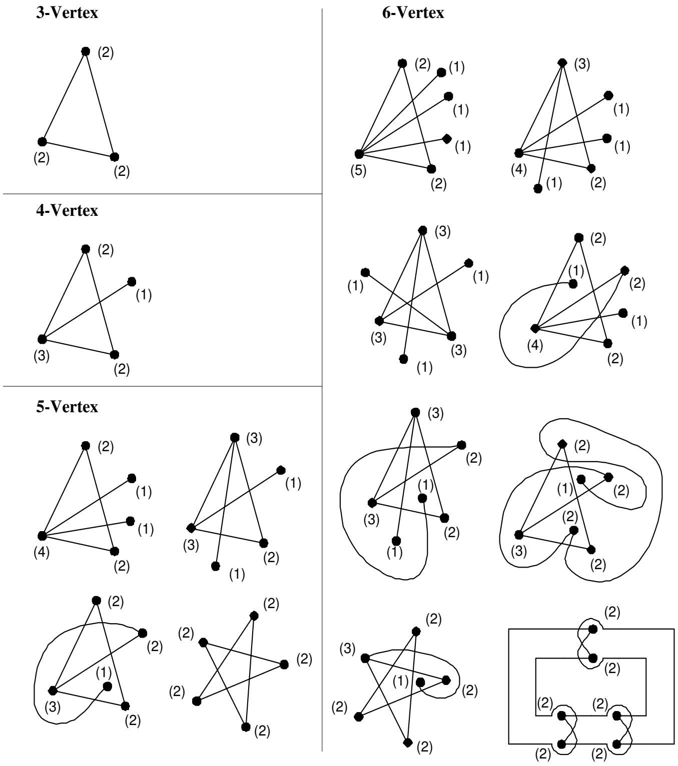

Fig. 8. Sample full thrackles and vertex degrees

Induction step: For an incidence type in the n-Vertex class ($n > 3$), if it is the incidence type of a full thrackle, then the thrackle is either an n-cycle thrackle or it has at least one vertex of degree 1 (Lemma 3).
Case 1: If it is an n-cycle thrackle, it can be β-reduced ($n \geq 5$ and $n \neq 6$) or γ-reduced ($n = 6$). When it can be β-reduced, the reduced thrackle has $n-2$ vertices (Lemma 6), and it is in the $(n-2)$-Vertex class. When it can be γ-reduced, the result is the 3-cycle thrackle (2, 2, 2) (Lemma 7), which is in the 3-Vertex class.
Case 2: If it is not an n-cycle thrackle, then it can be α-reduced. The result is that of a full thrackle on $n-1$ vertices (Lemma 5), which belongs to the $(n-1)$-Vertex class.
Thus, for any possible incidence type in the n-Vertex class, if it is the incidence type of a full thrackle, it can be reduced to another full thrackle incidence type in a lower class. □

By Theorem 1, we can see that all full thrackle incidence types form a poset, with the incidence type (2, 2, 2) as the minimal element (Figure 9).

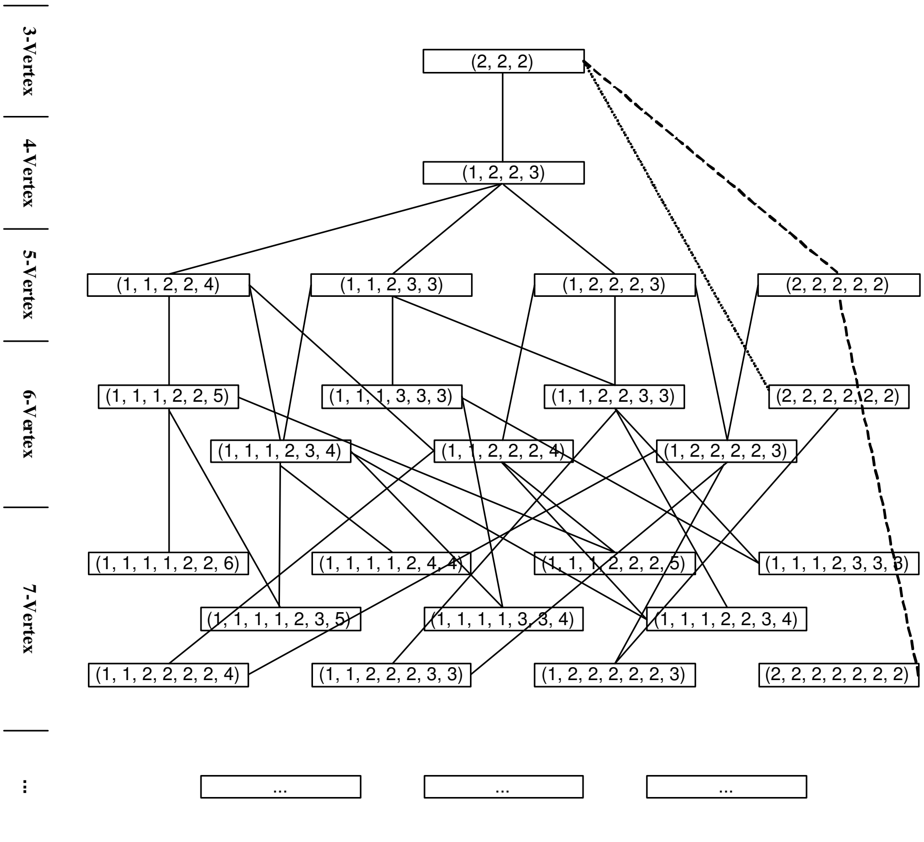

Fig. 9. Fragment of the poset of incidence types of full thrackles. Solid lines indicate α-reductions, dashed lines show β-reductions, and dotted lines show γ-reductions.

3 Extensibility of Full Thrackles

As we have seen, reduction operations induce a partial order on full thrackle incidence types. Namely, for full thrackle incidence types x and y, we set $x \preceq y$ if $x = y$ or if y can be reduced to x by a sequence of reductions. Incidence type (2, 2, 2) is the minimal element of this poset. Moreover, reductions may also be used as a tool to show that a full thrackle is not extensible if its reduced thrackle is not extensible, which will be studied in this section. The non-extensibility of a subset of full thrackles called the *1-2-3 set* is not proved using reductions. Hence, we reduce Conway's thrackle conjecture to the problem of proving the non-extensibility of the 1-2-3 set.

Definition 7. *Extensible Thrackle:* A thrackle T is called extensible if there exists a new line between a pair of vertices of T such that the resulting geometric graph is still a thrackle.

Lemma 8. The full thrackle (2, 2, 2) (the 3-cycle thrackle) is not extensible.
Proof: Since there are only three vertices, the degree range for a vertex is 1–2. Any addition of a new line will result in a vertex v of $deg(v) > 2$. Thus, the full thrackle (2, 2, 2) is not extensible. □

Next, we introduce a subset of full thrackles called the *1-2-3 set*. The significance of this set is that, unlike other full thrackles, a 1-2-3 full thrackle has a body and a tail, which will also be defined.

Definition 8. *1-2-3 Set:* The set of full thrackles where the incidence type of each thrackle is of the form $(1, \underbrace{2, ..., 2}_{n-2}, 3)$ $(n \geq 4)$.

Now we characterize the structure of a full thrackle in the 1-2-3 set. We start with the simplest, which is the full thrackle (1, 2, 2, 3) as shown in Figure 4 on the left. As we can see, this full thrackle can be decomposed into two parts: a 3-cycle thrackle, and a line between the vertices of degree 1 and 3. We call this 3-cycle thrackle the *body* and the line the *tail* of this 1-2-3 thrackle. If the tail contains just one vertex, it is said to have length 1. Next, we use induction to show that every full thrackle in the 1-2-3 set has a body and a tail.

Lemma 9. A full thrackle in the 1-2-3 set has a body and a tail, where the body is a simple cycle thrackle and the tail is a path between the vertices of degree 1 and 3.
Proof: By induction on the number of vertices.

Induction basis: The simplest 1-2-3 thrackle (1, 2, 2, 3) has a body and a tail (attached to the vertex of degree 3).
Induction hypothesis: Suppose each 1-2-3 thrackle on $n-1$ vertices has a body and a tail, which is a path between the vertices of degree 1 and 3.
Induction step: By Definition 8, the incidence type of a thrackle T on n vertices $(n \geq 5)$ from the 1-2-3 set is $(1, \underbrace{2, ..., 2}_{n-2}, 3)$. The vertex of degree 1 must be connected to either the vertex of degree 3 (Case 1) or one of the vertices of degree 2 (Case 2).
In Case 1, apply α-reduction to T; the incidence type of the resulting thrackle is $(\underbrace{2, ..., 2}_{n-1})$, which is an $(n-1)$-cycle thrackle. Thus, the body of T is an $(n-1)$-cycle thrackle and the tail of T has a length of 1. If the incidence type of T is (1, 2, 2, 2, 3) (when $n = 5$), it is not hard to see that the vertex of degree 1 must be connected to one of the vertices of degree 2 (since otherwise, T will contain a 4-cycle), which forces the α-reduction of T to fall into Case 2 instead of Case 1.
In Case 2, again, we can apply α-reduction to T; the incidence type of the resulting thrackle is $(1, \underbrace{2, ..., 2}_{n-3}, 3)$, which is a 1-2-3 thrackle T' on $n-1$ vertices. By the induction hypothesis, T' has a body and a tail which is a

path between the vertices of degree 1 and 3. The line removed from T by α-reduction was therefore attached to the only vertex of degree 1 in T', which is the tail of T'. Thus, the body of T is the same as that of T', and the tail of T is of length 1 more than that of T'. This completes the proof. □

For example, in Figure 8, there are two 1-2-3 thrackles in the 6-Vertex class (both of incidence type $(1, 2, 2, 2, 2, 3)$). One has a 3-cycle thrackle as its body and a tail of length 3, and the other has a 5-cycle thrackle as its body and a tail of length 1. Moreover, there should be no difficulty in seeing that the tail of a 1-2-3 thrackle is attached to the body at the vertex of degree 3.

Unlike other full thrackles, the non-extensibility of the 1-2-3 thrackles is not proved using reductions (as in Theorem 2 below). Next, we will show a special case which is related to the 1-2-3 thrackles (as in Lemma 10), since it will be used in Theorem 2 where reductions are used to prove the non-extensibility of full thrackles.

Lemma 10. Let T be a full thrackle that has exactly two vertices of degree 1. If adding a line l between these two vertices creates a thrackle T_l, then T_l contains a 1-2-3 thrackle as a subgraph.

Proof: Let us denote the two vertices of degree 1 in T by $\mathbf{p}$ and $\mathbf{q}$, and the two lines incident to them by p and q, respectively. Now we add line l between $\mathbf{p}$ and $\mathbf{q}$ to get T_l. By Lemma 2, the incidence type of T is either $(1, 1, \underbrace{2, ..., 2}_{n-3}, 4)$ (Case 1) or $(1, 1, \underbrace{2, ..., 2}_{n-4}, 3, 3)$ (Case 2) $(n \geq 5)$.

Case 1: If it is $(1, 1, \underbrace{2, ..., 2}_{n-3}, 4)$, w.l.o.g., $\mathbf{q}$ is connected to the vertex of degree 4 or a vertex of degree 2.

Case 1a: When $\mathbf{q}$ is connected to the vertex of degree 4, since T_l is a thrackle, then its incidence type is $(\underbrace{2, ..., 2}_{n-1}, 4)$. By Lemma 1, we can remove q from T_l and the result is still a thrackle, which we denote by T'. T' has incidence type $(1, \underbrace{2, ..., 2}_{n-2}, 3)$, which means T' is a 1-2-3 thrackle (by Definition 8). Therefore, T_l can also be produced by adding line q to the 1-2-3 thrackle T'.

Case 1b: When $\mathbf{q}$ is connected to a vertex of degree 2, since T_l is a thrackle, then its incidence type would again be $(\underbrace{2, ..., 2}_{n-1}, 4)$. By the thrackle properties, we know there must exist a path from $\mathbf{q}$ to the vertex of degree 4. In this path (denote it by w), all vertices are of degree 2. So for T_l, we can remove q (by Lemma 1), and repeatedly remove one line and a vertex at the end of the path w (by Lemma 5) until we reach the vertex of degree 4. Let's again call the result T'. Certainly, the incidence type of T' would be $(1, \underbrace{2, ..., 2}_{n-2-k}, 3)$, where k is the number of vertices of degree 2 in w. Therefore, by Definition 8, T' is a 1-2-3 thrackle, and adding a path w to the 1-2-3 thrackle T' can also form T_l.

Case 2: If it is $(1, 1, \underbrace{2, ..., 2}_{n-4}, 3, 3)$, then using similar techniques, we can show that when $\mathbf{q}$ is connected to a vertex of degree 3, or a vertex of degree 2, T_l will contain a 1-2-3 thrackle as a subgraph.

This completes the proof of Lemma 10. □

We do not see at present how to use reductions to prove the non-extensibility of the 1-2-3 thrackles. We are left with the following conjecture:

Conjecture 1. 1-2-3 non-extensibility conjecture: A 1-2-3 thrackle is not extensible.

Let T be a full thrackle with exactly two vertices of degree 1. In Lemma 10, we have shown that if adding a line l between these two vertices creates a thrackle T_l, then T_l contains a 1-2-3 thrackle as a subgraph, i.e., T_l can also be produced by adding a line to a 1-2-3 thrackle. Therefore, the non-extensibility problem of full thrackles that have exactly two vertices of degree 1 can be transformed into the problem of proving the non-extensibility of the 1-2-3 thrackles, which is covered by Conjecture 1.

Next, we show how to use reductions to prove the non-extensibility of full thrackles.

Theorem 2. Let T be a full thrackle not of incidence type (2, 2, 2). If the 1-2-3 non-extensibility conjecture is correct, then T can be reduced to a full thrackle T' with fewer vertices such that if T' is not extensible, then T is also not extensible.

Proof: By Theorem 1, T can be reduced to a full thrackle T' with fewer vertices. By contradiction, assume T can be extended to a thrackle Y by adding a new line l. Assuming that the reduction of T to T' does not delete the end-vertices of l, we can transform Y to Y' by applying to the sub-thrackle T in Y the same reduction that was used to reduce T to T'. Compare Y' to T'; the only difference between these two thrackles is line l, which indicates that T' can be extended to Y', contradicting the assumption that T' is not extensible.

Note that the above proof works only when the two vertices incident to line l both still exist in Y' after the reduction operation is applied to Y. What if one or both of the vertices incident to line l are removed during the reduction? We address this with the following three cases:

Case 1: The reduction applied to the full thrackle T is a γ-reduction.

In this case, we know the full thrackle T is the 6-cycle thrackle. Use the 6-cycle thrackle and the numbering scheme shown in Figure 7. Note that there are three pairs of vertices, #1 with #4, #2 with #5, and #3 with

#6. If line l is added between two vertices that are not in the same pair, for instance, between #1 and #3 or between #1 and #5, the above proof by contradiction is still valid. If line l is added between two vertices that are in the same group, for example, between #1 and #4, then vertices #1, #2, #3 and #4 will form a 4-cycle. Since a thrackle cannot contain 4-cycles, Y is not a thrackle, which is a contradiction. Similarly, line l cannot be added between vertices #2 and #5, nor #3 and #6. This completes Case 1.

Case 2: The reduction applied to the full thrackle T is a β-reduction.

In this case, we know the full thrackle T is a simple cycle thrackle on five or more vertices. Because β-reduction removes two adjacent vertices from a simple cycle thrackle, we can pick for the reduction from T to T' two adjacent vertices neither of which are incident to line l. Since T has at least five vertices, such two adjacent vertices are always available. Therefore, the above proof by contradiction can still be applied.

Case 3: The reduction applied to the full thrackle T is an α-reduction.

There are three subcases:

Case 3a: For those full thrackles that have ≥ 3 vertices of degree 1, we choose the vertex that is not incident to line l to apply α-reduction. Then, the above proof by contradiction is still valid.

Case 3b: For those full thrackles that have exactly two vertices of degree 1, if line l is connected to only one of the two vertices, we can use the vertex that is not incident to line l to come to a contradiction. If line l is connected to both vertices of degree 1, then the statement of our theorem reduces to Lemma 10, which was proven earlier. If the 1-2-3 non-extensibility conjecture is true, the result cannot be a thrackle.

Case 3c: By Lemma 2, the incidence type of a full thrackle with only one vertex of degree 1 must be in the form of $(1, 2, ..., 2, 3)$. That is, if it is a full thrackle on n vertices, its incidence type is $(1, \underbrace{2, ..., 2}_{n-2}, 3)$, where $n \geq 4$. All these full thrackles form precisely the 1-2-3 set. If the 1-2-3 non-extensibility conjecture is true, then, this completes Case 3c.

This completes the proof of Theorem 2. □

Corollary 1. Suppose the 1-2-3 non-extensibility conjecture is true. Then if a full thrackle T_1 can be reduced to T_2, T_2 can be reduced to T_3, ..., T_{n-1} can be reduced to T_n, and T_n can be reduced to the full thrackle (2, 2, 2), then T_i is not extensible for all $1 \leq i \leq n$.

Proof: Theorem 2 and Lemma 8 show that T_n is not extensible. By Theorem 2, T_{n-1} is not extensible. Similarly, using Theorem 2 recursively, we conclude T_i is not extensible for all $1 \leq i \leq n$. □

As shown by the proof of Theorem 2, reductions can be applied to prove the non-extensibility of full thrackles except for the 1-2-3 thrackles. Is there any relationship between the 1-2-3 thrackles and Wehner's hypothesis on the structure of potential counterexamples to Conway's conjecture? Let's further inspect the 1-2-3 thrackles. By Lemma 9, a full thrackle T in the 1-2-3 set has a body and a tail. Denote the vertex of degree 1 in T by $\mathbf{v}$ (the tip of its tail), and that of degree 3 by $\mathbf{u}$. If T is extensible, then there exists a new line l that can be added between two vertices of T without loss of the thrackle properties. If l is added between two vertices neither of which is $\mathbf{v}$, then a reduction argument leads to a contradiction (as shown in the proof of Theorem 2 above, we can use α-reduction to remove $\mathbf{v}$). Difficulty arises when vertex $\mathbf{v}$ is involved, which leads to the following three cases:

Case 1: Line l is between $\mathbf{v}$ and $\mathbf{u}$.
Case 2: Line l is between $\mathbf{v}$ and one of the vertices in the body (except $\mathbf{u}$).
Case 3: Line l is between $\mathbf{v}$ and one of the vertices on the tail (except $\mathbf{u}$).

In any of the above cases, the result contains two simple cycles. In Case 1, two simple cycles share one vertex in common, which is $\mathbf{u}$. In Case 2, they share a path in common, which is the path from $\mathbf{u}$ to the vertex in the body that line l is connected to. In Case 3, two simple cycles are connected by a single path, which is the path from $\mathbf{u}$ to the vertex on the tail that line l is connected to. Thus, we have just proved Wehner's hypothesis of what a minimal counterexample to Conway's thrackle conjecture may look like [4]. If the 1-2-3 non-extensibility conjecture is true, then none of these three cases exist.

Corollary 1 tells us that if the 1-2-3 non-extensibility conjecture is true, then in the poset of the full thrackle incidence types, every incidence type is not extensible, because, by Theorem 1, from any incidence type there is a reduction sequence which leads to the minimal incidence type (2, 2, 2). Thus, we can state the following theorem:

Theorem 3. If the 1-2-3 non-extensibility conjecture is true, then any full thrackle is not extensible and Conway's conjecture is true.

4 Summary and Future Work

We define α, β and γ-reductions for full thrackles and show that these reductions enable us to rule out the extensibility of many infinite series of incidence types of full thrackles. Theorem 3 tells us that in order to completely solve Conway's thrackle conjecture, we only need to show that the 1-2-3 thrackles are not extensible. Moreover, since we can use Theorem 2 to reduce an n-cycle thrackle to the 3-cycle thrackle, it may be possible to further reduce the 1-2-3 non-extensibility conjecture to the non-extensibility problem of the *kites*, where a kite stands for a 1-2-3 thrackle that has the 3-cycle as the body and a tail of arbitrary length. This may be a promising direction to completely prove Conway's thrackle conjecture.

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