Polyhedral Partitions and Stresses

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Abstract

The main focus of this thesis is on the geometry of polyhedral partitions of Euclidean spaces and other orientable manifolds and its connections to rigidity theory, convexity, splines, and stochastic geometry. In the first part of the work we introduce a general notion of stress on cell-complexes and explore connections between stresses and liftings (generalization of C_1^0-splines) for d-dimensional manifolds realized in $\mathbb{R}^d$. For example, we offer new sufficient conditions for the existence of a sharp lifting for a "flat" piecewise-linear realization of a manifold. As an application, two algorithms are given that determine whether a piecewise-linear realization of a d-manifold in $\mathbb{R}^d$ admits a lifting to $\mathbb{R}^{d+1}$ which satisfies given constraints. In the thesis we discuss two generalizations of the famed Maxwell correspondence between stresses on planar frameworks and projections of spatial polyhedra. The former says that the spaces of liftings and d-stresses of a manifold are isomorphic under rather general conditions on the topology of the manifold. The latter is a partial analog of the Maxwell correspondence for spatial frameworks. We also demonstrate connections between stresses and Dirichlet-Voronoi diagrams. In the probabilistic part of the thesis we investigate geometric bootstrap percolation models suggested by Connelly for describing the rigidity / flexibility properties of molecular systems. In these models local rules are of geometric nature as opposed to simple counts used in standard bootstrap percolation models. Both models deal with a relaxation of tension in a 2-dimensional medium. We also discuss possible applications of these results to mathematical chemistry. As a consequence of Maxwell correspondence these models can also be interpreted in terms of the geometry of convex surfaces and polygonal tilings. We find the exact value of the critical probability for both models. In fact, we obtain somewhat stronger results showing that in both cases the relaxation of tension occures in finite time almost surely.

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This thesis has been prepared in the manuscript format, and, overall, follows the general guidelines of the School of Graduate Studies and Research. All of the results in Chapter 2 were obtained by the candidate. Chapter 3 contains results obtained by Robert Erdahl, Sergei Ryshkov, and K. R. The candidate played a leading role in these investigations. Chapter 4 is the product of joint efforts of Mikhail Menshikov, Stanislav Volkov and the candidate. The results were obtained and written up during their visit to the Fields Institute. The results of Chapter 5 were produced by Robert Connelly, Stanislav Volkov and the candidate.

All results of the thesis are original.

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Chapter 1

Introduction and General Discussion

This thesis is devoted to some problems of discrete, computational and stochastic geometry dealing with polyhedral partitions of manifolds, especially Euclidean spaces. First, we obtain results characterizing geometrical, combinatorial, and complexity properties of piecewise-linear partitions of manifolds having certain regularity properties, for example, the existence of a convex piecewise-linear surface that projects on the partition. Second, we link geometric properties of polyhedral partitions and arrangements to rigidity theory, obtaining some results on geometric statics of planar and spatial graphs. Third, we analyze two tension percolation problems motivated by applications of rigidity theory to molecular chemistry, and find the critical probability threshold in both cases.

Chapter 2 is devoted to the geometry of polyhedral partitions of homology manifolds, mainly $\mathbb{R}^d$. In this chapter the emphasis is on connections between d-stresses and the geometric and combinatorial properties of the partitions. Chapter 3 deals with the theory of higher-dimensional stresses on cell-complexes and especially with generalizations of Maxwell theory of stresses, reciprocals and liftings. Chapter 4 is devoted to a continuous tension percolation model motivated by Maxwell theory of stresses and mathematical models of molecular structures based on large random networks of Hooke springs. Chapter 5 is devoted to a lattice tension percolation model which is a discrete alternative to the model analyzed in Chapter 4. *Each chapter can be read independently of the others.*

In the 1860-70's J. C. Maxwell, using geometrical methods, studied the distribution of forces on a planar framework that is in static equilibrium. He considered frameworks that are vertical projections of 1-dimensional skeletons of polyhedral spheres with no vertical facets. Maxwell discovered that if the polyhedral surface is a convex polytope, then the frameworks admits an equilibrium stress (understood as in physics, but see Section 2.3 for a formal definition) that is tension on all interior edges and is compression on all boundary edges. In fact, Maxwell formulated the converse theorem too [53, 54]. Some of Maxwell's results were also discovered by Rankine (1864) and Cremona (1872). Crapo and Whiteley put Maxwell's observations on a rigorous basis. They proved that there is a natural isomorphism between the linear spaces of liftings (lifting is inverse to projecting; see Section 2.5 for a formal definition) and the linear space stresses for a piecewise-linear realization of a simply connected 2-manifold in the plane. This isomorphism is called *Maxwell correspondence.* The Maxwell papers gave the first characterizations of the partitions that admit convex liftings and, more generally, *sharp* liftings, i.e., liftings with no flat dihedral angles.

A Dirichlet-Voronoi domain of a point lattice in an affine space $\mathbb{R}^d$ is the set of points of $\mathbb{R}^d$ no further from some point point of the lattice than from the other points of the lattice. In early 1900's G. F. Voronoi proved that any simple polytope (where each vertex is incident to exactly d facets) that tiles $\mathbb{R}^d$ face-to-face is the affine image of the Dirichlet-Voronoi domain of some lattice [82]. In general, a polytope that tiles the space face-to-face by its translates is called a *parallelohedron* (or *space-filler*). Central to Voronoi's proof is the construction of a *generatrise* for the tiling. The *generatrise* (French) is an infinite convex polyhedral surface in $\mathbb{R}^{d+1}$ that projects onto the tiling. In this century the problem of the existence of a convex lifting for both general tilings and lattice tilings was studied by many. For lattice tilings it is a part of a big open problem—the Voronoi conjecture on parallelohedra. The Voronoi conjecture says that any parallelohedron is an affine image of the Dirichlet-Voronoi domain of some lattice. The progress in this problem was slow. Delaunay [33] proved the Voronoi conjecture for $d = 4$ (the case of $d = 2, 3$ is due to the famous crystallographer Feudorov). The same year, using the result of Delaunay, Zhitomirski proved the Voronoi conjecture for parallelohedra whose lattice tilings have exactly three tiles at the star of each $(d-2)$-dimensional face. Delaunay considered the Voronoi conjecture as one of the most interesting conjectures in the geometry of numbers [32, 33]. McMullen proved that any space-filling zonotope (the vector sum of line segments) is combinatorially equivalent to a Dirichlet-Voronoi domain. Erdahl [38] improved upon this by showing that in fact, in the case of zonotopes, there is an affine equivalence. It is worth to mention that all 2- and 3-dimensional parallelohedra are zonotopes, but there is a non-zonotopal parallelohedron in dimension 4 [33]. The general case of the Voronoi conjecture remains open.

Although Voronoi was interested in liftings possessing some special symmetries, the most difficult part of his theorem was the construction of the generatrise, and a part of his proof dealing with the *existence* of a convex lifting does not essentially depend on the lattice structure of the tiling (see Delaunay (1947) and Ryshkov, Rybnikov (1997) for detailed analysis of this work). It is clear that Maxwell and Voronoi were interested in the same property of a polyhedral partition, namely the existence of a convex lifting. Such partitions are called *regular*, and there are good reasons for this. In Chapter 2, using our method of *quality transfer* and the d-dimensional analog of the Maxwell-Crapo-Whiteley theory, we prove that a partition of $\mathbb{R}^d$ is the projection of a strictly convex surface if and only if any one of the following conditions holds:

1) it admits a strictly positive d-stress,

2) it is an additively weighted Dirichlet-Voronoi partition,

3) it is an additively weighted Delaunay partition,

4) it is the section of a $(d+1)$-dimensional Dirichlet-Voronoi partition,

5) it has a dual (projectively) partition,

6) it has a convex reciprocal.

Some of these remarkable equivalences appeared prior to the author's work in papers and preprints of Aurenhammer, Ash, Bolker, Crapo, Paschinger and Whiteley.

It turns out that Maxwell's theorem has analogs for the case of $\mathbb{R}^d$ which were first discovered for $d = 3$ by Crapo and Whiteley [85]. In the case of $\mathbb{R}^d$ the role of self-stresses is played by so-called d-stresses. Let $\mathbf{n}(F, C)$ denote the inner unit normal to cell C at its facet F, and $vol_k(C)$ denotes the Euclidean k-volume of a cell C. The star of a cell in a cell-complex is the union of all cells whose closure contains this cell.

Definition 1.0.1 *A real-valued function $s(\cdot)$ on the $(k-1)$-cells of a polyhedral cell-complex K in $\mathbb{R}^d$ is called a k-stress if at each internal $(k-2)$-cell F of K*

$$\sum_{\{C \mid F \subset C\}} s(C) vol_{k-1}(C)\, \mathbf{n}(F, C) = \mathbf{0},$$

where the sum is taken over all $(k-1)$-cells in the star of F. The quantities $s(C)$ are the coefficients of the k-stresses, a tension if the sign is strictly positive and a compression if the sign is strictly negative.

Thus, the notion of 2-stress on a framework is the same as the notion of self-stress adopted in civil engineering. There $s(e)$ is force per unit length, and the static force applied at the end points of edge e is $s(e)\|e\|$. For a $(k-1)$-cell C a k-stress μ is force per unit relative $(k-1)$-volume (area) of C, and the static force applied at a $(k-2)$-face of C is $\mu\, vol_{k-1}C$. The notion of k-stress was first introduced by Lee (1996) for simplicial complexes only and in a different manner. Lee based his definition on the notion of the Stanley-Reisner ring of a simplicial complex. Note, that the Stanley-Reisner ring is not defined for non-simplicial cell-complexes. Lee used k-stresses to geometrically explain the g-theorem conjectured by McMullen and proved by Stanley (see Lee (1996) and McMullen (1996) for the discussion of the g-theorem). Lee's results were also used by McMullen (1996) to give a geometric proof of the g-theorem. Our definition (first suggested in Rybnikov (1999)) seems to the first general definition of k-stress for cell-complexes, although Tay, White, and Whiteley (1995) mentioned in their paper the possibility of defining generalized stresses for cell-complexes.

In Section 2.5 we show that for a homology d-manifold Δ realized in $\mathbb{R}^d$ with $H_1(\Delta, \mathbb{Z}_2) = 0$ (the homologies are computed not for the embedding, but for the manifold), the space of d-stresses and the space of liftings are isomorphic under the Maxwell correspondence. This result improves upon earlier results by Crapo and Whiteley. In fact, using the idea employed in a construction given in Section 2.5 (see Figure 2.3), one can show that for a homology d-manifold Δ the space of d-stresses and the space of liftings are isomorphic under Maxwell correspondence for any $\mathbb{R}^d$-realizable cell-decomposition of Δ if and only if $H_1(\Delta, \mathbb{Z}_2) = 0$ (watch the order of quantifiers here!).

The notion of reciprocal is intimately related to the notion of d-stress. Let $\mathcal{M}^d$ be a homology manifold, and let M^d be its piecewise-linear realization in $\mathbb{R}^d$. Consider a graph where the vertices correspond to the d-cells of $\mathcal{M}^d$, and the edges correspond to the $(d-1)$-cells of $\mathcal{M}^d$. Let two vertices share an edge in this complex if and only if the corresponding d-cells are adjacent. Such a graph (which is always locally-finite, although it can be infinite) is called the combinatorial dual graph of $\mathcal{M}^d$. A reciprocal R is a rectangular realization in $\mathbb{R}^d$ of the combinatorial dual graph such that the edges of R are orthogonal to the corresponding $(d-1)$-cells. The notion of reciprocal was introduced by Maxwell in his studies of geometric statics. Consider now the planar case. Instead of a reciprocal, one can attempt to redraw the 1-skeleton $Sk^1(M^2)$ so that the edges of the redrawn graph are parallel to the corresponding edges of $Sk^1(M^2)$. It was the idea of Italian mathematician L. Cremona (1870) to draw parallel diagrams for understanding stresses in frameworks. Cremona [27] showed that parallel drawings are equivalent to stresses for 1-skeletons of polyhedral spheres.

The question of the existence of a reciprocal for a planar framework and methods for finding reciprocals drew the attention of engineers at the second half of the last century. Many engineering computations for the Eiffel Tower were carried out by Koechlin, an assistant of Eiffel, by drawing reciprocal diagrams manually (see [85]). This problem of finding methods for drawing orthogonal reciprocals was stated in the well-known book "Higher geometry" by F. Klein [49]. There, F. Klein described a mechanical tool used by engineers for drawing reciprocals on the plane. The above mentioned connections between reciprocals, liftings and stresses were first found by J.C.Maxwell [53] in the 1860's. However, he did not give complete and rigorous proofs. In a rigorous form, these connections were formulated and proved by Crapo and Whiteley in the 1980's [83].

Theorem 1.0.2 *(Maxwell, Cremona, Crapo, Whiteley) For a piecewise-linear realization in $\mathbb{R}^2$ of a simply connected 2-dimensional manifold the spaces of self-stresses, reciprocals, parallel drawings and liftings are all isomorphic.*

It turns out that in the case of general dimension there are similar strong connections between reciprocals, parallel drawings, liftings, multivariate splines and d-stresses. This thesis contains a few new theorems on such connections. The results of Sections 2.4-2.6 can be summarized in the following theorem.

Theorem 1.0.3 *Let $M^d \subset \mathbb{R}^E$ be a PL-realization of a homology manifold $\mathcal{M}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ (the manifold need not be compact). The spaces of d-stresses, reciprocals, and liftings are all isomorphic under Maxwell correspondence.*

It is natural to ask what tilings of $\mathbb{R}^d$ can be obtained as projections of piecewise-linear surfaces, not necessarily convex. In a way, this is the weakest degree of regularity of a tiling. A more natural question would be to inquire about the dimension of the linear space of liftings. A lifting is obviously a generalization of the notion of C_1^0-spline to piecewise-linear realizations that are not embeddings. Denote by S_1^0 the linear space of C_1^0-splines, and by $Lift$ the linear space of liftings considered up to addition of a hyperplane. Since we consider liftings up to addition of a hyperplane, $dim\, Lift + d + 1 = dim\, S_1^0$. For example, in the case where there is no non-trivial lifting $dim\, Lift = 0$, but $dim\, S_1^0 = d + 1$.

Let Δ be an embedding in $\mathbb{R}^d$of a d-manifold with boundary. Consider a linear space whose elements are piecewise-polynomial functions on Δ that are represented by polynomials of degree at most m on each d-simplex and are smooth of order at least r on all of Δ. This space is called the space of C_m^r splines over Δ. Denote it by $S_m^r(\Delta)$.

If Δ is a triangulation, then the dimension of $S_m^0(\Delta)$ is, according to Billera (1989),

$$\sum_{i=0}^{d} f_i(\Delta) \binom{m-1}{i}$$

where $f_i(\Delta)$ is the number of i-simplexes in the complex Δ. When $m = 1$ the above formula gives $f_0(\Delta)$. This can be also seen using the basis of the Courant functions on Δ. Courant was the first to propose the use of C_1^0-splines in numerical methods involving approximations of multivariate functions. The Courant function c_i corresponding to the vertex v_i is defined by the Kronecker symbol δ_i^j, where v_j

ranges over all vertices of the triangulation Δ. Obviously, the Courant functions are linearly independent and form a basis for the space of S_1^0-splines.

When Δ is not a triangulation or $r > 0$, the dimension of $S_m^r(\Delta)$ depends not only on the f-vector of Δ, but also on the combinatorics and the geometry of cell-partition Δ. In all dimensions there are cell-partitions over which there are only trivial C_m^0 splines—affine functions. For the planar case an example is as follows. The crown of a tile is defined as the set of all tiles sharing at least one vertex with this tile. Consider a tiling of the plane by a centrally-symmetric hexagon, and slightly perturb a vertex a of this partition. This involves only 3 hexagons. After this, one cannot lift the crown a hexagon which has a as a vertex (see [28] or [66, 67] for a proof), and, therefore, the perturbed partition has only trivial liftings. Now, let $d > 2$. Let St be the star of a $(d-3)$-cell of a cell-partition of $\mathbb{R}^d$ whose orthogonal section is congruent to the regular octahedron triangulated by putting a vertex at the center. Perturb the vertex of this star. It is easy to see that for almost all perturbations a non-trivial lifting of the resulting configuration is no longer possible. If, in addition, the stars of all other vertices of the partition have exactly $d+1$ tiles coming together, the whole partition has only trivial liftings (see Section 2.9). The non-existence of non-trivial C_1^0-splines clearly implies the non-existence of non-trivial C_m^0-splines.

As was mentioned above, Voronoi [82] proved that a lattice tiling T by translated copies of a simple (0-primitive in Voronoi terminology) convex polytope can be lifted to a convex surface and $dim\, S_1^0(T) = 1$. The condition of simplicity is essential for his proof. Voronoi's proof works for non-lattice tilings too (see [67]). In general, Delaunay calls a manifold k-primitive if the star of each internal k-cell has $(d-k+1)$ d-cells (this is the minimum possible number for triangulations) and each d-cell has at least one internal k-face. In Chapter 2 we prove the following theorem developing the ideas of Voronoi and Delaunay.

Theorem 1.0.4 *Let M^d be a PL-realization of a closed $(d-2)$-primitive d-dimensional manifold in $\mathbb{R}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ $(d > 2)$. A sharp lifting exists if and only if there exists an all non-zero $(d-1)$-stress for the star of each $(d-3)$-cell. If a sharp lifting exists, then $dim\, Lift(M^d) = 1$ and the lifting is unique up to the choice of a dihedral angle.*

Note that $(d-2)$-primitive decompositions have the following interesting property discussed in Chapter 2.

Lemma 1.0.5 *Any sharp lifting of a closed $(d-2)$-primitive manifold is either locally convex or concave.*

Theorem 1.0.6 *Let M^d be a PL-realization of a closed $(d-3)$-primitive d-dimensional manifold in $\mathbb{R}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ $(d > 2)$. Then a sharp lifting exists and $dim\, S_1^0(M^d) = dim\, Lift(M^d) + d + 1 = d + 2$.*

In fact in both theorems and the lemma the conditions on M^d can be weakened and the requirement of closedness can be removed (see Chapter 2). The above theorems are improvements on results by Davis [28], Aurenhammer [6], Crapo and Whiteley [83].

An important observation due to Wang is that a spline $f \in S_m^r(\Delta)$ is represented by polynomials p_1 and p_2 on adjacent d-cells if and only if $p_1 - p_2 \in \mathcal{I}(a^{r+1})$, the ideal (over $\mathbb{R}$) generated by the $r+1$ power of an affine function vanishing on the common facet of these d-cells.

Definition 1.0.7 *A real-valued function $c_r^{r-1}(\cdot)$ on the $(d-1)$-cells of Δ is called a C_r^{r-1}-cofactor if for each internal $(d-2)$-cell F of Δ, and for each non-negative integer vector $(i_1, \ldots, i_n)$ such that $i_1 + \ldots + i_n = r$*

$$\sum_{F \subset \partial C} c_r^{r-1}(C)\, \mathbf{n}_1^{i_1}(C) \cdots \mathbf{n}_d^{i_d}(C) = 0$$

where C ranges over all $(d-1)$-faces making contact at F and $\mathbf{n}_1(C), \dots, \mathbf{n}_d(C)$ are the coordinates of the unit normal to the hyperplane spanned by C with orientation induced by the orientation of the link of F.

Comparing the definition of a d-stress and of a C_r^{r-1}-cofactor, one sees that C_r^{r-1}-cofactors of multivariate splines are natural generalizations of d-stresses for polyhedral partitions of a manifold Δ embedded into $\mathbb{R}^d$(but not for other realizations). In Section 2.5 we show that if $H_1(\Delta, \mathbb{Z}_2) = 0$, then $dim\, S_1^0(\Delta) = d + 1 + dim\, Lift(\Delta) = d + 1 + dim\ Stress_d$ and $dim\ S_r^{r-1}(\Delta) = dim\ P_m[x_1, \dots, x_d] + dim\ Cof_r^{r-1}\ (\Delta) = \begin{pmatrix} m+d \\ d \end{pmatrix} + dim\, Cof_r^{r-1}(\Delta)$.

From an algorithmic/computational point of view the most interesting questions about $S_m^r(\Delta)$ are: How does one compute the dimension? Is there is a basis with nice properties (easy to compute; locally supported); Does $S_1^0(\Delta)$ contain strictly convex polyhedral surfaces? For a PL-realization of a closed manifold, it is also interesting to know if this realization can be interpreted as the projection of a convex polyhedral sphere. From the practical point of view the planar case is the most interesting one (automotive industry, computer graphics, computer vision, robotics). However, the ability to compute $dim\ S_m^r(\Delta)$ and a basis is useful for all dimensions (for the finite elements method and for constructive approximations of functions).

For any cell-decomposition of an arbitrary region in $\mathbb{R}^2$, $dim\ S_1^0(\Delta)$ can be found by standard methods of linear algebra and graph theory in time at most $O(f_1^3)$. However one can inquire about *combinatorial* algorithms finding $dim\ S_1^0(\Delta)$ for generic decompositions. If Δ is a generic decomposition of a domain in $\mathbb{R}^2$ there is a combinatorial algorithm that determines if there is a sharp lifting for Δ and that finds $dim\ S_1^0(\Delta)$ in time $O(f_0^2)$ (Sugihara 1986, Whiteley 1996, 1988). This algorithm was implemented in software programs for computer aided design (see [74, 85, 87]) that reconstruct a polyhedral object in $\mathbb{R}^3$ from a generic planar projection. When each interior vertex of Δ is incident to only three cells, results of Davis (1958) and Ryshkov, Rybnikov (1994, 1996) imply that *dim* $S_1^0(\Delta)$ can be found in $O(f_0)$ operations with a very small factor. A partition of a d-manifold is called k-primitive if each k-cell is incident to exactly $d-k+1$ d-cells. As for higher-dimensional stresses and liftings we prove the following.

Theorem 1.0.8 *The dimension and a basis of the space of liftings of a $(d-2)$-primitive partition (not necessarily generic) of a closed d-manifold can be found in time $O(f_{d-1})$.*

A similar algorithm for 0-primitive decompositions (also called simple) of $\mathbb{R}^d$ was found by Aurenhammer [5]. The above theorem holds also for non-closed manifolds, but it requires $H_1(\Delta, \mathbb{Z}_2) = 0$ and some additional combinatorial assumptions (see Section 2.11). In Section 2.12 we also offer a polynomial algorithm that determines if a cell-decomposition of a region in $\mathbb{R}^3$ with $H_1(\Delta, \mathbb{Z}_2) = 0$ admits a convex lifting (this is equivalent to the existence of a d-tension). For generic realizations, the complexity of this algorithm does not depend directly on the dimension of the space. The complexity of the algorithm is $O(f_{d-1}^3 L)$, where L is the binary size of the numerical input (first two coordinates of the covectors determining facets of the manifold). Since the possibility of convex lifting is closely connected to other geometrical properties of decompositions, the latter algorithm also establishes whether an arbitrary decomposition of $\mathbb{R}^d$ (or of a convex region in $\mathbb{R}^d$) can be regarded as a weighted Dirichlet-Voronoi decomposition or as a weighted Delaunay decomposition. This may have some applications in the natural sciences.

In Chapter 3 we show how a d-stress on a piecewise-linear realization of an oriented (non-simplicial, in general) d-manifold in $\mathbb{R}^d$ naturally induces stresses of lower dimensions on this manifold, and discuss implications of this construction to the analysis of self-stresses in spatial frameworks. (There was an earlier work in this direction by Crapo and Whiteley [25] covering a special case of our construction.) The constructed mappings are not linear, but polynomial. It has been mentioned that in the 1860-70s J. C. Maxwell and W. J. M. Rankine described an interesting relationship between self-stresses in planar frameworks and vertical projections of polyhedral 2-surfaces. We offer a partial analog of Maxwell's correspondence for self-stresses in spatial frameworks and vertical projections of 3-dimensional surfaces based on our construction of polynomial mappings. Applying this theorem, we derive a class of three-dimensional spider webs similar to the family of two-dimensional spider webs described by Maxwell. In addition, we conjecture an important property of our mappings which is supported by a heuristic count based on the lower bound theorem ($g_2(d+1) = dim\ Stress_2 \geq 0$) for d-pseudomanifolds generically realized in $\mathbb{R}^{d+1}$ (Fogelsanger).

As was explained above, a tension on the 1-skeleton of a tiling corresponds to a convex surface that projects onto this tiling. Therefore if a random planar network admits an equilibrium tension, it can be lifted to the 1-skeleton of a convex surface. In Chapter 4, using rigidity theory and convexity, we study the effect of punching polygonal holes in a large tensed membrane. For example, if one randomly punches holes in an infinite tensed membrane, when does the tension cease to exist? We outline a mathematical theory of tension based on graph rigidity theory and introduce several probabilistic models for this problem. We show that if the "centers" of the holes are distributed in $\mathbb{R}^2$ according to Poisson law with parameter $\lambda > 0$, and the distribution of sizes of the holes is independent of the distribution of their centers, the tension vanishes on all of $\mathbb{R}^2$ for any value of λ. In fact, when the initial configuration of subsets is distributed according to a Poisson law and the sizes of the elements of the original configuration are independent of this Poisson distribution, this result follows from a more general result on the behavior of iterative convex hulls of connected subsets of $\mathbb{R}^d$. For the latter problem we establish the existence of a critical threshold in terms of the number of iterative convex hull operations required for covering all of $\mathbb{R}^d$. The processes described in the paper are somewhat related to bootstrap and to rigidity percolation models.

A grid based on a regular triangle obviously supports an all-positive equilibrium stress. Suppose each edge is removed from such a lattice independently with probability $1-p$, $p > 0$. Can we still construct an equilibrium tension? In Chapter 5 we prove that for any value of $p > 0$ there is no equilibrium stress on the damaged lattice. In fact, the proof can be modified to show that this is the case for a large class of regular planar graphs. Whether or not a similar theorem holds for spatial grids—for example, the grid of the shortest vectors of a point lattice based on a regular triangle—remains an open question.

Chapter 2

Stresses and Liftings

2.1 Introduction

In this thesis we develop a variety of geometrical and algorithmic methods that are useful for studying piecewise-linear surfaces, weighted Dirichlet-Voronoi and Delaunay diagrams, self-stresses in frameworks and geometrical cell-complexes and parallel drawings of polyhedral pictures. The space of d-stresses on a cell-complex introduced in this paper plays an important role in investigations of many affine and projective properties of a broad class of geometrical cell-complexes. Our notion of stress generalizes the notion of affine stress introduced by Lee [51].

In Sections 2.3-2.7 we prove that for a piecewise-linear realization in $\mathbb{R}^d$ of a homology manifold $\mathcal{M}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, and for an arbitrary decomposition of $\mathbb{R}^d$ by convex polyhedra the linear space of d-stresses is isomorphic to:

1. the linear space of liftings (with one fixed d-cell);

2. the linear space of reciprocals (with one fixed vertex).

These results can be considered as generalizations of similar results by Crapo and Whiteley for $d = 2, 3$ [23, 24, 25, 83]. For the first time this equivalence is proved under the general condition $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ for $d > 3$. Thus, there are non-spherical closed compact manifolds for which the equivalence between stresses and liftings holds. If Δ

is a decomposition of $\mathbb{R}^d$ by convex polyhedra, then the cone of d-tensions is equivalent to: 1) the cone of *additively weighted* Dirichlet-Voronoi diagrams, representing Δ; 2) the cone of convex liftings; 3) the cone of convex reciprocals. The equivalence between the last three objects was proved earlier by Aurenhammer [6] and McMullen [58]. In Section 2.8 we consider a general notion of duality for decompositions of $\mathbb{R}^d$ by convex polyhedra, and show that the classes of weighted Dirichlet-Voronoi and Delaunay diagrams (decompositions) coincide.

New sufficient conditions for the existence of a sharp lifting of a piecewise-linear realization of a cell-decomposition of a manifold in $\mathbb{R}^d$ are given in Sections 2.8 and 2.9. For instance, we show that any closed $(d-3)$-primitive manifold $\mathcal{M}^d$ (where the star of each $(d-3)$-cell has only four d-cells) with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ has a sharp lifting, but the existence of a sharp lifting for a $(d-2)$-primitive manifold $\mathcal{M}^d$ (where the star of each internal $(d-3)$-cell has only three d-cells) with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ requires the existence of a non-trivial d-stress on the star of each $(d-3)$-cell. In the same sections the problems of convexity and uniqueness of a lifting are analyzed. These results are improvements upon the well-known theorem of Davis [28] on the existence and uniqueness of a sharp convex lifting for a simple cell-decomposition of $\mathbb{R}^d$, $d > 2$, and upon similar results of Crapo and Whiteley on liftings of simple piecewise-linear spheres [25, 85]. As an application of the developed geometrical approach, two algorithms are given that determine whether a piecewise-linear realization of a d-manifold in $\mathbb{R}^d$ admits a lifting to $\mathbb{R}^{d+1}$ which satisfies given constraints, and find the dimension of the space of liftings. For decompositions of $\mathbb{R}^d$ by convex polyhedra, often referred to as tilings, these algorithms also recognize whether a decomposition is a weighted Dirichlet-Voronoi diagram and determine whether there is a convex surface which projects onto this decomposition. The first algorithm (Section 2.11) applies only to $(d-2)$-primitive manifolds, and has linear running time in the number of $(d-1)$-faces, which is optimal. This algorithm is similar to an algorithm of Aurenhammer [5], but is more general and can be applied to a broader class of complexes. The second algorithm (Section 2.12) applies to general cell-decompositions of homology manifolds (including arbitrary cell-decompositions of $\mathbb{R}^d$), but has worse time complexity, although it is polynomial. This algorithm uses a linear programming routine. Sugihara (1986) suggested a similar approach for studying liftings of more general objects than homology manifolds—polyhedral pictures (see Chapter 2.10 for definitions). Our contribution here is an adaptation of his method to homology manifolds and a thorough complexity analysis of this particular case. All theorems and algorithms are also interpreted for spherical cell-complexes (Section 2.13). These interpretations give new criteria and algorithms for recognizing whether a spherical complex is the radial projection of a convex polytope.

2.2 Polyhedral cell-complexes

All complexes which we shall consider are simplicial complexes from the topological point of view. However, all theorems and algorithms in this paper are stated for fixed decompositions of simplicial complexes into polyhedral *cells* (also called blocks or simplicial stars in combinatorial topology, see [52, 61]) which are not necessarily simplexes. We assume that all complexes have at most a countable number of cells and are locally finite. A homology k-sphere (k-disk) is a polyhedron with the homology groups of a standard k-sphere (k-disk). A compact k-cell is understood to be a polyhedron (simplicial complex) which is a cone with a homology k-sphere as base. Cells of co-dimension 1 are referred to as *facets*. To include consideration of general cell-decompositions of Euclidean spaces into convex polyhedra we allow non-compact cells in a realization of a cell-complex in $\mathbb{R}^d$; i.e. a polyhedral k-dimensional subset of the realization may be considered as a whole cell from the geometrical point of view, if it lies in a k-dimensional affine subspace, and is homeomorphic to a linear half-space of $\mathbb{R}^k$. We also assume that each non-compact cell has finitely many faces.

We denote the star of a k-dimensional cell C^k by $St(C^k)$, and the k-dimensional skeleton of a complex $\mathcal{K}^d$ by $Sk^k(\mathcal{K}^d)$. For us the relative boundary $\partial\mathcal{K}^d$ of a complex $\mathcal{K}^d$ is a sub-complex of $\mathcal{K}^d$ which consists of the closures of all $(d-1)$-cells which are not shared by at least two d-cells. We shall refer to cells that belong to the relative boundary as *boundary* cells, and to the other cells as *internal.* We denote the number of k-cells of $\mathcal{K}^d$ by $f_k(\mathcal{K}^d)$ and the number of internal k-cells by $f_k^\circ(\mathcal{K}^d)$. A *(combinatorial) path* in a complex $\mathcal{K}^d$ is a finite ordered sequence $\mathfrak{p} = [C_1, \ldots, C_k]$ of d-cells, where every two consecutive d-cells share a common $(d-1)$-cell. A *circuit* is a path where the first and last cells coincide.

We shall consider a somewhat more general construction than an embedding or an immersion of a cell-complex into Euclidean space, such as a *piecewise-linear (PL-) realization* of a cell-complex in Euclidean space. In all geometric discussions cell-complexes will be considered as fixed piecewise-linear realizations, rather than abstract combinatorial objects.

Such a general construction can be helpful, for example, for studying frameworks with bar intersections, polyhedral scenes, Schlegal diagrams, splines over triangulations (in the planar case this point of view was adopted in [23, 84, 87, 88]; in the three-dimensional case such PL-realizations were considered by Crapo and Whiteley in [23, 83]). For example, a Schlegal d-diagram (see Ziegler's book [93] for the theory of Schlegal diagrams) is a PL-realization of a $(d+1)$-polytope P^d in $\mathbb{R}^d$ obtained by radial projection of P^d onto one of its facets (a Schlegal diagram of a 4-cube is drawn in *Fig. 2.1*).

One can identify an abstract combinatorial cell-complex $\mathcal{K}^d$ with its embedding into

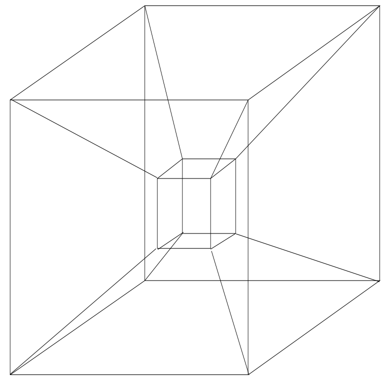

Figure 2.1: Schlegal diagram of 4-cube

$\mathbb{R}^{2d+1}$ (since it can be triangulated). A PL-realization of a combinatorial simplicial complex $\mathcal{K}^d \subset \mathbb{R}^{2d+1}$ with a fixed decomposition into polyhedral cells is a continuous PL-mapping r of $\mathcal{K}^d$ in $\mathbb{R}^N$ $(N \geq d)$ such that *the closure of each k-cell, $k = 0, \dots, d$ is embedded by r into $\mathbb{R}^N$ as a "flat" (lying in a k-subspace) k-polyhedron.* For a PL-realization $(\mathcal{K}^d, r)$, we reserve an upper case Roman font $K^d = (\mathcal{K}^d, r)$, and for an abstract combinatorial structure an upper case script $\mathcal{K}^d$. The body of $r(\mathcal{K}^d)$ is denoted by $|K^d|$.

In fact, for our purposes the cells need not be embedded. Recall, that each cell has un underlying structure of a simplicial star, i.e., there is a fixed baricentric triangulation for each cell of the complex. *All theorems on stresses and liftings in this Chapter still hold, if we to assume that each k-cell of a manifold is realized as a simplicial complex lying in an affine k-subspace—the cells need not be embedded.* Note, that in this case a cell has no interior and exterior, and therefore the notions of interior and exterior normals cannot be used. If a cell is not embedded, we have to fix a combinatorial orientation for the boundary of the cell (regarded as a simplicial complex) and define the positive direction of normal to the cell at each *simplex* of the boundary using the "right hand" rule. Notice, that if a $(k-2)$-cell C^{k-2} from the boundary of a $(k-1)$-cell C^{k-1} is not embedded, one cannot talk about the positive or negative direction of a normal to C^{k-1} at C^{k-2}.

If we refer to the metric, projective, or affine properties of a cell-complex, these should be understood as the properties of its fixed PL-realization. However, when we consider the combinatorial or homological properties of a cell-complex, we are referring to its abstract combinatorial structure. A PL-realization of the star of a k-cell in $\mathbb{R}^d$ is called general (for us) if all its $(d-1)$-cells lie on different $(d-1)$-planes.

For basic notions of algebraic and combinatorial topology see Munkers (1984) and Seifert, Threlfall (1980). We shall consider only strongly connected, pure dimensional polyhedral complexes. A simplex s belongs to the *link* of a simplex S if the closure of s does not intersect with the closure of S, and there is a simplex B such that both s and S belong to the closure of B. A homology k-sphere (k-disk) is a polyhedron with the homology groups of a standard k-sphere (k-disk). A *homology d-manifold* (with boundary) is a cell-complex such that the link (in the case of a non-simplicial cell-decomposition, the link of a cell can is defined through the baricentric triangulation) of each k-cell, is either a homology $(d-k-1)$-sphere or a homology $(d-k-1)$-disk. A manifold is closed if each facet is adjacent to exactly two d-cells. A closed manifold is called *orientable* if its cells can be oriented so that they form a cycle over $\mathbb{Z}$. A manifold with boundary is orientable if its cells can be oriented so that they form a chain over $\mathbb{Z}$ whose boundary belongs to the module generated by the boundary cells. All statements in the paper are formulated for both closed manifolds and for manifolds with a boundary, unless stated otherwise. Since we consider manifolds only from the combinatorial point of view, a manifold is always understood to be a *homology* manifold. Throughout the paper we include "good" decompositions of $\mathbb{R}^n$ (like, for example, weighted Dirichlet-Voronoi diagrams) into the class of homology manifolds.

The star of an internal k-cell in a d-manifold is called $(d-k)$-primitive if it has $d-k+1$ d-cells. A cell-decomposition of a d-manifold is referred to as *k-primitive* if the star of each internal k-dimensional cell has $d-k+1$ d-cells (some authors call 0-primitive decompositions *simple*; our terminology goes back to Voronoi [82]). For decompositions of $\mathbb{R}^d$ by convex polyhedra $d-k+1$ is the minimum possible number of tiles in the star of a k-face. If a PL-realization of a homology sphere $\mathcal{S}^d$ can be lifted onto a convex polytope in $\mathbb{R}^{d+1}$, then 0-primitive vertices of $\mathcal{S}^d$ correspond to simple vertices of this convex polytope. When a k-primitive cell-decomposition of $\mathcal{M}^d$ is assumed to be fixed, we will refer to this k-primitive decomposition of $\mathcal{M}^d$ as *k-primitive manifold* $\mathcal{M}^d$. The notion of k-primitive decomposition naturally arises in studies of space-fillers, lattice polytopes and stereohedra [82, 92, 32, 33]. For example, the affine equivalence between space-fillers and Dirichlet domains of lattices was proved by Voronoi only for 0-primitive (simple) tilings. Later Zhitomirski [92] proved that this equivalence holds for $(d-2)$-primitive tilings, and Erdahl [38] for zonotopes, but it still remains unknown whether there are more general sufficient combinatorial conditions on a space filler to be the Dirichlet domain of a lattice (for details see [32, 33]). The existence of a lattice Dirichlet domain which is affinely isomorphic to a space-filler Π is equivalent to the existence of a convex lifting with some special symmetries for the lattice tiling $T(\Pi)$ by Π (Voronoi [82]).

2.3 Stresses on cell-complexes

If (V,E) are the vertices and edges of a framework in $\mathbb{R}^d$, then a self-stress (or simply stress) is an assignment of real numbers $s_{ij}=s_{ji}$ to the edges, a tension if the sign is positive or a compression if the sign is negative, so that the equilibrium conditions $\sum_j s_{ij}(\vec{v}_j-\vec{v}_i)=0$ hold at each vertex $\vec{v}_i$. The notion of stress can be naturally generalized to k-stresses on cell-complexes. This generalization proves to be useful in the theory of space-fillers, the combinatorics and geometry of piecewise-linear manifolds, and the rigidity theory.

Consider a PL-realization K^d in $\mathbb{R}^N$ of a cell-complex $\mathcal{K}^d$, and let $\vec{n}(C^{k-2}, C^{k-1})$ be the inner unit normal to a $(k-1)$-cell C^{k-1} at its $(k-2)$-face C^{k-2}.

Definition 2.3.1 *A real-valued function $s(\cdot)$ on $(k-1)$-cells of K^d is a k-stress if for each internal $(k-2)$-cell C^{k-2} of K^d*

$$\sum_{C^{k-1}} s(C^{k-1})\, \vec{n}(C^{k-2}, C^{k-1}) = 0,$$

where C^{k-1} ranges over the $(k-1)$-cells such that $C^{k-2} \subset \partial\, C^{k-1}$. The quantities $s(C^{k-1})$ are the coefficients of k-stresses, a tension if the sign is positive and a compression if the sign is negative.

Let us consider the case where the $(k-2)$-cells of $\mathcal{K}^d$ are not embedded, but realized with self-intersections in their affine $(k-2)$-subspaces. Suppose $\mathcal{K}^d$ is a cell-complex where a baricentric triangulation is fixed for each cell. Pick a (combinatorial) orientation for $(k-1)$-each cell of $\mathcal{K}^d$. To define the notion of k-stress we have to formulate the equilibrium conditions for each simplex of the baricentric triangulation of each $(k-2)$-cell. However, it is easy to see that if the equilibrium condition holds for one simplex of C^{k-2}, it holds for all other simplexes of C^{k-2}: when we pick another $(k-2)$-simplex from the triangulation of C^{k-2} all normals either change their direction to the opposite, or stay the same. Below we give the definition of stress for general PL-realizations of polyhedral cell-complexes.

Denote by $\vec{n}(S^{k-2}, C^{k-1})$ the unit normal to oriented cell C^{k-1} at its simplicial facet S^{k-2} whose orientation is induced by the orientation of C^{k-1}.

Definition 2.3.2 *A real-valued function $s(\cdot)$ on (generally non-embedded) oriented $(k-1)$-cells of K^d is a k-stress if for each $(k-2)$-simplex S^{k-2} of each internal $(k-2)$-cell C^{k-2} of K^d*

$$\sum_{C^{k-1}} s(C^{k-1})\, \vec{n}(S^{k-2}, C^{k-1}) = 0,$$

where C^{k-1} ranges over all oriented $(k-1)$-cells such that $C^{k-2} \subset \partial\, C^{k-1}$.

This definition has only one disadvantage: it does not directly generalize the definition for a framework and Definitions 2.3.2, for the interior normal to a $(k-1)$-simplex at its facet might not correspond to the chosen combinatorial orientation of the simplex. Of course, if we change the combinatorial orientations of some simplexes, the space of stresses of the reoriented complex will be isomorphic to the original one. An alternative approach to Definition 2.3.2 is to work with inner normals to simplexes. However, in this case a stress on a $(k-1)$-cell changes sign as we go from one simplex of its baricentric triangulation to another; only if the cell is embedded, the sign is the same on all simplexes.

Notice, that our definition works for cell-complexes in spherical space $\mathbb{S}^N$ and in hyperbolic space Λ^N too. Each k-cell is realized in a k-dimensional subspace of constant curvature. A space of constant curvature has a natural global projective structure which allows us to introduce the notion of projective normal. Thus, in the case of $\mathbb{S}^N$ or Λ^N, instead of the notion of Euclidean normal we have to use the notion of projective normal (the observation that the theory of liftings, reciprocals and stresses is equally applicable to $\mathbb{S}^N$, Λ^N, and $\mathbb{R}^d$ is due to Walter Whiteley). If not all $s(C^{k-1})$ are zero, the k-stress s is called non-trivial. *Fig. 2.2* illustrates the geometry of the equilibrium condition from the Definition 3.1 for a 3-stress on the star of an edge of a cell-complex in $\mathbb{R}^3$.

By the relationship between self-stresses in a planar framework and liftings to $\mathbb{R}^3$ of the cell-decomposition induced by this framework (this was originally discovered by Maxwell: see [53, 54, 66, 67]), d-stresses in d-dimensional manifolds can be thought of as a generalization of self-stress in frameworks. To complete the analogy with stresses on frameworks the stress coefficients assigned to faces should be divided by their volumes, but for our purposes this would be inconvenient since our formulas

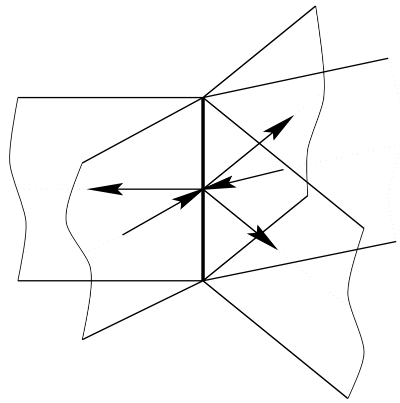

Figure 2.2: 3-stress on the star of an edge in $\mathbb{R}^3$

would then become more cumbersome, and because we admit cells of infinite volume.

Our generalization of the notion of stress is geometrical and does not involve coordinates, whereas that of Lee [51] is algebraic and more restrictive (for finite simplicial complexes). If $b_a(C^{k-1})$ denotes Lee's *affine* k-stress on a $(k-1)$-simplex C^{k-1}, then $s(C^{k-1}) = b_a(C^{k-1})\, vol\, (C^{k-1})$ (it is a matter of taste whether to include the volumes of $(k-1)$-cells into the formula defining a k-stress or not). It is more natural from the geometrical point of view to assign coefficients of k-stresses to k-cells, rather than to $(k-1)$-cells, as it is done in our definition, but this shifted notation was already used by several authors [51, 76, 25] and we will keep to this convention. Tay, White, and Whiteley remarked in [76] that the notion of linear stress can be extended to general cell-complexes. Note, that our notion of stress coincides with the notion of stress introduced by Tay and coauthors for simplicial complexes. McMullen also uses the language of outer normals in [59], where he defines weights (on simple polytopes), a notion dual to stresses.

It easy to see that k-stresses form a linear space, and that k-tensions (where all coefficients of k-stresses are positive) and k-compressions (where all coefficients of k-stresses are negative) form congruent polyhedral cones in this linear space. We will denote the space of all k-stresses on M^d by $Stress_k(M^d)$, and the cone of all

k-tensions in this space by $Tension_k(M^d)$.

Theorem 2.3.3 *Let M^d be a PL-realization of an orientable manifold $\mathcal{M}^d$ in $\mathbb{R}^d$. For each $k = 1, \ldots, d-1$, there is a polynomial mapping of degree $d-k+1$ from the space $Stress_d(M^d)$ to the space $Stress_k(M^d)$. In the case of tiling, i.e., when the manifold is $\mathbb{R}^d$ itself and all cells are embedded, a d-tension (compression) is mapped to a k-tension (compression).*

The proof of this theorem is based on the construction of a local Euclidean reciprocal (see Section 7) for the star $St(v)$ of a vertex v of $\mathcal{M}^d$. Then define a function (polynomial in the linear and angular parameters of the reciprocal) on all subcomplexes of this reciprocal corresponding to the stars of faces, and show that for each $(k-2)$-face of $St(v)$ this function represents a k-stress which is well-defined on all the manifold. This theorem is proved in the next chapter.

2.4 Quality transfer

In this section, we introduce the notion of quality transfer which allows us to make a connection between stresses and the geometry of PL-manifolds. Let Q be a set of *qualities* and let $\mathfrak{G}$ be a group acting on Q. Here we consider the problem of assigning an element of Q, a quality, to each of the d-cells of M^d so that qualities assigned to adjacent cells are governed by rules associated with the common facet. If an arbitrary quality is assigned to some d-cell C_0, a quality can be assigned to any other d-cell C_k by translating qualities along a path connecting C_0 to C_k, using the rules associated with facets. Suppose that $\mathfrak{f}$ maps every ordered pair of adjacent cells into a group element, so that the reversed pair is mapped into the reciprocal group element. If $\mathfrak{p} = [C_0, \ldots, C_k]$ is a combinatorial path and q is the quality assigned to C_0, then the quality assigned to C_k via $\mathfrak{p}$ is given by the formula.

$$q \circ \mathfrak{f}(\mathfrak{p}) = q \circ \mathfrak{f}([C_0, C_1]) \cdot \mathfrak{f}([C_1, C_2]) \ldots \mathfrak{f}([C_{k-1}, C_k])$$

The qualities assigned to cells are well defined (i.e. independent of path) if and only if every circuit lies in the kernel of $\mathfrak{f}$.

Consider a 1-complex (graph) $\mathcal{G}$ where the vertices are d-cells, and the edges are the $(d-1)$-cells of $\mathcal{M}^d$ (therefore $\mathcal{G}$ may be infinite). Two vertices share an edge if, and only if, the corresponding d-cells are adjacent. The graph $\mathcal{G}$ is called the combinatorial dual of $\mathcal{M}^d$, and it is somewhat easier to consider quality transfer on $\mathcal{G}$ rather than on $\mathcal{M}^d$. In this model, the edges are associated with elements of $\mathfrak{G}$, and qualities are assigned to the vertices. Assigning qualities to the vertices of $\mathcal{G}$ is well defined if it is well defined over all cycles of $\mathcal{G}$. A more manageable criterion that the qualities assigned to d-cells are well defined is the following:

Lemma 2.4.1 *Let $\{c_i\}$ be a generating system for $H_1(\mathcal{G}, \mathbb{Z}_2)$. Then quality transfer is well defined on $\mathcal{G}$ if and only if it is well defined over all cycles from $\{c_i\}$.*

Proof. Fix a quality for a vertex v_0 of $\mathcal{G}$, and denote by $Q(v_i)$ the set of qualities that can be assigned to a vertex v_i by translating qualities from v_0. Consider the 1-complex $cov\ \mathcal{G}$, where the vertices are the pairs $(v_i, q(v_i))$ where v_i is a vertex of $\mathcal{G}$ and $q(v_i) \in Q(v_i)$. Vertices (v_i, q) and (v_j, q') of $cov\ \mathcal{G}$ share an edge in $cov\ \mathcal{G}$ if and only if v_i and v_j are adjacent in $\mathcal{G}$ and there is $g \in \mathfrak{G}$ ($\mathfrak{G}$ is the group acting on Q) such that $g(q) = q'$. It is easy to see that $cov\ \mathcal{G}$ is a covering of $\mathcal{G}$. Since $\mathcal{G}$ is a 1-complex and $\{c_i\}$ is a generating system of $H_1(\mathcal{G}, \mathbb{Z}_2)$, then $\{c_i\}$ is also a generating system for the fundamental group $\pi(\mathcal{G})$. The covering map p from $cov\ \mathcal{G}$ onto $\mathcal{G}$ induces a monomorphism $p^* : \pi(cov\ \mathcal{G}) \to \pi(\mathcal{G})$. Every cycle from $\{c_i\}$ is lifted onto $cov\ \mathcal{G}$ in the trivial way (i.e. the lifting of c_i is a single cover of c_i). Since $\{c_i\}$ is a generating system for $\pi(\mathcal{G})$, p^* is an epimorphism. Therefore, p^* is an isomorphism, and the covering map p is one-to-one. As the covering is trivial, the quality transfer is well-defined on $\mathcal{G}$. $\Box$

For the remaining portion of this paper we shift our attention from general cell-complexes to d-dimensional homology manifolds realized in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$. PL-manifolds with non-compact cells are permitted, if in the PL-realization non-compact cells can be retracted onto (new) compact cells (which are subsets of old non-compact cells) whose internal faces span the same affine subspaces as replaced non-compact cells. For instance, it allows the inclusion of decompositions of $\mathbb{R}^d$ by convex polyhedra (for example, weighted Dirichlet-Voronoi diagrams) into the class of manifolds where the theory of d-stresses and liftings works. We will consider all homologies with coefficients in the group of two elements: $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Lemma 2.4.2 *Let $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. If quality transfer is well defined over the links of all $(d-2)$-cells, then it is well defined on $\mathcal{M}^d$.*

Proof. Since $\mathcal{M}^d$ is a homology manifold, one can consider a cell-decomposition $\mathcal{D}$ of $\mathcal{M}^d$ (in the combinatorial sense) which is dual to the original (for a description of this construction see [61, 52]). By the definition of a polyhedral cell-complex, the cells of the decomposition of $\mathcal{M}^d$ can be triangulated so that all cells of the original decomposition become triangulated in a baricentric fashion. A k-cell of the dual decomposition is defined as the union of the k-simplexes that share a common vertex in the baricentric triangulation (which is the same for both the original, and the dual). If M^d has non-compact cells, we understand $\mathcal{D}$ to be a cell-decomposition of a manifold (with boundary) obtained from M^d by contraction of all its non-compact cells onto their compact subsets (in section 2 we assumed that it is always possible). This operation does not change the homotopical class and therefore preserves the homology too. Since $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, $Sk^2(\mathcal{D})$ has trivial $H_1(Sk^2(\mathcal{D}), \mathbb{Z}_2)$. In other words, a cycle on $Sk^1(\mathcal{D})$ can be represented as the sum of the boundaries of 2-cells of $\mathcal{D}$. It is easy to see that the dual graph $\mathcal{G}(\mathcal{M}^d)$ is a subgraph of $Sk^1(\mathcal{D})$ and coincides with $Sk^1(\mathcal{D})$ when $\mathcal{M}^d$ is a closed manifold. The d-cells having facets that belong to $\partial\mathcal{M}^d$ require a more detailed consideration. If $\mathcal{M}^d$ has a boundary, $Sk^2(\mathcal{D})$ has edges and vertices corresponding to boundary $(d-1)$-cells, and 2-cells corresponding to boundary $(d-2)$-cells. Indeed, any cycle of $\mathcal{G} \subset Sk^1(\mathcal{D})$ is the boundary of a chain whose carrier does not contain such 2-cells. Thus, the cycles of $\mathcal{G}$ that correspond to the stars of internal $(d-2)$-dimensional cells form a generating system for $H^1(\mathcal{G}, \mathbb{Z}_2)$. By Lemma 2.4.2, quality transfer is well defined on $\mathcal{M}^d$. □

Note, that for any homology manifold $\mathcal{M}^d$, $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ is equivalent to $H_1(\mathcal{M}^d, \mathbb{Z}) = 0$. Homologies can be substituted by cohomologies. Also, due to Poincare duality we can take H_{d-1} instead of H_1.

The technique of quality transfer on Euclidean tilings goes back to Voronoi [82]. However, his technique was essentially homotopical, not homological (see [32, 66]). The homological version of this procedure was later implicitly used by several authors, including Crapo, Whiteley [25, 83], and McMullen [57].

If M^d is a fixed realization in $\mathbb{R}^d$ of a manifold $\mathcal{M}^d$, one can attempt to assign $+1$ or -1 to each cell (recall that every d-cell is *embedded* into $\mathbb{R}^d$) so that two adjacent d-cells have the same orientation if and only if their outer normals at their common facet have opposite directions. By the definition of a cell-complex, $\mathcal{M}^d$ can be baricentricly triangulated. One can prove that $\mathcal{M}^d$ is orientable in the sense defined above, if and only if it is orientable as a simplicial complex (in the usual sense). Since we need our geometrical notion of orientation in the following section, we now show that any manifold $\mathcal{M}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ is orientable in this sense.

Assuming that for a d-cell C_0 of M^d the orientation of the embedding is o, we define the orientation for an adjacent cell C_1 as follows. If outer normals to C_1 and C_0 at their common facet have opposite directions (when C_1 and C_0 are convex, it means that they point to different half-spaces determined by the facet), assign orientation $+o$ to C_1; if the normals are cooriented, assign orientation $-o$ to C_1. Since $\mathcal{M}^d$ is strongly connected, an orientation can be assigned to any cell of M^d by transferring the orientation along paths of adjacent cells in this fashion. We have to show that the orientation assigned to a cell in this manner is independent of a particular path used for the transfer, and is therefore well defined. By Lemma 4.2, this requires that the transfer of orientation be well-defined over the links of the internal $(d-2)$-cells which are all $\mathbb{S}^1$. Since the closure of each cell of $\mathcal{M}^d$ is *embedded* into $\mathbb{R}^d$, the orientation is correctly defined over the links of all $(d-2)$-cells, so, by Lemma 4.2, the orientation can be properly introduced on all of M^d. Throughout the paper we will denote the orientation of a cell C by $o(C)$.

2.5 Stresses, splines and liftings

Let K^d be a PL-realization of a combinatorial cell-complex $\mathcal{K}^d$ in $\mathbb{R}^d$. A family of affine functions $L(\mathbf{x}; C^d)$ on $\mathbb{R}^d$ corresponding to the d-cells of K^d is referred to as a lifting of K^d if for any cell C of K^d the affine span of C is the vertical projection of the intersection of all the hyperplanes defined by the affine functions of the d-cells incident to C. In other words, each cell of K^d is the vertical projection of the corresponding cell of the realization L^d determined by the affine functions $L(\mathbf{x}; C^d)$. A lifting is called *sharp* if each pair of adjacent d-cells of K^d is lifted onto distinct hyperplanes in $\mathbb{R}^{d+1}$. $L(\mathbf{x}; C^d)$ is called locally convex (concave) if on each sub-complex of K^d which is embedded into $\mathbb{R}^d$, $L(\mathbf{x}; C^d)$ is a convex (concave) PL-function. We refer to the angle between two adjacent d-cells of a lifting as a dihedral angle. Denote the linear space of liftings defined up to the choice of transferring a supporting plane by $Lift(K^d)$. Locally convex liftings form the polyhedral cone $CLift(K^d)$ in this space. Lifting is a natural generalization of the notion of continuous PL-function (C_1^0-spline) on a cell-decomposition of a polyhedral region. A locally convex lifting is a (locally) convex PL-function on the region. The conception of lifting is very convenient in studies of stresses in planar frameworks, and in the analysis of polyhedral scenes

[4, 23, 25, 84, 88].

Let M^d be a PL-realization in $\mathbb{R}^d$ of a *manifold* $\mathcal{M}^d$. A real-valued function c_1^0 on the ordered pairs of adjacent d-cells of M^d is called a C_1^0-cofactor if

1) $c_1^0([C, C']) = -c_1^0([C', C])$

2) for each internal $(d-2)$-cell C^{d-2}

$$\sum_{i=1}^{i=n} c_1^0([C_i, C_{i+1}])\vec{n}([C_i, C_{i+1}]) = 0$$

where $[C_1, \ldots, C_n, C_{n+1} = C_1]$ is a cyclical order of the d-cells making contact in C^{d-2}, and $\vec{n}([C_i, C_{i+1}])$ is the outer unit normal to C_i at its facet shared with C_{i+1}. We will also use the term C_1^0-*cofactor* for referring to the value of a function $c_1^0(\ )$ on a facet.

A lifting $L(\mathbf{x}; C) = \langle \vec{a}(C), \mathbf{x}\rangle + a_0(C)$ determines a C_1^0-cofactor c_1^0 by $c_1^0([C, C']) = \langle \vec{a}(C') - \vec{a}(C)), \vec{n}([C, C'])\rangle$. In theorem 5.1 we show that if $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, then a cofactor determines a lifting up to the choice of an affine function. The following theorem connects stresses and liftings for PL-realizations of a manifold with $H_1 = 0$.

Theorem 2.5.1 *Let M^d be a PL-realization of a manifold $\mathcal{M}^d$ with trivial $H_1(\mathcal{M}^d, \mathbb{Z}_2)$. The linear space $Stress_d(M^d)$ is isomorphic to the linear space $Lift(M^d)$. All-non-zero stresses correspond to sharp liftings. If M^d is an embedding, then $Tension_d(M^d) \cong CLift(M^d)$.*

Proof. Take the set A of affine scalar valued functions on $\mathbb{R}^d$ to be the set of qualities. This set has a linear space structure and acts on itself by transfer. We will denote this transferal group by $\mathfrak{G}$. If each ordered pair of adjacent cells is associated with a group element from $\mathfrak{G}$, and an affine function is assigned to any particular d-cell, affine functions can be assigned to adjacent cells using the action of $\mathfrak{G}$ on A. Lemma 4.2 gives sufficient conditions that the quality assigned to a cell in this manner be independent of the path used for the transfer, and is therefore well defined. As an application of this lemma, we shall establish a linear correspondence between d-stresses and liftings.

Let s be a non-trivial d-stress on M^d, and fix the orientations of the d-cells of M^d so that the orientations agree for pairs of adjacent cells as it was described above. If d-cells C_1, C_2 are adjacent, denote by $E_{1,2}(\mathbf{x}) = \langle \vec{n}, \mathbf{x} \rangle - c = 0$ the equation of their common facet C^{d-1} which is adjusted so that the vector $\vec{n}$ is an outer unit normal to C_1 at C^{d-1} (if C_1 is convex, this is the same as saying that $E_{1,2}(\mathrm{x}) \leq 0$ on C_1). If $s(C_1, C_2) = s(C_2, C_1)$ is the stress on C^{d-1}, let $g(\mathbf{x}, [C_1, C_2]) = s(C_1, C_2)o(C_1)(\langle \vec{n}, \mathbf{x} \rangle - c)$ be the affine function associated with the ordered pair $[C_1, C_2]$ (as an element of $\mathfrak{G}$). If $q(\mathbf{x}, C_1)$ is the affine function for C_1, then the affine function for C_2 is given by $q(\mathbf{x}, C_1) + g(\mathbf{x}, [C_1, C_2])$. It is easy to see that this definition is symmetric and for the path $[C_1, C_2, C_1]$ we have $g(\mathbf{x}, [C_1, C_2]) + g(\mathbf{x}, [C_2, C_1]) = 0$. Since $g(\mathbf{x}, \mathfrak{p})$ is defined on two-element paths, it is also defined on arbitrary paths. Since $\mathcal{M}^d$ is a manifold, the link of each internal $(d-2)$-cell is $\mathbb{S}^1$. Consider an ordered circuit $[C_1, \dots, C_n, C_{n+1}]$, where C_i are the d-cells from the star of an internal $(d-2)$-cell C^{d-2} and $C_{n+1} = C_1$. It is easy to see that $g(\mathbf{x}, [C_1, \dots, C_n, C_1]) = 0$ if and only if

(2.1)
$$\vec{v} = \sum_{i=1}^{i=n} s([C_i, C_{i+1}])o(C_i)\vec{n}([C_i, C_{i+1}]) = 0$$

Rotate $\vec{v}$ by $90°$ in the orthogonal complement to *span* (C^{d-2}) in such way that the result of the rotation of $(o(C_1)\vec{n}([C_1, C_2]))$ becomes the inner unit normal to the common facet of C_1 and C_2 at C^{d-2}. Denote this rotation operator by r. Consider instead of $St(C^{d-2})$ the section of $St(C^{d-2})$ with a perpendicular 2-plane. By convention we will refer to the direction of rotation from C_1 to C_2 through their common facet as clockwise. For each C_i, $i = 1, \dots, n$, call the facet where C_i contacts C_{i-1} (C_n if $i = 1$) the first facet and the facet where C_i contacts C_{i+1} the second facet. If $o(C_i) = o(C_1)$, then the direction of rotation from the first facet of C_i to the second is clockwise. In this case, the rotation transforms vector $o(C_1)\vec{n}([C_i, C_{i+1}])$ into the inner unit normal to the common facet of C_i and C_{i+1} at C^{d-2}. If $o(C_i) = -o(C_1)$, then the direction of rotation from the first facet of C_i to the second is counter-clockwise. It is easy to see that, in this case, $o(C_1)\vec{n}([C_i, C_{i+1}])$ is also transformed into the inner unit normal to the common facet of C_i and C_{i+1} at C^{d-2}. Thus $r(\vec{v}) = 0$, for $r(\vec{v})$ is the vector sum of d-stresses at C^{d-2}. It therefore follows that formula 2.1 holds.

By Lemma 2.4.2, since the quality transfer is well defined over the links of all $(d-2)$-cells, it is well defined on all of $\mathcal{M}^d$. Our construction provides continuous gluing over facets of M^d. If an affine function is fixed for any d-cell (the "origin"), affine functions can be assigned to the other cells using the group $\mathfrak{G}$ and the quality transfer. The resulting family of affine functions is a lifting by construction and Lemma 4.2. It is easy to see that if the coefficients of d-stresses are all-non-zero, then the lifting is sharp.

Conversely, let $L^d = L(\mathbf{x}, C)$ be a lifting of M^d. If $L(C)$ is the lifting of a d-cell C of M^d, define $o(L(C)) := o(C)$ (recall that the orientations of embedding $o(C)$ have been adjusted for the d-cells of M^d). The d-stress on M^d corresponding to L^d can be found in the following way. Let $L(C_1)$ and $L(C_2)$ be adjacent d-dimensional cells of the lifting projecting onto cells C_1 and C_2 of the "flat" realization M^d, and defined by affine functions $L(\mathbf{x}, C_1) = \langle \vec{a}_1, \mathbf{x} \rangle + c_1$ and $L(\mathbf{x}, C_2) = \langle \vec{a}_2, \mathbf{x} \rangle + c_2$. Let $\vec{n}_1$ be the outer unit normal to C_1 at facet C^{d-1} shared with C_2. Now, we can assign the coefficient of stress $o(L(C_1))\langle \vec{a}_2 - \vec{a}_1, \vec{n}_1 \rangle$ to the facet C^{d-1}. Reversing the arguments in the proof of formula (2.1) from the first part of the theorem, one can see that introduced quantities $o(L(C_1))\langle \vec{a}_2 - \vec{a}_1, \vec{n}_1 \rangle$ are actually coefficients of d-stresses. Notice that these arguments have local character and do not employ the condition that $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. $\square$

The above theorem still holds if the cells of $\mathcal{M}^d$ are not embedded, but realized as simplicial stars with self-intersections (see Section 2.3 for details). *Proof.* One can extend a d-stress s on M^d to a stress on the PL-realization of its baricentric triangulation $D(M^d)$: set $s(S^{d-1}) = 0$ for any $(d-1)$-simplex S^{d-1} which does not belong to the triangulation of a $(d-1)$-cell of M^d, and set $s(S^{d-1}) = s(C^{d-1})$ if S is a $(d-1)$-simplex of a $(d-1)$-cell C^{d-1}. All simplicial cells of the baricentric triangulation are, indeed, embedded. Reorient (if necessary) all $(d-1)$-simplexes in the baricentric triangulation so that the positive direction of normal is always inwards. The space of d-stresses of the reoriented complex is isomorphic to the original space of d-stresses. This reorientation is required, because we want to use the definition of stress for complexes with embedded cells. Now, we can apply Theorem 2.5.1. For any d-cell of M^d, the lifting corresponding to our stress s is defined by the same hyperplane because of the way we extended s to the baricentric triangulation. Conversly, if a lifting of the baricentric triangulation is flat on each cell, the corresponding stress is can be interpreted as a stress on the initial cell-partition. $\square$

Since, in the second part of Theorem 2.5.1, the homological condition that $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ is not used, the above theorem constructs a monomorphism from $Lift(M^d)$ into $Stress_d(M^d)$ for an arbitrary orientable $\mathcal{M}^d$. This monomorphism maps a C_1^0-cofactor c_1^0 to a d-stress s via formula $s(C^{d-1}) = c_1^0(C^{d-1})$, where C^{d-1} is an internal facet of M^d. Notice that in the proof we did not assume the convexity of cells. Theorem 5.1 was independently proved by Crapo and Whiteley for homology 3-spheres in preprint [25], although the topological part of the proof was only sketched and the condition $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ was not mentioned as sufficient for the existence of the isomorphism between $Stress_d(M^d)$ and $Lift(M^d)$.

Although the space of lifting is always a subspace of the space of stress, the converse is not necessarily true. A realization of the torus depicted in Fig. 2.3 demonstrates that in general one cannot drop the condition $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ (triangles

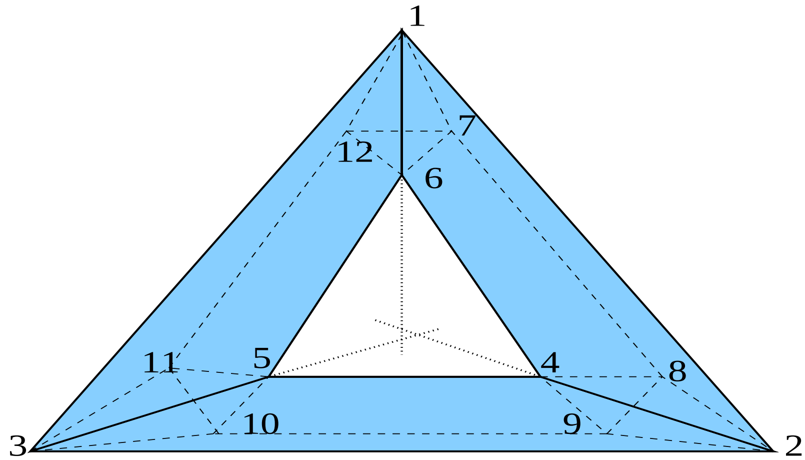

Figure 2.3: Non-liftable torus in the plane

(123), (456) and hexagon (7 8 9 10 11 12) are not loaded; edges (16), (24), (35) can have arbitrary tensions; all other edges are under compression; since lines 16, 24, 35 do not pass through a common point, the torus does not lift).

If Δ is a cell-decomposition of a region in $\mathbb{R}^d$, then a lifting of Δ is a continuous PL-function on Δ, i.e. a C^0_1-spline. More generally, a C^r_m-spline over Δ is a C^r-smooth function which is represented by polynomials of degree at most m on each d-cell. Such functions form a vector space over $\mathbb{R}$, which is denoted by $S^r_m(\Delta)$. The theory of splines on manifolds embedded into $\mathbb{R}^d$ is similar to the theory of stresses and liftings. Recall that coefficients of d-stresses define gluing of affine functions over facets of a decomposition in the way shown in Section 5. Regarding smooth splines as generalizations of PL-functions, one can say that C^{r-1}_r-*cofactors* play the same role for C^{r-1}_r-splines as stresses for C^0_1-splines. They define the smooth gluing of polynomial patches over facets (the core of the theory of cofactors of smooth splines can be found in [85].) Let's fix orientations for the links of all internal $(d-2)$-cells. The orientation of the link of a $(d-2)$-cell induces the orientations of normals to supporting hyperplanes of facets making contact in this $(d-2)$-cell.

Definition 2.5.2 *A real-valued function $c_r^{r-1}(\ )$ on the $(d-1)$-cells of Δ is called a C_r^{r-1}-cofactor if for each internal $(d-2)$-cell C^{d-2} of Δ, and for each non-negative integer vector $(i_1, \ldots, i_n)$ such that $i_1 + \ldots + i_n = r$*

$$\sum_{C^{d-2} \subset \partial F} c_r^{r-1}(F)\; n_1^{i_1}(F) \cdots n_d^{i_d}(F) = 0$$

where F ranges over all facets making contact at C^{d-2} and $n_1(F), \ldots, n_d(F)$ are the coordinates of the unit normal to the hyperplane spanned by F whose orientation is induced by the fixed orientation of the link of C^{d-2}.

Clearly the C_r^{r-1}-cofactors form a linear space, which we denote by $COF_r^{r-1}(\Delta)$. The following lemma [85] explains the importance of the above definition.

Lemma 2.5.3 *Let c_r^{r-1} be a C_r^{r-1}-cofactor for Δ. Then, adopting the notation from definition 6.2, for every internal $(d-2)$-cell of Δ we have*

$$\sum_{C^{d-2} \subset \partial F} c_r^{r-1}(F)\; (n_1(F)x^1 + \cdots + n_d(F)x^d + n_{d+1}(F))^r = 0$$

where $(x_1, \ldots, x_d)$ are the usual Euclidean coordinates in $\mathbb{R}^d$ and $n_1(F)x_1 + \cdots + n_d(F)x_d + n_{d+1}(F) = 0$ is an equation for F.

This statement is known as the C_r^{r-1}-cofactor reduction lemma [85]. The following lemma explains why C_r^{r-1}-cofactors determine the smooth gluing of polynomial patches (for the proof see [12]).

Lemma 2.5.4 *Let function f be represented by polynomials of at most degree m on each d-cell of Δ. Then f is a C_m^r-spline if and only if for each pair of adjacent d-cells the difference between polynomials corresponding to these cells is divided by the $(r+1)$-power of an affine function vanishing on the common facet of these d-cells.*

The following theorem, which generalizes a Billera's theorem [14, 85] for $d > 2$ is a consequence of Lemma 4.2 on quality transfer. It underlines the analogies between stresses and cofactors of C_r^{r-1}-splines.

Theorem 2.5.5 *If* $H_1(\Delta, \mathbb{Z}_2) = 0$, *then*

$$dim\ S_r^{r-1}(\Delta) = \binom{r+d}{d} + dim\, COF_r^{r-1}(\Delta)$$

Proof. A proof of this theorem can be obtained by direct substitution of C_1^0-cofactors by C_r^{r-1}-cofactors in the proof of Theorem 5.1. □

2.6 Reciprocals, liftings and stresses

Consider a planar framework (V, E) that is in a state of static equilibrium, and assume that the framework determines a cell-decomposition $D(V, E)$ of $\mathbb{R}^2$ (assuming that the framework has vertices at infinity), or of a simply-connected region in $\mathbb{R}^2$. Consider a vertex of (V, E). The sum of vectors of stresses applied to this vertex is equal to zero. Therefore, when rotated on 90° clockwise they form a polygon (self-intersecting in general). It was noticed by Maxwell (for a proof see [84]) that the positions of rotated edges of (V, E) can be adjusted so that they form a reciprocal graph (or simply reciprocal). Each edge of this reciprocal corresponds to an edge of (V, E) and each vertex to a cell of $D(V, E)$ (one vertex corresponds to the complement of $D(V, E)$ if any). We introduce and explore a similar notion for d-manifolds in $\mathbb{R}^d$ (see also [83, 25].

The *combinatorial dual graph* $\mathcal{G}(\mathcal{M}^d)$ of a manifold $\mathcal{M}^d$ is a (multi)graph where the vertices are the d-cells of $\mathcal{M}^d$, and the edges are the internal $(d-1)$-cells of $\mathcal{M}^d$.

A *reciprocal* of a PL-realization M^d of a manifold $\mathcal{M}^d$ in $\mathbb{R}^d$ is a rectilinear realization R in $\mathbb{R}^d$ of the combinatorial dual graph $\mathcal{G}(\mathcal{M}^d)$ such that the edges of R are perpendicular to the corresponding facets. If none of the edges of a reciprocal collapses into a point, the reciprocal is called non-degenerate. Reciprocals with one fixed vertex form a linear space. Denote it by $Rec(M^d)$.

For orientable manifolds one can introduce the notion of convex reciprocal. Let $v(C_1)$ and $v(C_2)$ be vertices of a reciprocal R corresponding to adjacent d-cells C_1

and C_2. If $\vec{v}(C_2) - \vec{v}(C_1)$ is cooriented with an outer normal to C_1 at the facet shared with C_2, then the edge $[v(C_2)v(C_1)]$ is called *properly oriented*. Otherwise it is called improperly oriented. If realization M^d is an embedding and all edges of R are properly oriented, R is called a convex reciprocal (since the cycles of R corresponding to the stars of the $(d-2)$-cells are convex in this case). Reciprocals were originally considered by Maxwell [53] in connection with stresses in plane frameworks. The linear space of planar reciprocals was studied in [23]. Convex reciprocals form a cone $CRec(M^d)$ in the linear space $Rec(M^d)$. Maxwell noticed that convex reciprocals corresponded to convex liftings of planar cell-complexes. Reciprocals were also studied in [6, 23, 24, 84, 66, 67].

The general theory of the relationship between reciprocals, liftings, and stresses for 2-dimensional orientable manifolds has been developed by Crapo and Whiteley [23, 24, 84]. For an orientable d-manifold the space of reciprocals always contains the space of liftings, and is contained in the space of d-stresses. Theorem 2.5.1 and the following theorem show that if $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, all these three spaces coincide.

Theorem 2.6.1 *If $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, $Rec(M^d)$ is isomorphic to $Lift(M^d)$.*

Proof. Let R be a reciprocal for M^d, let $v(C_1)$ and $v(C_2)$ be vertices of R corresponding to d-cells C_1 and C_2. Since $\vec{v}(C_2) - \vec{v}(C_1)$ is orthogonal to F, there is $c \in \mathbb{R}$ such that $\langle \vec{v}(C_2) - \vec{v}(C_1), \mathbf{x} \rangle + c = 0$ is the equation of F. A lifting corresponding to R can be constructed as follows. If $L(\mathbf{x}; C_1)$ is an affine function corresponding to C_1, then the affine function corresponding to C_2 is given by $L(\mathbf{x}; C_1) + \langle \vec{v}(C_2) - \vec{v}(C_1), \mathbf{x} \rangle + c$. Let's fix an arbitrary affine function for a d-cell C_0. Connect a d-cell C with C_0 by a combinatorial path. We define the affine function $L(\mathbf{x}; C)$ corresponding to a d-cell C via the use of the above formula. To prove that this construction defines a lifting of M^d we have to show that for each d-cell C affine functions defined via different paths coincide. Thus, we have to show that if we start from affine function $L(\mathbf{x}; C)$, a closed cell-facet circuit starting at C and ending at C defines the same function $L(\mathbf{x}; C)$. For each circuit $[C_1, \dots, C_{n-1}, C_1]$ we have $d+1$ linear equations: d equations saying that the sum of the covectors defining the linear parts of functions $L(\mathbf{x}; C_{i+1}) - L(\mathbf{x}; C_i)$ is zero, and one equation for the free terms. The sum of the covectors defining the linear parts of the differences is always zero, since every cycle of the dual graph of $\mathcal{M}^d$ is realized in $\mathbb{R}^d$ as a rectilinear cycle of R in $\mathbb{R}^d$. The sum of the free terms is, obviously, zero for any circuit around the star of a $(d-2)$-cells: if the origin lies on this $(d-2)$-cell, all the free terms are zero. Since $H_1(\mathcal{M}^d, \mathbb{Z}^2) = 0$, an application of Lemma 2.4.2 shows that for each d-cell affine functions defined via different paths coincide. It is easy to see that $L(\mathbf{x}; C)$ determines a PL-realization of $\mathcal{M}^d$ in $\mathbb{R}^{d+1}$ where non-degenerate dihedral angles correspond to non-vanishing edges of R.

Let $L^d = L(\mathbf{x}; C)$ be a lifting of M^d, and let $L(\mathbf{x}; C_1) = \langle \vec{a}_1, \mathbf{x} \rangle + c$ and $L(\mathbf{x}; C_2) = \langle \vec{a}_2, \mathbf{x} \rangle + c$ be the affine functions determining adjacent d-cells of L^d which correspond to d-cells C_1 and C_2 of M^d. If v_1 is a vertex of a reciprocal corresponding to C_1, then the vertex corresponding to C_2 is given by $v_1 + \vec{a}_2 - \vec{a}_1$. Fix a vertex corresponding to a d-cell C_0 at the origin and construct all other vertices and edges of the reciprocal using this formula. It is easy to see that the resulting rectilinear 1-complex is a reciprocal for M^d and non-degenerate dihedral angles of L^d correspond to non-degenerate edges of the reciprocal. □

Remark 2.6.2 *As in the case of Theorem 2.5.1, the above theorem holds for PL-realizations with non-embedded cells. The proof need not any adaptations.*

In the 2-dimensional case, the connection between liftings and reciprocals was first noticed by Maxwell. The isomorphism between spaces of liftings and reciprocals for manifolds in $\mathbb{R}^2$ was proved by Crapo and Whiteley [23, 24]. The relationship between convex liftings of cell-decompositions of $\mathbb{R}^d$ and convex reciprocals was also shown in [5]. Crapo and Whiteley found the exact connection between reciprocals and liftings for general manifolds. The following theorem was proved in [26] for orientable 2-manifolds, but the proof works for general dimension and for non-orientable manifolds.

Theorem 2.6.3 *(Crapo, Whiteley) Let M^d be a PL-realization of a manifold $\mathcal{M}^d$ in $\mathbb{R}^d$. There is a linear endomorphism from the space of liftings to the space of reciprocals. Let $\mathfrak{C}$ be a generating system of 1-cycles of $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ on the dual cell-decomposition of $\mathcal{M}^d$. A reciprocal R has a corresponding lifting if and only if for each cycle of $\mathfrak{C}$ the corresponding* oriented *cycle $c = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ of R satisfies the following condition*

$$\sum_c \langle \mathbf{v}_{i+1} - \mathbf{v}_i, \mathbf{p} \rangle = 0$$

where the sum is over all edges $\mathbf{v}_i\mathbf{v}_{i+1}$ of oriented cycle c, and $\mathbf{p}$ is an arbitrary point on the hyperplane of the $(d-1)$-cell corresponding to edge $\mathbf{v}_i\mathbf{v}_{i+1}$ of R.

The condition in Theorem 2.6.3 is called a *moment condition* (Crapo, Whiteley): it guarantees that the sum over the cell–facet cycle c of the free terms of the affine functions defined by the covectors $\mathbf{v}_i\mathbf{v}_{i+1}$ is zero.

2.7 Voronoi diagrams: duality, projections and stresses

The aim of this section is to show some important connections between stresses, liftings and weighted Dirichlet-Voronoi and Delaunay diagrams. A point s in Euclidean space $\mathbb{R}^d$ is said to be additively weighted if it is associated with a real number $w(s)$ referred to as the weight of s. The weighted distance from s is

$$\mathbf{d}^2(s, x) - w(s)$$

where $\mathbf{d}(s, x)$ denotes the Euclidean distance between points s and x.

Let S be a discrete set of additively weighted points in $\mathbb{R}^d$, such that all weights are bounded in absolute value. Points of S are called sites. A point $x \in \mathbb{R}^d$ belongs to the *Dirichlet-Voronoi domain* of a site $p \in S$ with weight $w(p)$ if and only if

$$\mathbf{d}^2(p, x) - w(p) \leq \mathbf{d}^2(s, x) - w(s)$$

for all $s \in S$.

It is easy to see that each non-empty domain of a weighted Dirichlet-Voronoi diagram is a convex polyhedron and that these polyhedra form a face-to-face decomposition of $\mathbb{R}^d$. Such a decomposition is called an *(additively) weighted Dirichlet-Voronoi diagram* (also often referred to as a weighted Dirichlet decomposition, or a power diagram). If all weights are equal, then the diagram is referred to as a Dirichlet-Voronoi diagram. Let Δ be a weighted Dirichlet-Voronoi diagram. Then Δ can be represented as a weighted Dirichlet-Voronoi diagram in different ways. The location and the weight of at least one site can be chosen arbitrarily. (Notice that weighted sites can lie outside their domains.) Weighted Dirichlet-Voronoi diagrams have many applications. (For more information on weighted Dirichlet-Voronoi diagrams see [7]). Algorithms constructing weighted Dirichlet-Voronoi diagrams and data structures related to these diagrams are described in [7, 36]. A discrete set of points S in $\mathbb{R}^d$ uniquely defines a special decomposition of the convex hull of S which is combinatorially and metrically dual to the Dirichlet-Voronoi diagram of S [30, 31]. Such decomposition is called a Delaunay decomposition. Every d-cell of that decomposition is inscribed into a sphere which does not contain points of S in its interior.

Let S be a discrete set of points in $\mathbb{R}^d$. A convex polytope P in $\mathbb{R}^d$ is called a Delaunay cell of the system of points S if:

1) all vertices of P belong to S;

2) there is a sphere circumscribed around P;

3) no points of S except the vertices of P lie inside or on the sphere.

Delaunay cells form a face-to-face decomposition of *conv* S. This decomposition is defined uniquely by S. Delaunay decompositions have many applications in computational geometry, mesh generation, the theory of lattices, mathematical crystallography, etc. One of the generalizations of Delaunay decompositions, the weighted Delaunay decompositions, are intensively employed in computational geometry [37]. For generically distributed sites and weights, the weighted Delaunay decomposition is a triangulation. Such triangulations are called *regular*. A sphere circumscribed around a cell of a Delaunay decomposition is called an empty sphere. Informally, a Delaunay decomposition consists of cells which sit in their own empty spheres. Generalizing Delaunay diagrams to weighted Delaunay diagrams we replace standard Euclidean spheres by virtual spheres defined through weighted distances.

Let S be a discrete set of points in $\mathbb{R}^d$. Let W be a set of real weights bounded in absolute value which are associated with these points. A polytope $P \subset \mathbb{R}^d$ is called a weighted Delaunay cell of the system of weighted points (S, W) if:

1) all vertices of P belong to S;

2*) there is a point $c(P) \in \mathbb{R}^d$ called the weighted center of P and $r(P) \in \mathbb{R}$ such that for every vertex v of P:

$$\mathbf{d}^2(c(P), v) - w(v) \ = \ r(P)$$

3*) for each $s \in S$ which is not a vertex of P:

$$r(P) < \mathbf{d}^2(c(P), s) - w(s)$$

Weighted Delaunay cells form a face-to-face decomposition of *conv* S.

The weighted vertices of a weighted Delaunay decomposition are the weighted sites for the dual weighted Dirichlet-Voronoi diagram. The duality between weighted Delaunay and Dirichlet-Voronoi decompositions can be proved exactly in the same way as the duality for ordinary Dirichlet-Voronoi and Delaunay decompositions (see [30, 31]). The duality between Dirichlet-Voronoi diagrams and Delaunay decompositions can be naturally generalized in the following way.

Let Δ be an arbitrary locally finite decomposition of $\mathbb{R}^d$ into compact convex polyhedra. A decomposition Δ^* of $\mathbb{R}^d$ is called dual of Δ if the following conditions hold:

1. *combinatorial duality:* there is a one-to-one correspondence between m-dimensional faces of Δ^* and $(d-m)$-dimensional faces of Δ, $0 \leq m \leq d$; this correspondence induces an isomorphism between the incidence graphs (infinite) of Δ and Δ^*;

2. *orthogonality and proper orientation:* 1-skeleton of Δ^* is a convex reciprocal for Δ.

It is easy to see that this relationship is reciprocal, i.e. if Δ^* is dual of Δ, then Δ is dual of Δ^*. The above definition captures all properties of the duality between ordinary (or weighted) Dirichlet-Voronoi and Delaunay decompositions except for the relationship between Voronoi sites and vertices and Delaunay centers and vertices. The property that Delaunay vertices are Voronoi centers and vice-versa does not find its generalization in this construction. One can ask what does the existence of a dual decomposition for an arbitrary Δ imply in terms of the geometry of Δ? In this section we will show that two decomposition are dual if and only if it is possible to present one of them as a weighted Voronoi diagram (S, W) and the other as the dual Delaunay decomposition (later we show that these two properties are equivalent). The notion of dual decomposition can also be regarded as a d-dimensional analog of the notion of planar convex reciprocal.

If Δ^* is a dual of Δ, then any decomposition obtained from Δ^* by scaling and translating is also a dual of Δ. It is obvious that the vector sum of any two dual decompositions of Δ is again a dual decomposition for Δ. Thus all decompositions (considered up to translation) which are dual for decomposition Δ form a cone $Dual(\Delta)$.

Theorem 2.7.1 *A decomposition Δ of $\mathbb{R}^d$ has a dual decomposition if and only if Δ has a d-tension. The cone of dual decompositions $Dual(\Delta)$ is isomorphic to the cone of tensions $Tension_d(\Delta)$.*

Proof. Let Δ^* be a dual for Δ. By definition, $Sk^1(\Delta^*)$ is a non-degenerate convex reciprocal for Δ. Thus, by Theorem 6.1 we have a monomorphism from the cone of dual decompositions into the cone of tensions. In fact, these cones are isomorphic. The cone of tensions is isomorphic to the cone of convex liftings by the results of Section 5. Consider a convex lifting of Δ. Let $p \in \mathbb{R}^{d+1}$ be a point lying above this surface. Intersections of cones dual to the stars of vertices of the lifting with $\mathbb{R}^d$ are convex polyhedra which form a face-to-face decomposition of $\mathbb{R}^d$. This decomposition is a dual of Δ by construction. □

Paschinger [62] proved that a finite decomposition of $\mathbb{R}^d$ is the projection of a convex polyhedral surface if and only if it is a weighted Voronoi diagram. This theorem holds for infinite decompositions by polyhedra with finite number of vertices as well (adopting the proof is straightforward). The following theorem follows from Theorem 5.1, Theorem 6.1 and Aurenhammer's theorem. For $d = 2$ this theorem was proved in [4].

Theorem 2.7.2 *A decomposition of $\mathbb{R}^d$ by finite convex polyhedra is a weighted Voronoi decomposition if and only if it has a d-tension.*

If (S_1, W_1) and (S_2, W_2) are two diagrams representing a decomposition Δ, then there is a collection of weights W, such that $(S_1 + S_2, W)$ represents Δ. If a weighted diagram (S, W) represents a decomposition Δ, and $h(S)$ is a homothetic image of S, then there is a set of weights W' such that Δ is a weighted diagram with site set $h(S)$ and weight set W'. The same is true about translating the set of sites, i.e. if $t(S)$ is a translated copy of S, then there is a set of weights W' such that Δ is a weighted diagram with site set $t(S)$ and weight set W'. (All these statements can be proved via the use of the arguments from the first part of the proof of the Theorem 6.5; essentially it is the idea of quality translation over the links of $(d-2)$-cells). Hence, diagrams representing Δ constitute a cone $WVor(\Delta)$ whose elements are sets of weighted sites defined up to translation such that their diagrams give partition Δ. A d-tension on Δ defines an element of $WVor(\Delta)$ uniquely. The preceding theorems imply the following proposition.

Proposition 2.7.3 *$WVor(\Delta)$ is isomorphic to the cone of dual decompositions $Dual(\Delta)$ (up to translation) and to the cone of d-tensions $Tension_d(\Delta)$.*

Ash and Bolker [3] proved that a plane decomposition by finite convex polyhedra is a weighted diagram if and only if it is a section of a higher-dimensional Voronoi decomposition. As a non-trivial application of this theorem one can mention that any rhombic Penrose tiling can be represented by a weighted diagram, since it is a section of the standard decomposition of $\mathbb{R}^5$ by cubes. In fact, the Ash-Bolker theorem is d-dimensional, as their proof does not depend on d. There is an easy way to establish a correspondence between weighted Voronoi diagrams and sectional Voronoi diagrams. Assume that all weights are negative. (Since all weights are bounded in absolute value, we can always add to all weights an appropriate constant and make them negative.) A weighted site s with coordinates $(x^1, ..., x^d)$ and weight $w(s)$ corresponds to a not-weighted site in $\mathbb{R}^{d+1}$ with coordinates $x^1, ..., x^d, \sqrt{-w_s}$. Conversely, if h_s is the distance between $\mathbb{R}^d$ and a site s in $\mathbb{R}^{d+1}$ whose (not-weighted) Voronoi domain intersects $\mathbb{R}^d$, then the corresponding site on $\mathbb{R}^d$ has the same first d coordinates and weight $-|h_s|^2$. The following proposition is a direct consequence of the Ash and Bolker theorem and Theorem 5.1.

Proposition 2.7.4 *A decomposition of $\mathbb{R}^d$ by finite convex polyhedra has a d-tension if and only if it is the section of a $(d+1)$-dimensional Voronoi diagram.*

The following theorem shows that the classes of weighted Voronoi diagrams and Delaunay decompositions coincide (the author has not found this statement anywhere in literature).

Theorem 2.7.5 *A decomposition Δ of $\mathbb{R}^d$ by convex polyhedra is a weighted Voronoi diagram if and only if Δ constrainted to conv $Sk^0(\Delta)$ is a weighted Delaunay decomposition.*

Proof. If Δ is weighted Voronoi, it has the dual Delaunay decomposition Δ^* of *conv* $Sk^0(\Delta)$. A k-cell of this decomposition is the convex hall of the weighted Voronoi sites of d-cells making full contact in a $(d-k)$-face of Δ. $Sk^1(\Delta)$ is a reciprocal for Δ^*. We want to show that the vertices of $Sk^1(\Delta)$ can be taken for the sites of a weighted Voronoi diagram representing Δ^*. Fix zero weight for a vertex of $Sk^1(\Delta)$ and then define weights on the other vertices of $Sk^1(\Delta)$ by weight transfer via edge-paths on $Sk^1(\Delta)$. Let v and v_1 be adjacent vertices of $Sk^1(\Delta)$. If a vertex v is assigned weight $w(v)$ and x is the intersection point of the line spanned by $[vv_1]$ with the plane spanned by the corresponding facet F of Δ^*, then the site v_1 is assigned weight:

$$w(v_1) = \mathbf{d}^2(x, v_1) - \mathbf{d}^2(x, v) + w(v).$$

This assignment of weights makes the plane spanned by F equidistant (in terms of weighted distances) from the weighted sites $(v, w(v))$ and $(v_1, w(v_1))$. A simple check shows that such transfer of weights is well defined over the cycles of $Sk^1(\Delta)$ corresponding to the stars of all $(d-2)$-cells of Δ^*. Therefore by Lemma 4.2 weights can be assigned to all vertices of $Sk^1(\Delta)$ turning Δ^* into a weighted Voronoi diagram. Sites of a weighted Voronoi diagram are vertices for the dual weighted Delaunay diagram. Therefore vertices of Δ can be associated with weights so that D constrainted to *conv* $Sk^0(\Delta)$ can be regarded as a weighted Delaunay decomposition.

Let D be weighted Delaunay. It has the dual weighted Voronoi decomposition V. By Theorems 7.1 and 7.2 D is weighted Voronoi. (In the case when $convSk^0(\Delta) \neq \mathbb{R}^d$, D is a constrained weighted Voronoi diagram). □

The straightforward adaptation of the Euclidean theory of weighted diagrams, duality and tensions given in this section for the spherical case gives a somewhat more symmetric form of the theory for spherical complexes (see also Section 12 and [24] for the 2-dimensional case).

Theorem 2.7.6 *The following properties of a spherical d-complex Δ with convex cells are equivalent:*

1) Δ is a weighted Voronoi diagram in $\mathbb{S}^d$;

2) Δ is a weighted Delaunay decomposition of $\mathbb{S}^d$;

3) Δ is the central projection of a convex $(d+1)$-polytope in $\mathbb{R}^{d+1}$;

4) Δ has a "spherical" d-tension.

2.8 Combinatorics of $\mathcal{M}^d$ and $Lift(M^d)$.

The following obvious observations give important implications for the analysis of stresses and liftings of $(d-k)$-primitive manifolds.

Proposition 2.8.1 *Assume that the star $St(C^{d-k})$ of a $(d-k)$-cell C^{d-k} in a d-manifold contains $k+1$ $(d-1)$-cells. If $St(C^{d-k})$ is realized in $\mathbb{R}^d$ generically (no pair of $(d-1)$-cells lie on the same hyperplane) and the coefficient of d-stress for one of its $(d-1)$-cells is fixed, then the values of d-stresses for the other $(d-1)$-cells are uniquely determined. If the d-cells of $St(C^{d-k})$ are convex and $St(C^{d-k})$ is embedded, then all coefficients of stresses have the same sign.*

Proposition 2.8.2 *Assume the star $St(C^{d-k})$ of a $(d-k)$-cell C^{d-k} contains $k+1$ $(d-1)$-cells. If $St(C^{d-k})$ is realized in $\mathbb{R}^d$ generically and a lifting of the star of a*

$(d-1)$-cell of $St(C^{d-k})$ is fixed, then the lifting of $St(C^{d-k})$ is uniquely determined. If the d-cells of the star are convex and the star is embedded, then all dihedral angles of the lifting are either convex or concave.

The star of a $(d-2)$-cell can be either embedded or realized with a self-intersection. If the star is $(d-2)$-primitive, then there is only one type of PL-realization with a self-intersection (under our definition of a PL-realization), namely, where two d-cells have the same orientation, and the third cell has the opposite one. In the latter case we will say that the star is folded.

Let $[x, y]$ be a segment in $\mathbb{R}^d$. A combinatorial path $[C_1, \ldots, C_n]$ of d-cells of a PL-realization M^d is *strung* on $[x, y]$ if:

1. $x \in C_1$, $y \in C_n$;

2. $$[x, y] \subset \bigcup_{i=1}^{n} |C_i|$$

3. for any $i \neq j$ $|C_i| \bigcap |C_j|$ is either a common facet of C_i and C_j or empty.

Let $L(\mathbf{x}; C)$ be a lifting of M^d. If for any $x, y \in |M^d|$ a combinatorial path strung on $[x, y]$ is lifted convex up (down), then $L(\mathbf{x}; C)$ is called locally convex (concave).

Theorem 2.8.3 *Let $\mathcal{M}^d$ be $(d-2)$-primitive, and each d-cell of $\mathcal{M}^d$ has an internal $(d-2)$-face. Assume that for each d-cell C of $\mathcal{M}^d$ set $(\partial C \backslash \partial \mathcal{M}^d)$ is a strongly connected $(d-1)$-pseudomanifold. Let M^d be a PL-realization of $\mathcal{M}^d$ where all cells are convex. If there is a sharp lifting of M^d, then for any strongly connected subcomplex of M^d which is actually embedded the lifting is either locally concave or convex. The lifting is unique up to the choice of a d-face and a dihedral angle.*

Proof. Let $[C_0, C_1, C_2]$ be a path of d-cells on $\mathcal{M}^d$ such that in the realization M^d $\cup_{i=0}^{2} C_i$ is embedded into $\mathbb{R}^d$. Let $L(\mathbf{x}; C)$ be a sharp lifting. Assume $L(\mathbf{x}; C)$ is convex on $([C_0, C_1])$. It is enough to show that the convexity of $l([C_0, C_1])$ implies the convexity of $l([C_1, C_2])$. Denote by E_1 a $(d-2)$-cell shared by C_0 and C_1. Consider a d-cell sharing E_1 with C_0 and C_1. Let C_1 and this d-cell share a facet F_1 and let F_2 be the facet which is common for C_1 and C_2. Connect F_1 and F_2 by a combinatorial path $\mathfrak{p}$ on $\partial\, C_1$ which consists of internal facets of $\mathcal{M}^d$ sharing internal $(d-2)$-cells (such path exists by the conditions of the theorem). Let E_2 be a $(d-2)$-cell shared by F_2 and the preceding $(d-1)$-cell in $\mathfrak{p}$. Since $\mathcal{M}^d$ is $(d-2)$-primitive, the type of the realization of a star cannot switch on the surface of C_1. Therefore $St(E_2)$ has the same type of realization as $St(E_1)$. If $St(E_1)$ is embedded in $\mathbb{R}^d$, then the stars of all $(d-2)$-cells from $\mathfrak{p}$ are embedded in $\mathbb{R}^d$. Therefore, $St(E_2)$ is also embedded and both $St(F_1)$ and $St(F_2)$ are lifted in the same way. If $St(E_1)$ is folded, then the stars of all $(d-2)$-cells from $\mathfrak{p}$ are folded. An application of Proposition 8.2 shows that if $St(F_1)$ is lifted onto a convex dihedral angle, then $St(F_2)$ is lifted onto a convex dihedral angle too. In both cases $L(\mathbf{x}; C)$ has necessarily the same type on $[C_1, C_2]$ as on $[C_0, C_1]$. Therefore, $L(\mathbf{x}; C)$ is convex on $[C_0, C_1, C_2]$. $\square$

Corollary 2.8.4 *[67] If a $(d-2)$-primitive decomposition of $\mathbb{R}^d$ by convex polyhedra has a non-trivial lifting, then this lifting is sharp and globally convex or concave. The lifting is unique up to the choice of a supporting plane and a dihedral angle.*

As an implication of the above theorem we have the following.

Corollary 2.8.5 *Let M^d be a PL-realization in $\mathbb{R}^d$ of a $(d-2)$-primitive manifold and $L^d = L(\mathbf{x}; C)$ be its sharp lifting to $\mathbb{R}^{d+1}$. If*

1) $|M^d|$ is convex;

2) for any $\mathbf{x}$ from $|M^d|$ there are exactly two points in L^d which project onto $\mathbf{x}$;

3) all cells of M^d are convex,

then L^d is a convex sphere in $\mathbb{R}^{d+1}$.

An analog of theorem 8.3 for stresses can be formulated as follows.

Proposition 2.8.6 *Let $\mathcal{M}^d$ be $(d-2)$-primitive, and each d-cell of $\mathcal{M}^d$ has an internal $(d-2)$-face. Assume for each d-cell C of $\mathcal{M}^d$ the set $\partial C \backslash \partial \mathcal{M}^d$ is a strongly connected $(d-1)$-pseudomanifold. If the stars of all internal $(d-2)$-cells of M^d are generic, then dim $Stress_d(M^d)$ is equal to either 1 or 0.*

Notice that even though $Stress_d(M^d)$ and $Lift(M^d)$ are isomorphic to either $\mathbb{R}$ or 0 for any realization of a closed $(d-2)$-primitive manifold $\mathcal{M}^d$ with generic stars of $(d-2)$-cells, they need not coincide.

2.9 Sharp liftings of $(d-k)$-primitive manifolds

It is natural to ask when there is a sharp lifting of a "flat" PL-realization M^d, especially when M^d is a decomposition of $\mathbb{R}^d$ or a PL-realization of a sphere in $\mathbb{R}^d$. In this section we give an improvement on a well-known theorem of Davis [28] on the existence and uniqueness of a sharp lifting for a 0-primitive (simple) cell-decomposition of $\mathbb{R}^d$, $d > 2$. We also improve a theorem by Whiteley [83] on the existence and uniqueness of a lifting for PL-realizations of 0-primitive (simple) d-spheres ($d > 2$) in $\mathbb{R}^d$. If $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, then the existence of a sharp lifting is equivalent to the existence of an all-non-zero stress (see Section 5). As before, we assume that the stars of all $(d-2)$-cells are generic. We also impose a quite natural restriction on the combinatorics of $\mathcal{M}^d$.

Let C be a d-cell in a d-manifold $\mathcal{M}^d$ with boundary. We understand by *recl* $(\partial C \setminus (\partial \mathcal{M}^d \backslash \partial Sk_{d-3}(\mathcal{M}^d)))$ a cell-complex which is obtained from ∂C by removing open $(d-1)$ and $(d-2)$-cells which belong to $\partial \mathcal{M}^d$, and then augmenting all remaining $(d-1)$-cells of C by their relative boundaries. Informally it means that after removing boundary (with respect to $\mathcal{M}^d$) $(d-1)$-cells from ∂C we tear along those $(d-2)$-cells which occur on the boundary of $\mathcal{M}^d$ (e.g. see figure 2.4; in this example the manifold is a 3-cube decomposed into five simplexes and $C = (abcd)$).

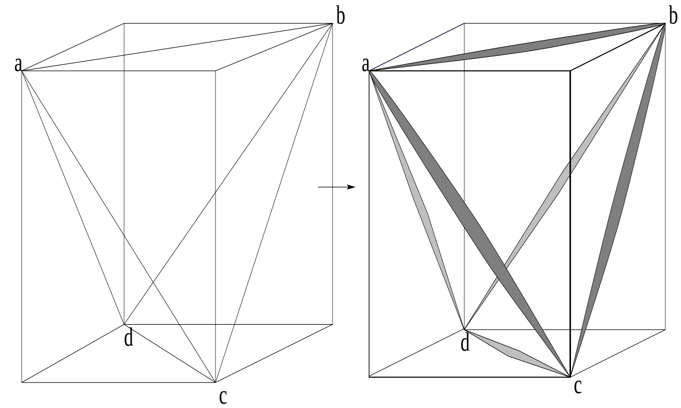

Figure 2.4: Illustration for Condition 1.9.1

Condition 2.9.1 *For any d-cell C the complex recl $(\partial C \setminus (\partial\mathcal{M}^d \backslash \partial Sk_{d-3}(\mathcal{M}^d)))$ can be represented as $\cup F_i(C)$, where*

1) each $F_i(C)$ is a $(d-1)$-manifold (possibly with boundary) with $H_1(F_i(C), \mathbb{Z}_2) = 0$.

2) $F_i(C)$ and $F_j(C)$ do not share internal $(d-2)$-cells of $\mathcal{M}^d$, if $i \neq j$;

Notice that some important classes of manifolds such as *cell-partitions of* $\mathbb{R}^d$, *closed compact manifolds, and convex tilings of convex regions in* $\mathbb{R}^d$ satisfy the above condition. If $d > 2$, the local geometric properties of M^d (i.e. geometry of the stars) have a stronger effect on $Stress_d(M^d)$ and $Lift(M^d)$ than in the planar case, as illustrated by the following theorem.

Theorem 2.9.2 *A PL-realization of a $(d-2)$-primitive manifold $\mathcal{M}^d$ $(d > 2)$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, satisfying Condition 9.1, admits an all-non-zero d-stress if and only if the star of each internal $(d-3)$-face has an all-non-zero d-stress. If the realization is an embedding and an all-non-zero stress exists, then there is a global tension.*

Theorem 2.9.3 *A PL-realization of a $(d-3)$-primitive manifold $\mathcal{M}^d$ $(d > 2)$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ and satisfying Condition 9.1 has an all-non-zero d-stress.*

It is easy to see that Theorem 9.3 easily follows from Theorem 9.2, since a generic realization of the star of a $(d-3)$-cell with only four d-cells always has an all-non-zero stress which is unique up to scale.

Proof of Theorem 9.2 We call a subset of facets of $\mathcal{M}^d$ an independent component if stresses on the facets of this subset do not depend on the choice of stresses for the facets which do not belong to this subset. All internal facets can be partitioned into independent components I_k. Choose a facet in each component and fix an arbitrary non-zero stress for each chosen facet. We have to show that the fixed set of stresses uniquely determines an all-non-zero stress on all of M^d. To prove this we will translate stress via chains of adjacent facets. The stress transfer has to be defined independent of path. By Proposition 8.2 a stress on a facet belonging to the primitive star of a $(d-2)$-cell determines the coefficients of stresses of the other two facets. Therefore, if F and F' are two adjacent facets, then there are two non-trivial reciprocal linear maps $l([F, F'])$ and $l([F', F])$ associated with pairs $[F, F']$ and $[F', F]$. Consider a graph G whose vertices are internal facets of $\mathcal{M}^d$, and whose edges are internal $(d-2)$-cells of $\mathcal{M}^d$, where an edge exists between facet F and F' precisely when F and F' share a common internal $(d-2)$-cell. (Notice that G is not the dual graph $\mathcal{G}(\mathcal{M}^d)$.) The process of assigning coefficients of stresses to facets of $\mathcal{M}^d$ can be regarded as the process of assigning real numbers to vertices of G via the use of functions $l([F, F'])$. In this model, each edge of G is associated with a pair of reciprocal linear functions. G consists of connected components G_k corresponding to I_k (see above). For each G_k, Let's assign a real number $s_k \in \mathbb{R}$ to an arbitrarily chosen vertex of G_k. We have to show that real numbers can be assigned to all vertices of G_k so that the numbers corresponding to adjacent vertices are connected by the pair of linear maps assigned to their common edge. This is the same as to show that quality transfer over each cycle is well-defined. We can regard G as a sub-graph of the 1-skeleton of the dual

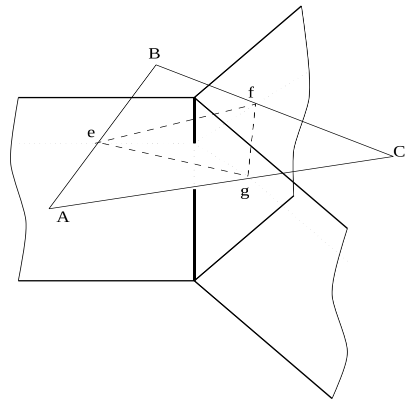

Figure 2.5: Illustration for Theorem 1.9.3

(combinatorial) cell-decomposition of $\mathcal{M}^d$. Denote by $\mathcal{D}$ this dual decomposition. Since $\mathcal{M}^d$ is $(d-2)$-primitive, all 2-cells of $Sk^2(\mathcal{D})$ corresponding to internal $(d-2)$-cells are (combinatorially) triangles. Such 2-cell of $Sk^2(\mathcal{D})$ can be decomposed into four smaller triangles, one of them being a cycle of G. Let (ABC) be a triangle of $\mathcal{D}$ where the vertices correspond to d-cells A, B and C forming the star of an internal $(d-2)$-cell and the edges correspond to the facets of this star. If e, f and g are the "centers" (in the combinatorial sense) of the sides AB, BC and CA, then the sub-triangulation of (ABC) consists of (efg), (eAg), (eBf), (fCg), where (efg) is a cycle of G (see Figure 2.5).

In the similar way we decompose all 2-cells of $Sk^2(\mathcal{D})$ corresponding to boundary $(d-2)$-cells of $\mathcal{M}^d$. If the star of a boundary $(d-2)$-cell contains m d-cells, then the subdivision of the corresponding 2-cell of $Sk^2(\mathcal{D})$ consists of m triangles and one $(1+m)$-gon. Consider an arbitrary 1-cycle $\mathfrak{c}$ on G. Since $H_1(Sk^2(\mathcal{D}), \mathbb{Z}_\in) = \prime$, then $\mathfrak{c} = \partial\, \Delta^2$, where Δ^2 is a 2-chain from $\mathcal{C}_2(Sk^2(\mathcal{D}), \mathbb{Z}_\in)$. The chain Δ^2 can be represented as the sum of 2-cells from the subdivision of $Sk^2(\mathcal{D})$ described above. Cells of this subdivision of $Sk^2(\mathcal{D})$ can be conveniently partitioned into two groups, namely those which are of type (efg) (the boundaries of such cells are the links of internal $(d-2)$-cells of $\mathcal{M}^d$), and those which are of type (eAg) (see above). Therefore $\Delta^2 = \Delta_1^2 + \Delta_2^2$, where Δ_1^2 is the sum of 2-cells of the first type and Δ_2^2 is the sum of 2-cells of the second type. The boundaries of 2-cells of the first type are actually cycles of G corresponding to internal stars of $(d-2)$-cells of M^d (clearly, 2-cells that correspond to boundary $(d-2)$-cells of $\mathcal{M}^d$ do not occur in Δ_1^2). We will refer to such cycles as *link-cycles.* Let $\partial\Delta_2^2 = \sum_k \mathfrak{c}(C_k^d)$, where $\mathfrak{c}(C_k^d)$ is a cycle connecting only the facets of the d-cell C_k^d. We will call such cycles *surface cycles*, for they can be thought of as lying on the surfaces of d-cells. Since the choice of the cycle $\mathfrak{c}$ was arbitrary, the link-cycles and the surface cycles form a generating system of $H_1(G, \mathbb{Z}_2)$. We only need to prove that quality transfer is well-defined over surface cycles. One can think of quality transfer over a surface cycle connecting facets of a d-cell C^d as of quality transfer over paths of $(d-1)$-cells of $\partial\, C^d$. Lemma 4.2 provides sufficient conditions that quality transfer on a manifold with $H_1 = 0$ over $\mathbb{Z}_2$ is well-defined. Before applying Lemma 4.2 we need to obtain some information on circuits over the links of $(d-3)$-cells on ∂C^d. By Condition 9.1 *recl* $(\partial C^d \backslash (\partial \mathcal{M}^d \backslash \partial Sk^{d-1}(\mathcal{M}^d))) = \cup F_m(C^d)$, where for each manifold $F_m(C^d)$ all $(d-3)$-cells internal with respect to $F_m(C^d)$ are internal with respect to $\mathcal{M}^d$. By hypothesis, there is an all-non-zero stress for the star of each internal $(d-3)$-cell of $\mathcal{M}^d$ which is, of course, unique up to scale. It implies that the quality transfer over the link of each internal $(d-3)$-cell of $\partial\, C^d \backslash \partial\mathcal{M}^d$ is well-defined. Quality transfer is well-defined on $\partial\, C^d \backslash \partial\mathcal{M}^d$, if and only if it is well-defined on each $F_m(C^d)$. By Condition 9.1, all such components are manifolds with $H_1 = 0$ over $\mathbb{Z}_2$. By Lemma 4.2, quality transfer over every circuit of $(d-1)$-cells of $F_m(C^d)$ is well-defined. Therefore, quality transfer is well-defined over surface cycles. Thus it is well-defined on G. It therefore follows that an all-non-zero stress on M^d exists and is unique up tao scale for each independent component I_k. $\square$

Corollary 2.9.4 *A PL-realization of a $(d-2)$-primitive $\mathcal{M}^d$ $(d > 2)$ satisfying Con-*

dition 9.1 and $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, admits a sharp lifting if and only if the star of each $(d-3)$-face has a sharp lifting. If $\mathcal{M}^d$ is closed, then any non-trivial lifting is sharp and unique up to the choice of an affine function and a dihedral angle.

Assume all cells of M^d be convex. Under the conditions of the preceding theorem, any sharp lifting is either convex or concave on each sub-complex of $\mathcal{M}^d$ which is embedded into $\mathbb{R}^d$.

Corollary 2.9.5 *Any PL-realization of a $(d-3)$-primitive $\mathcal{M}^d$ $(d > 2)$ satisfying Condition 9.1 and $H_1(\mathcal{M}^d, \mathbb{Z}^2) = 0$ has a sharp lifting. If the realization is an embedding and all cells are convex, then there is a convex lifting.*

These results improve a well-known theorem of Davis [28] on the existence and uniqueness of a convex lifting for a 0-primitive (simple) finite decomposition of $\mathbb{R}^d$ into convex polyhedra as well as a similar theorem by Whiteley [83] for spheres. Combining results of Theorem 8.3 and the above corollaries we obtain the following result generalizing a theorem by Whiteley [83], where the manifold is required to be 0-primitive.

Theorem 2.9.6 *Let M^d $(d > 2)$ be a PL-realization of a $(d-3)$-primitive manifold. If*

1) $|M^d|$ is convex;

2) the realization M^d covers each point of the interior of $|M^d|$ twice;

3) all cells of M^d are convex,

then M^d has a unique lifting (up to the choice of a supporting d-plane and a dihedral angle) onto a convex sphere in $\mathbb{R}^{d+1}$.

2.10 Algorithmic analysis of stresses and liftings.

It is useful to consider liftings from a somewhat more general point of view, adopted in a field of artificial intelligence called *scene analysis of polyhedral pictures* (see [74]).

A polyhedral incidence structure is a triple $S = (V, F, I)$: an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subset V \times F$, where $V \times F$ denotes the set of all ordered pairs whose first elements are taken from V and second elements from F.

A polyhedral d-picture is a pair $(S, \mathbf{p})$: an incidence structure S, and a location map $\mathbf{p} : V \to \mathbb{R}^d$ assigning coordinates to vertices. A lifting is a pair of "lifting" maps (LF, LP): a map $LF : F \to \mathbb{R}^{d+1}$, $f \mapsto A^f$ assigning affine functions on $\mathbb{R}^d$ to faces, and a map $LP : V \to \mathbb{R}$, $\mathbf{p}(v) \mapsto z^{\mathbf{p}(v)}$ assigning points in $\mathbb{R}^{d+1}$ to vertices of V, such that for every $v \in V$ and for every incidence (v, f) $z^{\mathbf{p}(v)} = A^f(\mathbf{p}(v))$. Liftings of $(S, \mathbf{p})$ form a linear space $Lift(S, \mathbf{p})$. A lifting is *sharp* if any two affine functions corresponding to faces with at least d common vertices are distinct.

It is worth mentioning that the problem of recognizing whether a planar polyhedral picture can be a interpreted as the projection of a spatial polyhedral scene satisfying given constraints is one of the central problems in computer line drawing [74]. Similar problems for parallel drawings occur in computer aided geometric design (see [85]). A parallel drawing of a polyhedral picture is another realization of the incidence structure of this picture such that the hyperplanes of the corresponding faces are parallel. Since parallel drawings in $\mathbb{R}^d$ are equivalent to liftings from $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ (see [85]), the ability to find a lifting of a given type allows us to construct a parallel drawing having prescribed qualitative properties. Thus algorithms constructing liftings may be useful to computer aided design and computer vision [85, 74]. As an example of another application for algorithms of this sort one can mention perturbing a non-simple decomposition by convex polyhedra of a region in $\mathbb{R}^d$ $(d > 2)$. If we do not know of any sharp lifting for the decomposition, then we may not be able to perturb the decomposition so that it will become 0-primitive (simple). However, if there is a method to find a sharp lifting, one can always perturb such a lifting and project it back to $\mathbb{R}^d$. The perturbed decomposition will be 0-primitive and the convexity of the cells will be preserved. By the results of Sections 7 and 9 a 0-primitive decomposition of $\mathbb{R}^d$ $(d > 2)$ is always a weighted Voronoi diagram; thus algorithms constructing sharp liftings can be used to approximate decompositions admitting sharp lifting by weighted Voronoi diagrams (see [75, 74] for applications of such approximations). Since $Tension_d(M^d) \cong WVor(M^d)$ for cell-decompositions of $\mathbb{R}^d$ by convex polyhedra, the problem of finding an all-non-zero stress is equivalent to the problem of recognition of weighted diagrams and sections of Voronoi diagrams. It can be useful in mathematical crystallography and other sciences dealing with space partitions [75, 74].

A picture $(S, \mathbf{p})$ is called *generic* if for $k \leq d+1$ no k points lie on a $k-2$-plane. All other realizations are called *singular*. The dimension of the space of liftings of a generic picture $(S, \mathbf{p})$ is minimal over all realizations $\mathbf{p}$ of the incidence structure S in $\mathbb{R}^d$. Generic realizations form an open dense subset of the space $\mathbb{R}^{|dV|}$ of all $\mathbb{R}^d$-realizations of S. Singular realizations correspond to points of an algebraic variety in the space of all $\mathbb{R}^d$-realizations. This variety is called the algebraic variety of singular realizations of S (for details see [87]).

There is a polynomial time combinatorial (i.e. not employing realization data) algorithm by Sugihara which determines whether a generic picture has a sharp lifting, and computes the dimension of the space of lifting in this case (see [45, 74, 86, 88]). In fact, one can modify this algorithm so that it will find $dim\ Lift(S, \mathbf{p})$ for any generic picture $(S, \mathbf{p})$. Imai's modification [45] of this algorithm has complexity $O(|I|^2)$. *It is an open question whether there is a polynomial algorithm which computes all maximal sub-pictures of* $(S, \mathbf{p})$ *which have sharp liftings.* Note that polyhedral pictures which occur in practical applications often have traces of symmetries, although up to round off errors all picture are generic (Sugihara, 1986).

In this paper we consider a special case, where S is the d-cell–vertex incidence structure of a fixed cell-decomposition of a manifold $\mathcal{M}^d$. Since we want every k-cells to be realized as a PL-ball living in an affine k-subspace of $\mathbb{R}^d$, it is natural to mean by the space of $\mathbb{R}^d$-realizations of $\mathcal{M}^d$ an affine subspace of $\mathbb{R}^{df_0}$ which consists of all realizations satisfying this requirement. If $\mathcal{M}^d$ is simplicial, then this space is all $\mathbb{R}^{df_0}$. Indeed, from the practical point of view the cases of orientable 2-manifolds and 3-manifolds are the most important ones (see [74, 85, 87]). Let's prescribe the type of dihedral angle of a lifting for some edges of a picture (e.g. convex versus concave, if two adjacent cells lie in different halfplanes) and ask whether there is a lifting which satisfies these conditions. More formally, the *lifting problem* is as follows.

1) Find $dim\, Lift(M^d)$.

2) Given a set of restrictions for some dihedral angles, determine whether a lifting of this type exists and construct such a lifting if it does.

For different generic realizations of $\mathcal{M}^2$ the answers to the lifting problem may be different, and therefore there are no purely combinatorial methods to answer the above questions for general instances of this problem. This is because the set of generic $\mathbb{R}^d$-realizations with the same type of lifting is a semialgebraic variety of full dimension in the space of all $\mathbb{R}^d$-realizations of $\mathcal{M}^d$. Notice that similar questions arise in the theory of tensegrity frameworks [19, 85].

One can express qualitative restrictions on dihedral angles in terms of C_1^0-cofactors (see Section 5). Let A and B be d-cells of M^d which make contact in a facet F. Notice that $c_1^0(A, B) > 0$ if and only if the affine function $L(A, \mathbf{x})$ corresponding to A is greater that the affine function $L(B, \mathbf{x})$ corresponding to B on the halfspace $< \mathbf{x}, \mathbf{n} > +c < 0$, where $< \mathbf{x}, \mathbf{n} > +c = 0$ is the supporting hyperplane of F and n is the outer normal to A at F. For example, Let's consider the case when d-cells A and B are convex (in fact, this is not important for our definitions). When A and B lie in different halfspaces with respect to the supporting hyperplane of F, $c_1^0(A,B) > 0$ if and only if the lifting of $St(F)$ is convex, and $c_1^0(A,B) < 0$ if and only if the lifting of $St(F)$ is concave. When A and B lie in the same halfspace, then $c_1^0(A,B) > 0$ if and only if the cell of the lifting corresponding to A lies above the cell of the lifting corresponding to B. The above restrictions (for $d = 2$) are standard in computer vision (see Sugihara [74]), and when $St(F)$ is embedded, are usually expressed with edge labels $+$ for convex lifting and $-$ for concave lifting (e.g. see figure 2.6). It is easy to see now that liftings which satisfy some fixed restrictions of the above type form an open convex polyhedral cone in the linear space of liftings.

If $\mathcal{M}^d$ is oriented, then the restrictions of the above type naturally give rise to dependent sets of an oriented matroid on internal facets of $\mathcal{M}^d$. Fix an orientation and a lifting for M^d. For a facet F shared by d-cells A and B define the sign of F in the *matroid of liftings* by $o(A)sgn(c_1^0(A,B))$. Facets with non-zero signs form a dependent set of this matroid. Hence, in the case of orientable $\mathcal{M}^d$ one can say that the task of our algorithms is to determine if a dependent set with given signing exists. Notice, that, in the same way, for any orientable manifold we can introduce the oriented matroids of d-stresses and reciprocals. Speaking of oriented matroids, it seems natural to add that all our results on the relationships between linear spaces of d-stresses, liftings, and reciprocals for orientable $\mathcal{M}^d$ can be reformulated in terms of morphisms between oriented matroids of stresses, liftings, and reciprocals.

Let $\mathcal{I}(\mathcal{M}^d)$ be the incidence graph of a *finite* cell-decomposition of a manifold $\mathcal{M}^d$ (see [36] for a description of incidence graphs), and let, as before, M^d be a PL-realization of $\mathcal{M}^d$ in $\mathbb{R}^d$. To store the parameters of the realization M^d with the incidence graph of combinatorial $\mathcal{M}^d$, $\mathcal{I}(\mathcal{M}^d)$ should be equipped with additional information which determines the positions of facets and d-cells of M^d (up to translation). Assume that

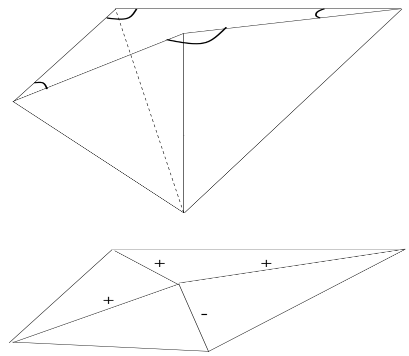

Figure 2.6: Convex versus concave

in the incidence graph $\mathcal{I}(M^d)$ storing the PL-realization M^d each node representing a facet of M^d stores the coefficients of a normal to the hyperplane on which the PL-realization of this facet lies. Since non-simplicial, non-convex and non-compact cells are allowed, this information is not sufficient for efficient algorithms. If C is a d-cell and F is its facet, the data structure should allow determining in constant time from which side of the facet F the cell C makes contacts with F (when C is a simplex, it can always be determined in constant time). Let's modify $\mathcal{I}(M^d)$ by restructuring the layer of the nodes representing the facets. For a node representing a facet with normal $(a_1, \ldots, a_d)$ we create a node-antipode representing the same facet, but with normal $(-a_1, \ldots, -a_d)$. Let $n(F; (a_1, \ldots, a_d))$ be a node of $\mathcal{I}(M^d)$ representing a facet F with equation $a_1 x^1 + \ldots + a_d x^d + c = 0$, and let F be shared by d-cells C_1 and C_2. The node $n(F; (a_1, \ldots, a_d))$ and a node that represents C_i are linked by an arc in the modified graph if and only if $\vec{a} = (a_1, \ldots, a_d)$ is an outer normal to C_i (i=1,2). In other words, in the modified incidence graph $\mathcal{I}(\mathcal{M}^d)$ (we do not change the notation) the layer which represents the facets contains two copies of each facet with different orientations of normals. In fact, the first algorithm uses only a subgraph of $\mathcal{I}(M^d)$ which stores the nodes representing d-cells, $(d-1)$-cells, and all incidences between them. Denote this sub-graph by $\mathcal{I}_{d-1}(M^d)$. The second algorithm uses $\mathcal{I}_{d-2}(M^d)$ which contains the nodes representing d-cells, $(d-1)$-cells,

$(d-2)$-cells and all incidences between them. In the description of the algorithms we will also use the combinatorial dual graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of $\mathcal{M}^d$ introduced in Section 6. Whenever it is appropriate, we identify in our descriptions d-cells of $\mathcal{M}^d$ with vertices of $\mathcal{G}$, and facets of $\mathcal{M}^d$ with edges of $\mathcal{G}$.

2.11 Algorithm for $(d-2)$-primitive partitions

2.11.1 Preliminaries to Algorithm 1

A reciprocal is called maximum if it has the maximum possible number of non-degenerate edges. The algorithm attempts to construct a maximum reciprocal for M^d and finds the dimension of $Rec(M^d)$ in linear running time in the number of internal facets (recall, that by the results of Section 6 $Rec(M^d) \equiv Lift(M^d)$).

Designing Algorithm 1 we were guided by the following analog of Theorem 2.8.3 for reciprocals. This proposition is a direct consequence of Section 2.6 and Theorem 2.9.3.

Proposition 2.11.1 *Let $\mathcal{M}^d$ be $(d-2)$-primitive, and each d-cell of $\mathcal{M}^d$ have an internal $(d-2)$-face. Assume for each d-cell C of $\mathcal{M}^d$ the set recl $(\partial C \backslash \partial \mathcal{M}^d)$ is a strongly connected $(d-1)$-pseudomanifold. If the stars of all internal $(d-2)$-cells of M^d are generic, then dim $Rec(M^d)$ is equal to either 1 or 0.*

Since the correctness of the algorithm can be guaranteed only for $(d-2)$-primitive decompositions, we assume that our input is a PL-realization of a finite $(d-2)$-primitive cell-decomposition of a manifold. We assume also that realization M^d has generic stars of $(d-2)$-cells and that the combinatorial $\mathcal{M}^d$ satisfies at least one of the following conditions:

1) $\mathcal{M}^d$ is a closed manifold,

2) $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$.

The question remains open if one can adopt the algorithm so that it handles the stars of $(d-2)$-cells that are not in general position. Of course, we can pass to a new block-partition of $\mathcal{M}^d$ by removing all facets whose realizations cannot support a non-zero stress, and amalgamating the cells neighboring at such facets into bigger blocks. This can be done on time $O(f_{d-2}(\mathcal{M}^d))$—since the manifold is $(d-2)$-primitive, $O(f_{d-2}(\mathcal{M}^d)) = O(f_{d-1}(\mathcal{M}^d))$. The resulting partition is still $(d-2)$-primitive and has all stars of $(d-2)$-cells in general position. *The problem is that the block of the new partition may not be homology disks.*

We pay attention to d-cells with parallel facets since such cells occur in tilings derived from lattices and other point systems with symmetries, and it seems quite natural to take them into account. The applicability of the algorithm to the above classes, its complexity and its robustness are discussed in Subsections 2.11.1-2.11.4.

Consider the case of $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. The above proposition implies that the space of liftings is determined by a system of at most $f^\circ_{d-1}(\mathcal{M}^d)$ equations like $x_i = cx_j$, where x_i, x_j are cofactors of facets, and c or $-c$ is the length of the corresponding edge of a maximal reciprocal. It is clear that one can read off such system of equations determining $Lift(M^d)$ from a maximal reciprocal and $\mathcal{I}_{d-1}(M^d)$ (cf. Section 10) in time $O(f^\circ_{d-1}(\mathcal{M}^d))$. In the coordinate system of x_i, the cone of liftings of a certain fixed type is determined by inequalities like $x_i < 0$ or $x_i < 0$, and therefore the question of the existence of a lifting of a certain type can be answered in time $O(f^\circ_{d-1}(\mathcal{M}^d))$.

If $H_1(\mathcal{M}^d, \mathbb{Z}_2) \neq 0$, we have to take care of the moment condition (see Section 2.6). The dimension of $Rec(M^d)$ is at most 1, and, therefore, the moment condition is satisfied for all non-trivial liftings if and only if it is satisfied for some non-trivial lifting. This can be checked in time $O(f^\circ_{d-1}(\mathcal{M}^d))$.

The overall complexity of the algorithm is $O(f_{d-1}(\mathcal{M}^d))$. The size of the input is proportional to the number of facets f_{d-1}. *Thus, Algorithm 1 solves the lifting* *problem (cf. Section 2.10) for* $(d-2)$*-primitive decompositions of closed manifolds and manifolds with trivial first homologies in optimal time.*

As in the previous section, we call a subset of internal facets of $\mathcal{M}^d$ an independent component if stresses on the facets of this subset do not depend on the choice of stresses for the facets from the complement of this subset. For example, if two facets in a $(d-2)$-primitive manifold can be connected by a chain of adjacent facets such that every two consecutive facets share an internal $(d-2)$-cell, then they belong to the same component. Often it is somewhat more convenient to describe the dependences of stresses via the use of the combinatorial dual graph $\mathcal{G}$. We will switch freely between vertices and edges of $\mathcal{G}$ and d-cells and facets of $\mathcal{M}^d$. Call a subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an independent component if the choice of stresses for facets from $\mathcal{E}'$ does not depend on the choice of stresses for facets from $\mathcal{E} - \mathcal{E}'$. All facets can be partitioned into disjoint independent components I_k in a unique way. This partition corresponds to an edge-partition of $\mathcal{G}$.

Starting with an arbitrary edge of $\mathcal{G}$, the algorithm processes an independent component of $\mathcal{G}$ to which this edge belongs. In the same way the algorithm processes all other independent components of $\mathcal{G}$ in a sequence. As the algorithm works, a dynamical data structure $LIST$ which contains all not-processed d-cells of $\mathcal{M}^d$ (i.e. their indexes) is maintained. Initially, $LIST$ has all d-cells of $\mathcal{M}^d$ in random order. We maintain links (implemented as pointers in both directions) between facets of $\mathcal{M}^d$ and corresponding elements of $LIST$. Thus, we can assume that any d-cell which is known to be in $LIST$ can be removed from $LIST$ in constant time, and that for any d-cell of $\mathcal{M}^d$ one can check in constant time whether its index is in $LIST$. (When an element of $LIST$ is deleted, its link is deleted too.) The algorithm uses $STACK$ where d-cells are pushed as the algorithm proceeds. Initially, $STACK$ is empty. The algorithm attempts to construct a reciprocal for each independent component of $\mathcal{G}$ using an inductive procedure which finds vertices of the reciprocal. Let $\mathcal{G}'$ be an independent component of $\mathcal{G}$, and $R'_k \subset R' \subset R$ ($k > 2$) be the subgraph of R corresponding to the first k constructed vertices of $\mathcal{G}'$. Let $v(A) \in \mathbb{R}^d$ denote the vertex of R'_{k+1} corresponding to a d-cell A. Assume that vertices $v(B)$ and $v(C)$ of R'_k corresponding to d-cells B and C that are adjacent to A have been constructed on the first k steps, and the construction can be extended to A. In this case one can find vertex $v(A)$ as the intersection point of the lines passing through $v(B)$ and $v(C)$, and perpendicular to the facets at which B and C contact A. Denote by $V(A; v(B), v(C))$ the function expressing this dependence (we write $V(A; v(B), v(C)) = \emptyset$ if these lines do not intersect). If $v(A)$ does not exist, R'_{k+1} and therefore reciprocal R' corresponding to $\mathcal{G}'$ collapses into a single point. After computing function $V(A; v(B), v(C))$ d-cell A is marked as *processed* (in both cases). If R' collapses into a point, all d-cells corresponding to its vertices are marked as processed. The algorithm produces the coordinates of vertices of a maximum reciprocal R and computes $dim\, Rec(M^d)$. It also formulates a system of linear equations Θ determining $Lift(M^d)$ whose variables are the C^0_1-cofactors (in our notation variable x_{AB} represents cofactor $c^0_1(A, B)$). Let $a(C)$ be the number of actually processed d-cells which are adjacent to C (this number changes as the algorithm proceeds). Initially, for all C the $a(C)$ are set equal to 0. Index j ranges over all independent components and is initialized by 1. The dimension of the j-th component is denoted by dim_j. Both $dim\, Rec(M^d)$ and dim_1 are initialized to zero.

2.11.2 Algorithm 1

INPUT: LIST containing the d-cells of $\mathcal{M}^d$ at random order; the cell–facet incidence graph $\mathcal{I}_{d-1}(\mathcal{M}^d)$: each facet having a pointer to the equation of the facet and the orientation information (see Section 2.10); mutual links between the elements of LIST and the d-cells in $\mathcal{I}_{d-1}(\mathcal{M}^d)$; empty STACK.

INITIALIZATION: Set $a(C) = 0$ for every d-cell C. Set $j = 1$. Set $dim Rec(M^d) = 0$. Set $dim_1 = 0$.

0 **while** $LIST$ is not empty;

1 **if** $STACK$ is not empty **then** take a cell A from $STACK$; mark A as processed; remove A from $LIST$;

2 **else if** $LIST$ is empty; **then** $dim\ Rec(M^d) := dim\ Rec(M^d) + dim_j$; **terminate**

else remove a cell A from $LIST$;

if there is a cell B adjacent to A in the $LIST$ **then** remove B from $LIST$; push A onto $STACK$; set $v(B)$ at the origin; mark B as processed;

else mark A as processed; $dim\ Rec(M^d) := dim\ Rec(M^d) + 1$;

j:=j+1; $dim_j := 1$; denote x_{AB} by x_j

endif

endif

endif

3 **if** $a(A) \geq 2$ **then** scan all cells adjacent to A and find two processed cells D and E that make contact with A in non-parallel facets;

if $v(A; v(D), v(E))$ exists **then** add $''x_{AD} = \langle v(A)v(D), n(A,D)\rangle x_j''$ and $''x_{AE} = \langle v(A)v(E), n(A,E)\rangle x_j''$ to Θ;

else $dim_j := 0$; add $''x_j = 0''$ to Θ;

for each processed C adjacent to A

add $''x_{AC} = 0''$ to Θ;

endfor

endif

else if $a(A) = 1$ **then** find the processed d-cell adjacent to A (denote it by F);

find the common facet C^{d-1} of A and F; find a point in $\mathbb{R}^d$

at unit distance from $v(A)$ so that $[v(A)v(F)]$ is orthogonal

to C^{d-1} and is cooriented with $n(A, F)$;

endif

endif

4 **for** all d-cells G adjacent to A **do**

if G has not been processed **then** $a(G) = a(G) + 1$

if G has two adjacent d-cells which make contact

with G in two non-parallel facets

then push G onto $STACK$

endif

elseif $v(A)$ does not exists **or** $[v(G)v(A)]$ is not orthogonal to its facet

then $dim_j := 0$; add $''x_j = 0''$ to Θ; add $''x_{AG} = 0''$ to Θ;

endif

endfor

5 **end**

2.11.3 Analysis of Algorithm 1.

The algorithm is supposed to process all independent components in a sequence, and can give wrong results only if it does not recognize independent components properly, i.e. if it treats edges of $\mathcal{G}$ which belong to a component as edges from different components. It is enough to prove that $STACK$ does not become empty until all d-cells of a current component $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ have been processed. The algorithm recognizes $\mathcal{G}_1$ as processed before all d-cells corresponding to vertices of $\mathcal{E}_1$ have been processed only if cannot find a non-processed cell which has two processed neighbours in non-parallel facets. The construction of the algorithm guarantees that in this case there is a d-cell C with one processed neighbour corresponding to a vertex of $\mathcal{G}_1$. Let B be a processed neighbour of C, and A be a processed neighbour of B from $\mathcal{G}_1$. Note that B must have a processed neighbour, because of the assumptions about $\mathcal{G}_1$ and C.

First, consider the case when $\mathcal{M}^d$ is closed. Then, A and C can be connected by a path which consists of d-cells making full contact with B. Therefore one can find a non-processed d-cell which contacts two processed d-cells (one of them is A) and shares an internal $(d-2)$-cell with them. This contradicts our assumption that the $STACK$ is empty.

Now assume that $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. Since $C \in \mathcal{V}_1$, there is a path $\mathfrak{p}$ in $\mathcal{G}_1$ which connects vertices B and C, but does not contain edge $[BC]$. We can assume that this path is the sum of two paths $\mathfrak{p}_1 = [B, \dots, C']$ and $\mathfrak{p}_2 = [C', \dots, C]$, where the first consists only of processed vertices except the last vertex, and the second contains only non-processed vertices. Therefore there is a non-trivial edge-cycle in $\mathcal{G}_1$ which contains $[BC]$. Since $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, $\mathfrak{p}_1 + \mathfrak{p}_2 + [CB] = \partial \sum_i \mathfrak{t}_i \ (mod\, 2)$, where the $\mathfrak{t}_i$ are triangles corresponding to the stars of internal $(d-2)$-cells of M^d (see Section 4). Let $\sum_i \mathfrak{t}_i = \sum^1 + \sum^2$, where $\sum^1$ is the sum modulo 2 of the $\mathfrak{t}_i$ with all processed vertices and $\sum^2$ is the sum modulo 2 of the $\mathfrak{t}_i$ with at least one non-processed vertex. Assume that none of the triangles of the second type contains two processed vertices. Thus $\partial \sum^1 = \mathfrak{p}_1$, which is impossible since $\mathfrak{p}_1$ is not a cycle. Therefore there is a non-processed vertex of $\mathcal{G}_1$ which forms a triangle in $\mathcal{G}_1$ with two processed vertices. This contradicts our choice of C. Since the choice of $\mathcal{G}_1$ was arbitrary, one can conclude that the algorithm processes all independent components properly.

It is easy to see that if L is the binary size of the numerical input (an array of the coordinates of normals to facets), then the number of arithmetical operations which have to be performed to a precision of $O(L)$ is $O(f_{d-1}(\mathcal{M}^d))$. This time is optimal, since of input is $\Omega(f_{d-1}(\mathcal{M}^d))$.

2.11.4 Robustness and approximations

Notice, that even when $\mathcal{M}^d$ ($d > 2$) satisfies all of the conditions for the successful performance of the algorithm (see Section 11.1), a real numerical experiment with not-exact machine arithmetic might give wrong results. When on the step 3 the algorithm checks whether two facet normals intersect (computing function $V(A; v(D), v(E))$, any loss of precision may result in the wrong conclusion (if $d > 2$). Therefore this algorithm is not numerically stable. When we know as a preliminary that a non-degenerate reciprocal exists, one can modify the algorithm so that it will construct an approximation of the reciprocal. Instead of computing the point of intersection of two lines (function $V(v(A); v(B), v(C))$, the modified version of Algorithm 1 should compute the point which is equidistant from both lines and minimizes the sum of the distances between a point and the lines. This procedure does not increase the complexity of the algorithm, because it can be done in time O(1). Such modification can be also used for approximation of cell-decompositions of $\mathbb{R}^d$ which have a high proportion of primitive stars of $(d-2)$-cells by weighted Voronoi diagrams. Suzuki and Iri report on how approximation of planar tesselations by Voronoi diagrams can be applied to urban planning and biological growth models [74]. However, we have no theorems on the numerical stability for the proposed modification.

What kind of manageable sufficient conditions can be used for determining whether a cell-decomposition of $\mathbb{R}^d$ ($d > 2$) can be represented by a weighted diagram? Some such conditions are:

1) The decomposition is $(d-3)$-primitive (see Theorem 9.3);

2) The decomposition is $(d-2)$-primitive and for the star of every $(d-3)$-dimensional face there is an all-non-zero d-stress (see Theorem 9.2).

The second condition is not locally robust, but can be used for analyzing tilings which consist of polytopes with symmetries arising from lattices (lattice polytopes, space-fillers) (see [67]).

2.12 Algorithm for general cell-decompositions

Let M^d be stored as it is described in Section 10. The algorithm determines whether a lifting of a given type exists. Our strategy is to formulate a system of homogeneous linear equations and inequalities whose feasibility set coincides with the set of liftings of the prescribed type. For the variables of this system we take the C_1^0-cofactors of the stars of facets (see Sections 2.5, 2.10, 2.11). The variables corresponding to those facets which are not involved in cycles of the dual graph are indeed free. Let $A(\mathbf{x}, C)$ be a family of affine functions corresponding to the d-cells of M^d. Let $\{\mathfrak{c}_j\}$ be some generating system of the cycle space of the dual graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ of $\mathcal{M}^d$. By the results of Sections 4 and 5, if for each d-cell-facet cycle $\mathfrak{c}_j = [C_1, \ldots, C_{k-1}, C_1]$ of $\{\mathfrak{c}_j\}$ the affine functions corresponding to the cells of this cycle determine a lifting of the sub-complex $C_1, \ldots, C_{k-1}$, then $A(\mathbf{x}, C)$ is a lifting of M^d. Affine functions $A(\mathbf{x}, C_1), \ldots, A(\mathbf{x}, C_{k-1})$ determine a lifting for a cycle $\mathfrak{c} = [C_1, \ldots, C_{k-1}, C_1]$ if and only if for some orientation of the cycle the sum of the differences $A(\mathbf{x}, C_i) - A(\mathbf{x}, C_{i-1})$ between the functions of consecutive cells along the cycle is zero. This is to say that the sum of the covectors defining the linear parts of functions $A(\mathbf{x}, C_i) - A(\mathbf{x}, C_{i-1})$ is zero, and the sum of the free terms is zero. The direction of the covector is fixed. If we fix the length of a covector, the free term is uniquely defined, since $A(\mathbf{x}, C_i) - A(\mathbf{x}, C_{i-1})$ must be zero on the facet between C_i and C_{i-1}. Therefore, there as many variables as there are facets in the generating set of cell-facet cycles $\{\mathfrak{c}_j\}$. We call these variables the *cofactors of facets*. They are the signed lengths of the covectors defining the linear parts of functions $A(\mathbf{x}, C_i) - A(\mathbf{x}, C_{i-1})$. If a cell-facet cycle lies on the star of a cell (of lower dimension), the sum of the free terms is, obviously, zero. Otherwise, we have to check that

$$\sum_{C_iC_{i+1}\in\mathfrak{c}} \langle \mathbf{v}_{i+1} - \mathbf{v}_i, \mathbf{p}_{i,i+1}\rangle = 0,$$

where $\mathbf{v}_i\mathbf{v}_{i+1}$ is the covector defining the linear part of $A(\mathbf{x}, C_{i+1}) - A(\mathbf{x}, C_i)$, and $\mathbf{p}_{i,i+1}$ is a vector pointing from the origin to some point of the common facet of C_i and C_{i+1}. Summing up, for each cycle of $\{\mathfrak{c}_j\}$ there are at most $d+1$ linear equations. In terms of reciprocals, our system of affine equations describes the linear space of all reciprocals which satisfy the moment condition (see Theorem 2.6.3).

It takes $O(f_{d-1})$ operations to construct the dual graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ from the graph of incidences of $\mathcal{M}^d$. The cost of finding a basis of the space of cycles for a graph with $|\mathcal{E}|$ edges is $O(|\mathcal{E}|) = O(f^\circ_{d-1})$ operations (first we find a spanning tree, then the basis associated with the tree). The number of cycles will be less than $|\mathcal{E}|$. Orienting these cycles and writing equations expressing the continuous gluing of d-cells of lifting over each cycle takes $O(|\mathcal{E}| dim\, H_1(\mathcal{G}, \mathbb{Z}_2)) = O(f^{\circ\,2}_{d-1})$ operations.

Note, that by Lemma 2.4.2 if $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, one can take the links of internal $(d-2)$-faces for a generating system of the cycle space of $\mathcal{G}(\mathcal{M}^d)$: in such case finding the oriented cycles spanning the cycle space of $\mathcal{G}(\mathcal{M}^d)$ requires $f^\circ_{d-2}f^\circ_{d-1}$ operations. For a manifold with trivial first homologies this strategy seems reasonable, since the number of vertices in a partition of a closed manifold is always less than the number of edges. For a d-cell–facet cycle around a $(d-2)$-cell, the covectors defining the linear parts of the differences $A(\mathbf{x}, C_i) - A(\mathbf{x}, C_{i-1})$ all lie in a linear 2-space. Thus, the vanishing of the vector sum of the covectors can be expressed with two homogeneous scalar equations which involve only two coordinates. Therefore, if the facets of M^d are situated in a general position relative to the coordinate system, we can take only $2f^\circ_{d-2}$ equations instead of df°_{d-2} .

Proposition 2.12.1 *If $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ and M^d is situated in general position with respect to the coordinate system of $\mathbb{R}^d$, the complexity of our algorithm does not depend* *on d at all.*

Now, let us return to the general case. Denote by Θ the following system of homogeneous equations and inequalities: $(d+1)\ dim\ H_1(\mathcal{G}, \mathbb{Z}_2)$ scalar homogeneous equations corresponding to the basis elements of the cycle space $H_1(\mathcal{G}, \mathbb{Z}_2)$ of $\mathcal{G}$, and at most f°_{d-1} inequalities expressing restrictions on the qualitative type of lifting (like $x_i > 0$ or $x_i < 0$ or $x_i = 0$). The complexity of constructing this system is $O(f^2_{d-1})$. Thus, the problem of finding a lifting of a given type has been reduced to the problem of finding a solution for a system of df°_{d-1} linear homogeneous equations and at most f°_{d-1} linear homogeneous strict inequalities with f°_{d-1} variables. These variables are the cofactors of facets.

System Θ is a homogeneous feasibility problem in the standard Karmarkar form. Solutions to our original geometrical problem are represented by *interior* points of the cone Θ. An interior point for the cone Θ can also be found by recasting this problem to an auxiliary linear programming problem in the standard format and solving this auxiliary problem by a polynomial projective method (see [81]). The fact that the inequalities of Θ are strict does not affect the formal time complexity.

The time complexity of the algorithm will be given as a function of the number of $(d-1)$- and $(d-2)$-cells. This estimation will also take into account the binary size of the numerical input. *Let L is the binary size of the array of coordinates of normals to the supporting hyperplanes of facets.* We assume that all operations are performed to a precision of $O(L\ lnL)$. By Vaidya's estimates [81] the complexity of finding a feasible solution for the resulting feasibility problem by a modified projective (Karmarkar-like) method is $O({f^{\circ}_{d-1}}^3 L)$.

This leads to:

Theorem 2.12.2 *Let M^d be a PL-realization of a finite cell-decomposition of a manifold in $\mathbb{R}^d$. Algorithm 2 establishes whether M^d has a lifting of a given type and finds* *such lifting in the case of the existence in time*

$$O(f^{\circ\,3}_{d-1}\,L\ +\ f_{d-1}).$$

If $H_1(\mathcal{M}^d, \mathbb{Z}^2) = 0$, and the links of $(d-2)$-cells are used as a generating system, then the number of equations f°_{d-2} can be asymptotically larger than the number of variables f°_{d-1}. Removing redundant equations by Gaussian elimination costs $O(f^{\circ}_{d-2} f^{\circ\,2}_{d-1})$ operations. After such reduction the number of equations does not exceed f°_{d-1}. In this version the construction of Θ requires $O(f^{\circ}_{d-2} f^{\circ\,2}_{d-1}\ +\ f^{\circ}_{d-2} f^{\circ}_{d-1}\ +\ f_{d-1})$ operations. Thus the total time complexity of this version is

$$O(f^{\circ\,3}_{d-1}\,L\ +\ f^{\circ}_{d-2} f^{\circ\,2}_{d-1}\ +\ f_{d-1}),$$

where L is the binary size of the array of the *first two coordinates* of normals to the supporting hyperplanes of facets.

Let HS^d be a realization of a homology sphere or a cell-decomposition of $\mathbb{R}^d$. It is interesting to estimate the complexity of the above algorithm in the case when the asymptotic upper bound theorem applies. Assume a sharp lifting for HS^d exists. (It is equivalent to the existence of an all-non-zero stress). In this case one can apply to $\mathcal{HS}^d$ the asymptotic upper bound theorem, because even if the cell-decomposition of $\mathcal{HS}^d$ is not simplicial or 0-primitive (simple), one can prove the asymptotic upper bound via the perturbation of a sharp lifting. (For example, the asymptotic upper bound can be always used in the case $d = 2$, since any cell-decomposition of a 2-manifold can be made simple by a small perturbation.) Let $n = f_d$ be the number of d-cells of $\mathcal{HS}^d$. We have $f_{d-2} = O(n^{min(\lfloor (d+1)/2 \rfloor, 3)})$, $f_{d-1} = O(n^{min(\lfloor (d+1)/2 \rfloor, 2)})$. Thus we have the following estimations for spherical manifolds: $O(n^3\,L)$ for $d = 2$, and $O(n^6\,L)$ for $d \geq 3$.

2.13 Fans

It is natural to interpret all obtained results in terms of fans. A *fan* in Euclidean space $\mathbb{R}^d$ is a finite collection of pointed polyhedral cones which is closed under taking faces and intersections, and which covers $\mathbb{R}^d$. A fan can be alternatively regarded as a spherical complex, namely a cell-decomposition of $\mathbb{S}^{d-1}$ where all cells are intersections of $\mathbb{S}^{d-1}$ and pointed polyhedral cones with centers at the center of $\mathbb{S}^{d-1}$. A convex polytope gives rise to its *normal* fan, whose cones are formed by normals to faces of the polytope. By analogy with the "flat" case, one can consider for a spherical complex K radial liftings, i.e. polytopes such that their radial projections from the center of $\mathbb{S}^{d-1}$ give K. A star polytope generates a special fan. The apex of this fan is visible from any point of the boundary, and the cones of this fan are based on the faces of the polytope. We call such fans *polytopical*. Notice that if a fan is the radial projection of a convex polytope if and only if this is a normal fan of a convex polytope. A fan is referred to as *k-primitive* if the star of each k-face has only $(d-k+1)$ d-cones (minimal possible number). Alternatively, a spherical complex on $\mathbb{S}^{d-1}$ is referred to as k-primitive if the star of each k-face has exactly $d-k$ $(d-1)$-cells.

Two questions are of our interest.

1. Under what conditions is a given finite spherical complex S^{d-1} the radial projection of the boundary complex of a convex polytope, or in other words, when is a fan normal?

2. Under what conditions is a given finite spherical complex S^{d-1} the radial projection of the boundary complex of a star polytope, or in other words, when is a fan polytopical?

Denote by $StPol(F)$ the manifold of star polytopes (up the choice of a hyperplane) that have F as their radial fan, and by $ConPol(F)$ the cone of convex polytopes (in the same sense) that have this F as their radial fan. These objects can be embedded into $\mathbb{R}^{f_{d-1}(F)-1}$.

All propositions in this section can be proved either via a straightforward adaptation of the arguments given for "flat" realizations to spherical complexes, or by regarding fans as finite cell-decompositions of $\mathbb{R}^d$ by convex polyhedra. In the latter case we lift a fan to a cone in $\mathbb{R}^{d+1}$, intersect this cone with a hyperplane parallel to $\mathbb{R}^d$, and project the intersection back onto $\mathbb{R}^d$. Depending on whether the cone is convex or not, the resulting polytope in $\mathbb{R}^d$ is either convex or star-like. Adopting the definition of a reciprocal (see Section 6) for the standard sphere $\mathbb{S}^{d-1}$ one can make use of this notion for spherical complexes.

Theorem 2.13.1 *A fan is polytopical if and only if the spherical complex of the fan has a non-degenerate reciprocal. Convex polytopes correspond to convex reciprocals.*

The theorem that a fan is normal if and only if the spherical complex of the fan has a convex reciprocal was proved earlier by McMullen [56]. We give here a characterization of fans of convex polytopes and star polytopes in terms of stresses. The definition of the k-stress for $k = 1, , \ldots , d-1$ (Section 3) works well for fans and spherical complexes. A k-stress on a fan corresponds to a $(k-1)$-stress on the corresponding spherical complex. Recall that for manifolds with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ the coefficients of facet stresses can be regarded as C_1^0-cofactors of facets. In other words, they define affine forms which are the differences between functions representing a lifting on adjacent d-cells. The coefficients of a d-stress on facets of a fan can be interpreted exactly in the same way (see Theorem 2.5.1). Adopting the arguments from the proof of Theorem 2.5.1 for the spherical case, we have the following proposition.

Theorem 2.13.2 *Let F be a polyhedral fan in $\mathbb{R}^d$ $(d > 1)$. Then dim $StPol(F) =$ dim $Stress_d(F)$. Cones $ConPol(F)$ and $Tension_d(F)$ are isomorphic.*

In other words a fan in $\mathbb{R}^d$ is the radial fan of a convex (star) polytope if and only if there is a d-tension (all-non-zero d-stress) on this fan. The following propositions are adaptations of the results of Sections 2.3-2.12 for fans.

Theorem 2.13.3 *Every $(d-3)$-primitive fan $(d > 2)$ is the radial (normal) fan of a convex polytope which is unique up to the choice of a hyperplane and a dihedral angle.*

Theorem 2.13.4 *Let fan F be $(d-2)$-primitive $(d > 2)$. If there is an all-non-zero tension for each $(d-3)$-dimensional face of F, then F is the radial (normal) fan of a convex polytope.*

Theorem 2.13.5 *Let F be a $(d-2)$-primitive fan. There is an algorithm which determines whether F is the radial (normal) fan of a convex polytope. This algorithm has linear time complexity in the number of $(d-1)$-cells of F. If such convex polytope exists, it is unique up to the choice of a facet and a dihedral angle.*

Theorem 2.13.6 *There is a polynomial algorithm which determines whether a fan F in $\mathbb{R}^d$ is the radial (normal) fan of a convex polytope. Let n be the number of d-cones in F, and L be the binary size of an array of normals to $(d-1)$-cones of F. The time complexity of this algorithm is $O(n^3 L)$ for $d = 3$, and $O(n^6 L)$ if $d > 3$ and the asymptotic upper bound theorem applies.*

In [70] Shephard gave necessary and sufficient conditions in terms of Gale diagrams that a given finite spherical complex S^{d-1} is the radial projection of the boundary complex of a convex polytope. Aurenhammer [8, 9] worked out applications of this approach to scene analysis and Voronoi diagrams. We can interpret the Gale diagram technique in terms of computational complexity. Our analysis will be based on the assumption that the algorithm works with a data structure which is well adapted for it. For example, we can assume that the nodes representing the rays of a fan F in the incidence graph of F contain their coordinates. Notice that Algorithm 2 requires inequalities determining $(d-1)$-cells of F, whereas an algorithm which can be extracted from the Shephard theorem requires coordinates of rays of F. If f_k is the number of k-cells of S^{d-1} ($(k+1)$-cones of F), then the Shepard's condition is the existence of an interior point in the intersection of a finite number of convex polytopes in $\mathbb{R}^{f_0-d}$ (see [70, 71]). Each polytope of this family corresponds to a $(d-1)$-cell of S^{d-1}. If a $(d-1)$-cell C_i^{d-1} has $f_0(C_i^{d-1})$ vertices, then the corresponding polytope $G(C_i^{d-1})$ is the convex hall of $f_0 - f_0(C_i^{d-1})$ vertices. Switching to dual polytopes $G^*(C_i^{d-1})$ one can see that the problem has polynomial complexity. Let $\cap_{i=1}^{f_{d-1}} G^*(C_i^{d-1})$ be the intersection of polytopes $G^*(C_i^{d-1})$ over all $(d-1)$-cells. Notice that unlike system Θ from Algorithm 2 affine equations determining the polytope $\cap_{i=1}^{f_{d-1}} G^*(C_i^{d-1})$ in the Shephard's criterion are not homogeneous. The complexity of solving the feasibility problem for $\cap_{i=1}^{f_{d-1}} G^*(C_i^{d-1})$ can be formally estimated as $O(((f_0 f_{d-1} + f_0) f_0^2 + (f_0 f_{d-1} + f_0)^{1.5} f_0) L)$ in the worst case (L is the binary size of numerical input) [81]. For example, for $d = 3$ it gives $O(f_0^4 L) = O(f_{d-1}^4 L)$ which is asymptotically worse than $O(f_{d-1}^3 L)$ given by Proposition 13.6.

Chapter 3

Traces of d-stresses in the spaces of stresses of lower dimensions on orientable manifolds

3.1 Introduction

If (E, V) are the edges and the vertices of a framework (possibly infinite) in $\mathbb{R}^d$, then a self-stress (or simply stress) is an assignment of real numbers $s_{ij} = s_{ji}$ to the edges, a tension if the sign is positive or a compression if the sign is negative, so that the equilibrium conditions

$$\sum_{\{j \mid (ij) \in E\}} s_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0$$

hold at each vertex $\mathbf{v}_i \in \mathbb{R}^d$. Obviously, self-stresses on (E, V) form a linear subspace of $\mathbb{R}^{|E|}$. In fact, the space of self-stresses is the left null-space of the *rigidity matrix* $RM(E, V)$ of the framework (E, V), which is constructed as follows. Let M be the incidence matrix for the edges and vertices of (E, V), where the rows are indexed by the edges and the columns by the vertices. Thus $M(n, m) = 1$ if and only if $v_m \subset \partial e_n$, but is equal to 0 otherwise. The matrix RM is obtained by replacing each unit entry (nm) of M by the corresponding edge vector pointing from vertex m to the other end of edge n, and zero entries by the zero vector in $\mathbb{R}^d$; these replacement vectors are taken to be row vectors. The left null-space of RM (which consists of vectors having $|E|$ coordinates) is the space of self-stresses. The dimension of this space is equal to $|E|$ minus the number of independent rows of RM, or in other words, the row corank of RM. The row rank of RM is exactly the dimension of the subspace of external loads that can be resolved by the framework. For examples of rigidity matrices for particular frameworks see [85]. If all external loads can be resolved, the framework is called *statically rigid.* Since the dimension of the space of external loads on a framework with V vertices in $\mathbb{R}^d$ is $dV - \binom{d+1}{2}$, the static rigidity implies that the dimension of the space of stresses is $E - dV + \binom{d+1}{2}$.

A *spider web* is a framework (with vertices at infinity usually allowed) which supports a self-stress which is strictly positive on all edges. Spider webs in $\mathbb{R}^2$ naturally appear from projections of convex surfaces. Planar and spatial spider webs serve as a tool for investigating various problems about dense packing of equal balls in $\mathbb{R}^2$ and $\mathbb{R}^3$ [4, 23, 24]. There are interesting applications of the theory of stresses in frameworks to physics, chemistry, and engineering (see [4, 10, 23, 24, 91]).

The notion of self-stress on a framework can be naturally generalized to k-stresses on cell-complexes. This generalization proves to be useful in the combinatorics and geometry of P.L.-manifolds, the rigidity theory, and the theory of Dirichlet-Voronoi diagrams. Such generalizations were offered by Lee [51], Tay et al. [76], Crapo and Whiteley [25], and Rybnikov [64].

J. C. Maxwell discovered that the projecting of the 1-skeleton of a piecewise-linear sphere onto a plane induces a self-stress on the projection [53, 54]. He also noticed that this relation is in some sense one-to-one. Later, Crapo and Whiteley [23, 84] proved that for piecewise-linear spheres realized in $\mathbb{R}^2$ (like on Figure 3.1) there is a natural linear isomorphism between the space of self-stresses on the 1-skeleton and the space of liftings (an operation inverse to the projecting of a spatial polyhedron onto the plane) considered up to the choice of a supporting plane for a facet of the

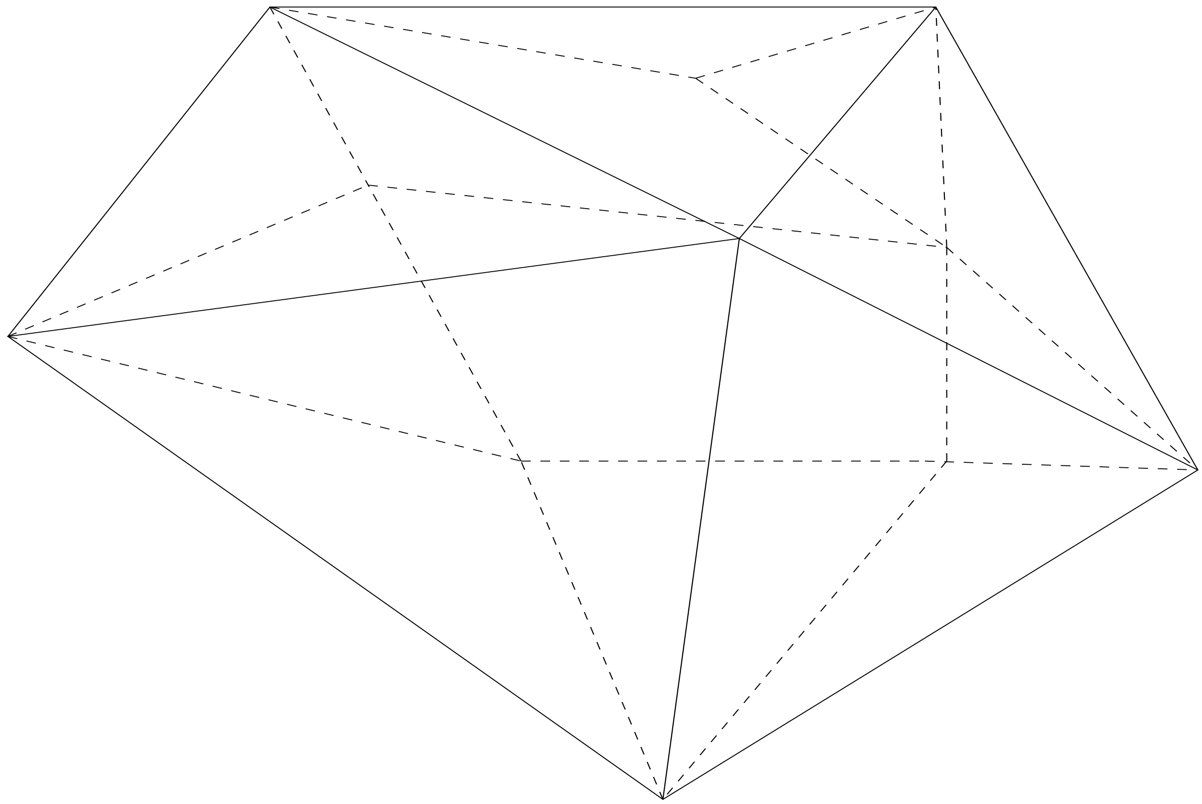

Figure 3.1: 2-sphere realized in $\mathbb{R}^2$

lifting.

In his studies of the relationships between stresses and projections Maxwell used a new geometrical tool, so called *reciprocals.* Roughly speaking, a reciprocal for a planar piecewise-linear realization of a sphere (or partition of $\mathbb{R}^2$) is a special planar realization of the dual combinatorial graph of the complex, such that its edges are perpendicular to the corresponding edges of the complex (see Figure 3.2). The relationship between a graph and its reciprocal is well illustrated by the relationship between the 1-skeletons of the Delaunay and Dirichlet-Voronoi decompositions of a planar point set. For spheres and $\mathbb{R}^2$ the space of reciprocals is also isomorphic to the space of self-stresses.

As it was shown by Crapo, Whiteley and Rybnikov for certain classes of d-

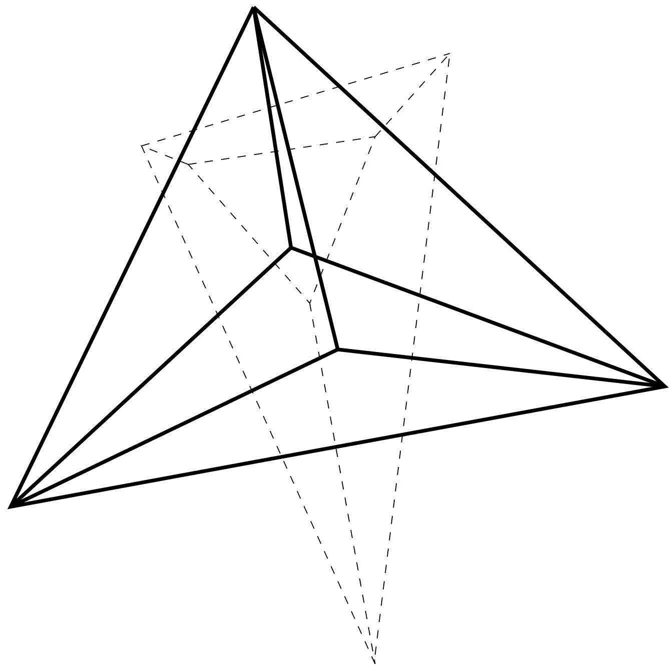

Figure 3.2: Reciprocal for a 2-sphere in $\mathbb{R}^2$

manifolds including homology spheres there is a similar connection between piecewise-linear d-manifolds realized in $\mathbb{R}^{d+1}$ and stresses defined on the $(d-1)$-cells of the realizations defined by the projections. In this case the equilibrium of forces is required not at the vertices, but at each $(d-2)$-cell [23, 84, 64]. Such stresses are called d-stresses because d is the lowest dimension of a manifold for which the space of d-stress is non-trivial in a sense that it depends on the combinatorics of the manifold. (The space of d-stresses of a closed $(d-1)$-manifold realized in $\mathbb{R}^d$ is $\mathbb{R}$ or 0 depending on whether this manifold is orientable or not.)

By an informal conjecture of J. Baracs and W. Whiteley there is an analogous correspondence between projections of the boundaries of 4-polyhedra from $\mathbb{R}^4$ onto $\mathbb{R}^3$ and self-stresses in spatial frameworks [90]. This idea is motivated by Minkowski theorem on the vanishing of the sum of normals to a convex polytope at its facets (see Section 3.5). Since the projections of d-manifolds with trivial $H_1(\cdot, \mathbb{Z}_2)$ from $\mathbb{R}^{d+1}$ onto $\mathbb{R}^d$ correspond to d-stresses (see Section 3.2 and [64, 85]), one can reformulate their conjecture as the existence of a natural correspondence between d-stresses on a d-manifold realized in $\mathbb{R}^d$ and self-stresses on its 1-skeleton (in the general theory of stresses such stresses are called 2-stresses). A theorem that we prove in Section 3.6 can be considered as a natural one-way connection between d-stresses and k-stresses, $k < d$ on oriented piecewise-linear manifolds realized in $\mathbb{R}^d$. Therefore, in some sense our theorem supports Baracs-Whiteley hypothesis. There are other canonical mappings between the spaces of stresses of different dimensions. According to Stanley [72] and Lee [51] for Cohen-Macaulay homology d-spheres in $\mathbb{R}^d$ the space of k-stresses has the same dimension as the space of $(d-k+1)$-stresses, $k \leq \lfloor \frac{d+1}{2} \rfloor$. These isomorphisms are a very important part of the Stanley's and McMullen's proofs of the g-theorem [59, 72]. These isomorphisms are linear (see [72, 51]), whereas our mappings are not; however they are polynomial. In general, our mappings are *not bijective*, since for a generic realization of a simplicial sphere in $\mathbb{R}^3$ the dimension of the space of 2-stresses may exceed the dimension of the space of 3-stresses. For example, using the results of Lee [51] one can show that for a generic realization in $\mathbb{R}^3$ of the boundary of the 4-dimensional cross-polytope $dim(Stress_2) = 6$, but $dim(Stress_3) = 4$.

Our mappings are well-defined not only for simplicial manifolds, but also for cell-partitions of manifolds. We conjecture that these mappings are *injective* for any generic realization of an orientable *simplicial* d-manifold in $\mathbb{R}^d$. However, we are primarily interested in applications of the general theory of stresses and liftings to three dimensional manifolds and spatial spider webs. From this point of view our construction can be regarded as a natural extension of Maxwell correspondence between stresses and liftings to spatial frameworks. In the last paragraph of this section we sketch the main ideas and concepts employed in our construction.

Let Δ be a homology d-manifold decomposed into cells, each of which being a simplicial star (see [61, 69] for discussion of such pseudo-dissections). The construction of our mapping can be divided into several steps. On the first step we establish a natural one-to-one correspondence between d-stresses and *reciprocals* (see Section 3.4). A concept of reciprocal basically generalizes the notion of Maxwell reciprocal (see above). In particular, a segment whose ends are the vertices of the reciprocal corresponding to two adjacent d-cells of Δ is perpendicular to their common facet. This one-to-one correspondence can be established only under certain homological restrictions on the manifold, for example $H_1(\Delta, \mathbb{Z}_2) = 0$. Notice that the star of a cell in a manifold satisfies this condition. Given a d-stress s, we can construct the corresponding reciprocal $R(s, v)$ for the star of each vertex v of Δ. If two cells C_1 and C_2 share a face F, the sub-reciprocals of $R(s, C_1)$ and $R(s, C_2)$ corresponding to F are congruent. Nevertheless, when $H_1(\Delta, \mathbb{Z}_2) \neq 0$, it is not possible, in general, to construct a global reciprocal (see [64]). One can consider for Δ the dual cell-decomposition Δ^*. The idea of such decomposition goes back to Poincaré (for details see [61, 69]). We construct a piecewise-linear realization of a baricentric triangulation TD of the dual cell-decomposition Δ^*. Note that the baricentric triangulation of the original cell-decomposition of Δ is isomorphic to the baricentric triangulation of Δ^* [69]. After such dissection a cell of Δ^* can be realized in $\mathbb{R}^d$ as a simplicial star. We consider only special realizations, namely, such that the baricentric triangulation of each k-cell is realized in its affine k-span. For instance, a sub-reciprocal $R(s, C)$ for the star of cell $C \subset \Delta$ defines such "flat" realization of $St(C^*)$ up to the choice of baricenters for the k-cells, $dim(\Delta) - dim(C) = dim(C^*) \geq k > 0$. In this case one can introduce a natural method of summation of the volumes of the oriented simplexes of the simplicial star $St(C^*)$ ($C^* \subset \Delta^*$), such that the result of the summation does not depend on the positions of baricenters of all cell of dimensions greater than 0. When $St(C^*)$ is embedded into $\mathbb{R}^d$ the result is the oriented volume of the simplicial star. That is why we call this function on "flat" realizations of oriented cells of the dual decomposition the *signed generalized volume*. Evidently, it can be equally thought of as a function on reciprocals. By a well-known Minkowski theorem the sum of outer (or inner) facet normals of a d-simplex scaled with the $(d-1)$-volumes of these facets equals to zero. Using the orientability of Δ , we will show in Section 3.6 that the generalized k-volumes of k-cells of Δ^* can be interpreted as the coefficients of $(d-k+1)$-stresses on Δ. As it can be seen from this informal description, the main ingredients of our construction are volumes, reciprocals, duality in homology manifolds, and the notion of orientability. In fact, we suspect that our construction can be generalized for any dimension n, $\lfloor \frac{d+1}{2} \rfloor \leq n \leq d$ thereby providing canonic polynomial mappings from the space of n-stresses to the spaces of k-stresses, $d-n<k<n$.

3.1.1 Notation

All complexes that we consider are polyhedra (simplicial complexes) from the topological point of view. However, all theorems in this paper are stated for fixed decompositions of simplicial complexes into polyhedral *cells* (also called blocks or simplicial stars in combinatorial topology [69, 61]) which are not necessarily simplexes. We assume that all complexes have at most countable number of cells. Cells of co-dimension 1 are referred to as *facets*. We denote the star of a cell C by $St(C)$, and the k-dimensional skeleton of a complex $\mathcal{K}$ by $Sk^k(\mathcal{K})$.

We shall consider a somewhat more general construction than an embedding or an immersion of a cell-complex into Euclidean space, such as a *piecewise-linear (PL-) realization* of a cell-complex in Euclidean space. In all geometric discussions cell-complexes will be considered as fixed piecewise-linear realizations, rather than abstract combinatorial objects. Such general construction can be helpful, for example, for studying frameworks with bar intersections, polyhedral scenes, splines over triangulations (in the planar case this point of view was adopted in [23, 76, 84]; in the three-dimensional case such PL-realizations were considered by Crapo and Whiteley in [23, 83]), and in the case of general dimension by Tay, White, and Whiteley [76]. For example, a Schlegel d-diagram is a PL-realization of a $(d+1)$-polytope P in $\mathbb{R}^d$ obtained by radial projection of P onto one of its facets.

Recall that one can identify an abstract combinatorial cell-complex $\mathcal{K}^d$ with its embedding into $\mathbb{R}^{2d+1}$ (since it can be triangulated). More formally, a PL-realization of a combinatorial simplicial complex $\mathcal{K}^d \subset \mathbb{R}^{2d+1}$ with a fixed decomposition into polyhedral cells is a continuous PL-mapping r of $\mathcal{K}^d$ in $\mathbb{R}^N$ $(N \geq d)$ such that *the closure of each k-cell, $k = 0, \ldots, d$ is embedded by r into $\mathbb{R}^N$ as a "flat" (lying in a k-subspace) k-polyhedron.*

If Δ is a piecewise-linear realization of a polyhedron with a specific cell-decomposition, we shall frequently abuse notation and make no distinction between the polyhedron, its cell-decomposition and the piecewise-linear realization. If we refer to the metric, projective, or affine properties of a cell-complex, these should be understood as the properties of its fixed PL-realization. However, when we consider the combinatorial or homological properties of a cell-complex, we are referring to its abstract combinatorial structure.

A homology k-sphere (k-disk) is a polyhedron with the homology groups of a standard k-sphere (k-disk). A *homology d-manifold* (with boundary) is a cell-complex such that the link (in the case of a non-simplicial cell-decomposition, the link of a cell can is defined through the baricentric triangulation) of each k-cell, is either a homology $(d-k-1)$-sphere or a homology $(d-k-1)$-disk. A manifold is closed if each facet is adjacent to exactly two d-cells. All statements in the paper are formulated for both closed manifolds and for manifolds with a boundary, unless stated otherwise. Since we consider manifolds only from the combinatorial point of view, a manifold is always understood to be a *homology* manifold. Throughout the paper we include "good" decompositions of $\mathbb{R}^n$ (like, for example, weighted Dirichlet-Voronoi diagrams) into the class of homology manifolds.

3.2 Stresses

The notion of a self-stress on a framework can be naturally generalized to a k-stress on a cell-complex of dimension at least $k-1$. This generalization appears to be useful in the combinatorics and geometry of homology spheres, in rigidity theory, and in Voronoi's theory of parallelohedra (see [51, 64]).

Consider a piecewise-linear realization K in $\mathbb{R}^N$ of a d-dimensional cell-complex $\mathcal{K}$. Denote by $\mathbf{n}(F, C)$ the inner unit normal to a cell C at its facet F. C need not be convex, but it is important that its boundary is a homology sphere.

Definition 3.2.1 *A real-valued function $s(\cdot)$ on the $(k-1)$-cells of K is a k-stress if at each internal $(k-2)$-cell F of K*

$$\sum_{\{C \mid F \subset C\}} s(C) vol_{k-1}(C)\, \mathbf{n}(F, C) = \mathbf{0},$$

where the sum is taken over all $(k-1)$-cells in the star of F. The quantities $s(C)$ are the coefficients of the k-stresses, a tension if the sign is strictly positive and a compression if the sign is strictly negative.

It is easy to see that k-stresses form a linear space, and that k-tensions and k-compressions form congruent positive cones in this linear space. We denote the space of all k-stresses on K by $Stress_k(M^d)$, the cone of all k-tensions by $Tension_k(M^d)$. If the coefficients $s(C)$ are not all zero the k-stress s is called non-trivial. Figure 3.3 illustrates the geometry of the equilibrium condition for the 2-stress at an edge of a cell-complex in $\mathbb{R}^3$.

In the case of a stress on a framework, $s(e)$ is force per unit length, and the static force applied at the end points of edge e is $s(e)\|e\|$. For a $(k-1)$-cell C a k-stress μ is force per unit relative $(k-1)$-volume (area) of C, and the static force applied at a

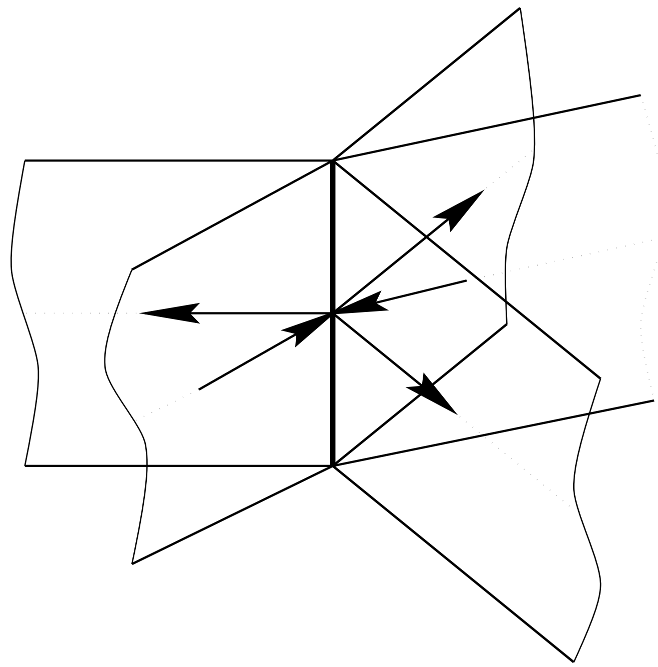

Figure 3.3: Equilibrium of forces at 1-cell

$(k-2)$-face of C is $\mu\, vol_{k-1}C$. The notion of stress is well defined for fans and cell-decompositions of $\mathbb{R}^d$ with non-compact cells. In this case the volumes of $(k-1)$-cells should be leaved out of the formula, and a coefficient of stress does not have meaning of force per unit relative volume (area).

The definition of a k-stress can be adjusted so that the equilibrium of forces is not required at the $(k-2)$-cells adjacent to just one $(k-1)$-cell. For instance, it makes sense when one is describing connections between splines and stresses [85]. Another example is analysis of stresses in frameworks with fixed vertices. In this case the equilibrium of forces is not required at the fixed vertices (called pinned vertices in the planar case).

As in the case of frameworks, the linear space of k-stresses can be characterized as the left null space of a geometric matrix RM_k which is constructed as follows. Let M_k be the incidence matrix for the k- and $(k-1)$-cells of K, where the rows are indexed by the k-cells and the columns by $(k-1)$-cells. Thus $M_k(i,j) = 1$ if and only if

$C_j^{k-1} \subset \partial C_i^k$, but is equal to 0 otherwise. The matrix RM_k is obtained by replacing unit entries of M_k by the corresponding positively oriented unit normal vectors, and zero entries by the zero vector; these replacement vectors are taken to be row vectors. The left null-space of RM_k which consists of vectors (with the number of components equal to the number of k-cells) is the space of $(k+1)$-stresses.

The notion of k-stress on simplicial complexes was introduced by Lee [51]. For a simplicial complex a k-stress can be interpreted as an element of a certain quotient of the face-ring of the complex K. Let K is a simplicial complex in $\mathbb{R}^d$, with vertex set $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then, in Lee's terminology the space of *affine* k-stresses on K is the linear subspace of polynomials of degree k of R/V, where R is the Stanley-Reisner ring of K, and V is the ideal generated by linear forms $\sum_{i=1}^n v_{ki}x_i$ $(k = 1, \dots, d)$, and $\sum_{i=1}^n x_i$ (see [51, 76]). For a simplicial complex K in $\mathbb{R}^d$ our k-stress on K is the same as Lee's affine k-stress on K. In fact, Lee considered two types of stress: linear and affine. Lee formulated most of his theorems in terms of so-called *linear* stresses. For generic realizations of K in $\mathbb{R}^d$ the space of our k-stresses is isomorphic to the space of Lee's *linear* k-stresses for K realized generically in $\mathbb{R}^{d+1}$. The equilibrium condition defining a linear stress says that the sum of normals $\mathbf{n}(F, C)$ weighted by $s(C)$ lies in the linear span of F. Higher-dimensional stresses were also considered by Tay, White, and Whiteley [76] and Rybnikov [64]. Our terminology is in good agreement with terminology in these papers.

If f_k denotes the number of simplexes of dimension k in K, then numbers g_k and h_k are defined as follows

$$g_k(K,d) = \sum_{j=-1}^{k-1} (-1)^{k+j-1} \binom{d-j}{d-k+1} f_j$$

$$h_k(K,d) = \sum_{j=0}^{k} (-1)^{j+k} \binom{d-j}{d-k} f_{j-1}$$

For a generic realization in $\mathbb{R}^{d+1}$ of a simplicial homology d-manifold Δ with homology groups of a standard sphere the dimension of the space of k-stresses is $g_k(\Delta, d+1)$ if $k \leq \lfloor \frac{d+1}{2} \rfloor$, and 0 if $k > \lfloor \frac{d+1}{2} \rfloor$ (see [51]). For a generic realization in $\mathbb{R}^d$ of a simplicial homology d-manifold Δ with homology groups of a standard sphere the dimension of the space of k-stresses is $h_k(\Delta, d+1)$ [76, 51].

There is no similar algebraic theory of stresses for non-simplicial manifolds. The main barrier is the absence of an analog of the notion of face ring for non-simplicial complexes.

Let $\mathcal{K}^d$ be a cell-complex where a baricentric triangulation is fixed for each cell. Consider a PL-realization K^d in $\mathbb{R}^N$ of $\mathcal{K}^d$ such that the triangulation of each cell C of $\mathcal{K}^d$ is realized in an affine subspace of dimension $dim(C)$. Pick a (combinatorial) orientation for $(k-1)$-each cell of $\mathcal{K}^d$. Denote by $\vec{n}(S^{k-2}, C^{k-1})$ the unit normal to oriented cell C^{k-1} at its simplicial facet S^{k-2} whose orientation is induced by the orientation of C^{k-1}. To define the notion of k-stress we have to formulate the equilibrium conditions for each simplex of the baricentric triangulation of each $(k-2)$-cell. However, it is easy to see that if the equilibrium condition holds for one simplex of C^{k-2}, it holds for all other simplexes of C^{k-2}: when we pick another $(k-2)$-simplex from the triangulation of C^{k-2} all normals either change their direction to the opposite, or stay the same.

Definition 3.2.2 *A real-valued function $s(\cdot)$ on (generally non-embedded) oriented $(k-1)$-cells of K^d is a k-stress if for each $(k-2)$-simplex S^{k-2} of each internal $(k-2)$-cell C^{k-2} of K^d*

$$\sum_{C^{k-1}} s(C^{k-1})\, \vec{n}(S^{k-2}, C^{k-1}) = 0,$$

where C^{k-1} ranges over all oriented $(k-1)$-cells such that $C^{k-2} \subset \partial\, C^{k-1}$.

3.3 Orientability and generalized volumes.

Let $\mathbb{R}^d$ be a Euclidean affine space with a fixed coordinate system. Consider an oriented, simplicial $(d-1)$-manifold Δ realized in $\mathbb{R}^d$. We introduce a generalized volume function, Vol_d, which assumes positive, negative or zero values on such manifolds. In the case where the manifold Δ bounds a d-dimensional body, and the orientation of Δ is chosen appropriately, $Vol_d(\Delta)$ is the standard Euclidean volume of the body. Let $F = (v_1, \dots, v_d)$ be an oriented simplex in $\mathbb{R}^d$. We denote by $[\mathbf{v}_1(F) - \mathbf{p}, \dots, \mathbf{v}_d(F) - \mathbf{p}]$ the matrix whose rows are d-vectors pointing from point $\mathbf{p} \in \mathbb{R}^d$ to the vertices of F.

Definition 3.3.1 *Let Δ be a closed oriented simplicial manifold of co-dimension 1 in $\mathbb{R}^d$. Then*

$$Vol_d(\Delta) = \frac{1}{d!} \sum_{F \subset \Delta} det[\mathbf{v}_1(F) - \mathbf{p}, \dots, \mathbf{v}_d(F) - \mathbf{p}]$$

where the summation ranges over all oriented $(d-1)$-faces of Δ.

It is well known that the generalized volume does not depend on the choice of point $\mathbf{p}$. That is why it is normally written for $\mathbf{p} = \mathbf{0}$. The above formula can be rewritten as

$$(3.1) \qquad Vol_d(\Delta) = \frac{1}{d} \sum_{F \in \Delta} d(\mathbf{p}, aff(F)) Vol_{d-1}(F, \mathbf{p})$$

where $d(\mathbf{p}, aff(F))$ stands for the distance between $\mathbf{p}$ and the hyperplane spanned by face F. The generalized $(d-1)$-volume $Vol_{d-1}(F, \mathbf{p})$ is computed with respect to the orientation of $aff(F)$ induced by vector $\mathbf{v}_i(F) - \mathbf{p}$ (i is arbitrary), i.e. with respect to an orthonormal coordinate frame $[\mathbf{e}_1 \dots \mathbf{e}_{d-1}]$ in $aff(F)$ such that $[\mathbf{v}_i(F) - \mathbf{p}, \mathbf{e}_1 \dots \mathbf{e}_{d-1}]$ is positively oriented in $\mathbb{R}^d$.

Let $\mathbb{S}^{d-1}$ be an oriented simplicial sphere, and let D be a cell-decomposition of $\mathbb{S}^{d-1}$ which is the result of an amalgamation of some of the simplexes of into blocks

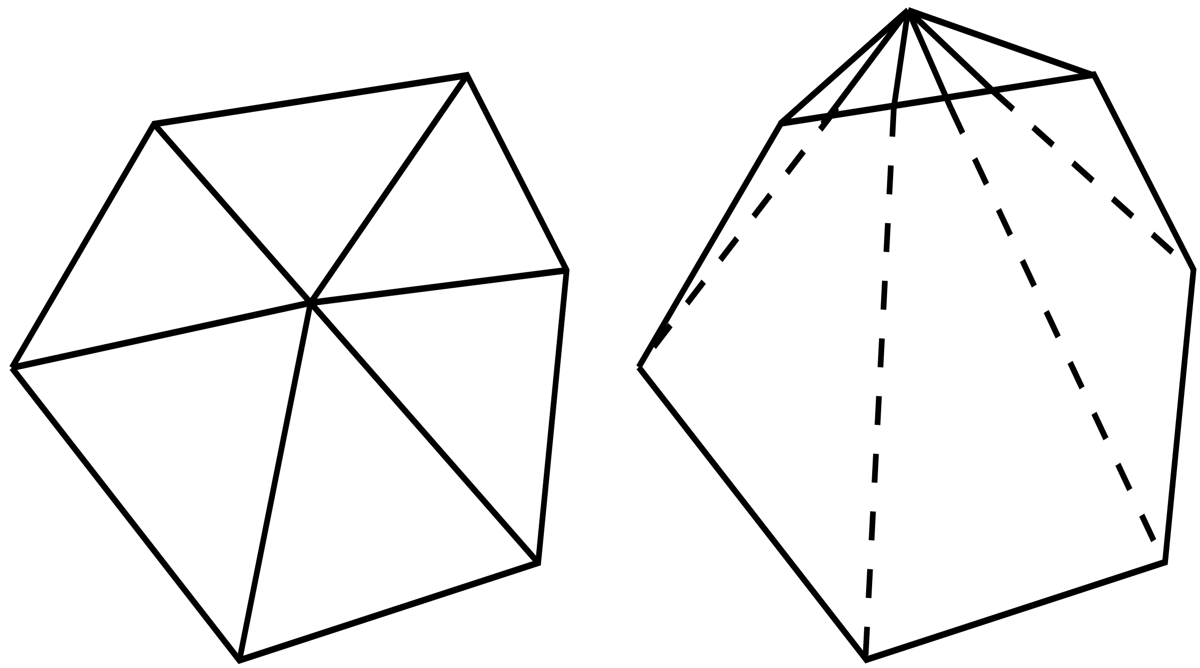

Figure 3.4: Two realizations of a star

$\mathbb{S}^{d-1}$ (see Section 3.1 and [61, 69]). Consider a realization of the simplicial complex $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ such that each block lies in the affine span of its vertex set. For example, a block can be realized as a convex polytope partitioned into simplexes or as a simplicial star with self-intersections (see Figure 3.4). Then $Vol_d(\mathbb{S}^{d-1})$ does not depend on the positions of the baricenters of the blocks of all dimensions greater than 0. It can be shown by induction in d. The case of $d = 1$ is obvious. The induction step follows from an application of Formula 3.1.

In Section 3.6 we will use the following observation.

Remark 3.3.2 *Let B be a d-dimensional cell-complex such that the closures of all its faces, including B, are cones over homology spheres. In other words B is a homology ball. An example of such complex would be a convex polytope. Suppose a baricentric triangulation of B is realized in $\mathbb{R}^d$ so that the affine dimension of the vertex set of each cell of B equals to the dimension of this cell (the cell structure of a convex polytope would serve as a simple example). Then the generalized volume of this simplicial* *sphere does not depend on the positions of the baricenters of its faces provided the baricenter of each face of B lies in the affine span of the vertex set of this face. We call the realizations of these baricenters in $\mathbb{R}^d$ virtual baricenters.*

For discussion of the algebraic properties of the generalized volume $Vol_d(\mathbb{S}^{d-1})$ as function of the edge lengths see [17].

3.4 Combinatorial dual graph and reciprocals

Let $F(V, E)$ be a framework realized in $\mathbb{R}^2$, and assume that graph (V, E) can be regarded as the 1-skeleton of a spherical complex Δ. Suppose that this framework is in a state of static equilibrium. Consider a vertex of (V, E). The sum of vectors of stresses applied to this vertex is equal to zero. Therefore, when rotated on $90°$ clockwise they form a polygon (self-intersecting in general). It was first noticed by Maxwell (and proved by Whiteley [84]) that the positions of rotated edges of $F(V, E)$ can be adjusted so that they form a reciprocal graph (often called simply *reciprocal*). Each edge of this reciprocal corresponds to an edge of $F(V, E)$ and each vertex to a cell of Δ. One can introduce a similar notion for piecewise-linear realizations of d-manifolds in $\mathbb{R}^d$ (for more information see [6, 64, 83, 25]). In this section we will explore connections between the d-stresses and the generalization of Maxwell reciprocal for d-manifolds.

The *combinatorial dual graph* $\mathcal{G}(\mathcal{M}^d)$ of a manifold $\mathcal{M}^d$ is defined as follows. The vertices of $\mathcal{G}$ are d-cells of $\mathcal{M}^d$, and the edges of $\mathcal{G}$ are internal $(d-1)$-cells of $\mathcal{M}^d$. Two vertices share an edge if and only if the corresponding d-cells are adjacent.

A *reciprocal* of a piecewise-linear realization M of a manifold Δ in $\mathbb{R}^d$ is a rectilinear realization R in $\mathbb{R}^d$ of the combinatorial dual graph $\mathcal{G}(\Delta)$ such that the edges of R are perpendicular to the corresponding facets. If none of the edges of a reciprocal collapses into a point, the reciprocal is called non-degenerate. Reciprocals were originally considered by Maxwell [53] in connection with stresses in plane frameworks. He, and almost at the same time, L. Cremona [27] noticed that reciprocals corresponded to equilibrium stresses on 1-skeletons of polyhedral spheres drawn in the plane. Reciprocals were later studied in [6, 23, 24, 84, 67]. Crapo and Whiteley gave an explicit treatment of the theory of reciprocals, stresses and liftings for 2-manifolds in [23, 24, 25].

To illustrate the concept of reciprocal let us consider the case where the realization M is an embedding. Let $v(C_1)$ and $v(C_2)$ be vertices of a reciprocal R corresponding to adjacent d-cells C_1 and C_2. Call the edge $[v(C_2)v(C_1)]$ *properly oriented* if $\mathbf{v}(C_2) - \mathbf{v}(C_1)$ is cooriented with an outer normal to C_1 at the facet shared with C_2. Otherwise call $[v(C_2)v(C_1)]$ improperly oriented. A hexagonal reciprocal for the embedded star of a vertex in a 2-manifold is shown on Figure 3.5. One can see that edges ef, cd are improperly oriented, and edges ab, cb, de, and fa are properly oriented). If all edges of R are properly oriented R is called a convex reciprocal (since the cycles of R corresponding to the stars of the $(d-2)$-cells are convex in this case).We refer to reciprocals of stars of the manifold as local reciprocals.

Evidently, reciprocals with one fixed vertex form a linear space. Denote it by $Rec(M)$. If M is an embedding, then convex reciprocals form a cone $CRec(M)$ in the linear space $Rec(M)$. The following theorem by Rybnikov [64] explains connections between reciprocals and stresses in the case of general dimension. We will utilize this theorem in the proof of our main theorem from Section 3.6.

Theorem 3.4.1 *Let M be a PL-realization of a homology d-manifold Δ in $\mathbb{R}^d$ with trivial first homology group over $\mathbb{Z}/2\mathbb{Z}$. Then there is an isomorphism between $Stress_d(M)$ and $Rec(M)$. Non-zero coefficients of stresses correspond to non-vanishing edges of a reciprocal. If M is an embedding of Δ into $\mathbb{R}^d$, then one can interpret properly oriented edges as corresponding to tensed facets, and improperly oriented edges*

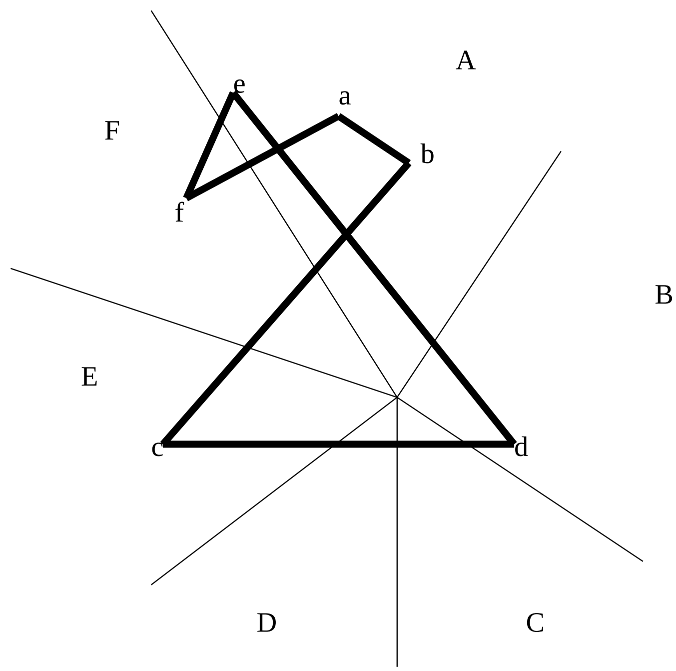

Figure 3.5: Non-convex reciprocal

as corresponding to compressed facets.

Let B be a d-dimensional cell-complex which is the cone over a homology sphere (not necessarily simplicial). Obviously, $B \setminus \partial B$ can be regarded as a star St. Let R be a reciprocal for St and denote by $R(C)$ a sub-reciprocal of R corresponding to a face $C \in St$. The vertex set of R is a realization of the vertex set of a complex dual to St. Denote it by St^*. For each cell C $(k = 1 \leq dim(C) \leq d)$ of St^* choose an arbitrary point $vbc(C, R)$ on each plane $aff(R(C))$, and call it the *virtual baricenter* of $R(C)$. The vertices of R and the points $vbc(C, R)$, $k = 1 \leq dim(C) \leq d$ define a piecewise-linear realization of St^*. We know from Remark 3.3.2 that if a baricentric triangulation of St^* is realized in $\mathbb{R}^d$ so that the affine dimension of the vertex set of each cell of St^* equals to the dimension of this cell (the cell structure of a convex polytope would serve as a simple example), then the generalized volume of oriented simplicial sphere $\partial\, St^*$ does not depend on the positions of the virtual baricenters of its faces provided the virtual baricenter of each face of St^* lies in the affine span of the vertex set of this face. We can sum up this observation in the following proposition which will be of great use in the following section.

Proposition 3.4.2 *Let R be a reciprocal for an oriented d-dimensional star St realized in $\mathbb{R}^d$. Then the generalized volume $Vol_d(R)$ is well defined.*

3.5 Minkowski theorem and stresses

In this section we give an application of a well-known Minkowski theorem (see, for example, [93] to stresses on polyhedral partitions of $\mathbb{R}^d$.

Theorem 3.5.1 *(Minkowski) Let P be a convex polytope in $\mathbb{R}^d$, and denote by $\{\mathbf{n}(F)\}$ the inner unit normals to facets of P. Then*

$$\sum_{F \subset \partial P} vol_{d-1}(F)\, \mathbf{n}(F) = \mathbf{0}.$$

The proof of this theorem can be found, for example, in [93]. Notice, that Minkowski theorem has a well-known physical interpretation: a convex polytope immersed floating in a fluid is in a static equilibrium if and only if the sum of inward forces applied at its facets is zero. This interpretation was already known to Rankine (1864).

If we choose a (combinatorial) orientation for P and denote by $Vol_{d-1}(F, \mathbf{n}(F))$ the generalized volume of an oriented facet F with respect to the orientation of $aff(F)$ induced by $\mathbf{n}(F)$, then the above formula can be rewritten as

$$\sum_{F \subset \partial P} Vol_{d-1}\left(F, \mathbf{n}(F)\right)\, \mathbf{n}(F) = \mathbf{0}.$$

Notice that in the last formula the directions of normals $\mathbf{n}(F)$ need not agree, since $Vol_{d-1}(F)$ is computed with respect to the orientation induced by $\mathbf{n}(F)$. Flipping the normal changes the sign of $Vol_{d-1}(F)$.

Let $St(v)$ be the star of a vertex v of a polyhedral partition of $\mathbb{R}^d$.

Definition 3.5.1 *A* dual convex polytope *for $St(v)$ is a d-dimensional polytope $D(St(v))$ in $\mathbb{R}^d$ satisfying the following conditions.*

1) There is a one-to-one correspondence $\mathcal{I}$ between the m-dimensional faces of $D(St(v))$ and the $(d-m)$-dimensional faces of $St(v)$ $(0 \leq m \leq d)$.

2) If $D^s \subseteq D^t$ are faces of $D(St(v))$ corresponding to faces F^{d-s} and F^{d-t} of $St(v)$, then $F^{d-t} \subseteq F^{d-s}$. In other words the mapping $\mathcal{I}$ induces an isomorphism between the face lattices of $D(St(v))$ and $St(v)$.

3) For $0 \leq m \leq d$ each m-dimensional face of $D(St(v))$ is perpendicular to the corresponding $(d-m)$-dimensional face of $St(v)$.

*4) $Sk^1(D(St(v)))$ is a convex reciprocal graph for the star $St(v)$ (*see Section 3.4*).*

The convexity of the dual polytope immediately follows from Conditions 1 - 4. Suppose that there is a d-tension on St(v) (all coefficients of d-stresses are strictly positive). By results of [64] and [67] there is a convex polytope D dual to $St(v)$. D is uniquely determined by this tension up to translation. By the Minkowski theorem cited above the sum of facet normals of a convex polytope scaled by the facet volumes is zero. Therefore, one can interpret the volumes of m-faces, $1 \leq m \leq d-1$ of D as coefficients of $(d-m+1)$-stresses on $(d-m)$-dimensional cells of $St(v)$. Thus a d-tension on the star $St(v)$ induces an $(d-m)$-tension on $St(v)$, $1 \leq m \leq d-1$. It is easy to see that the constructed mappings are polynomial.

Proposition 3.5.2 *Let Δ be a cell-decomposition of a polyhedral region in $\mathbb{R}^d$. For $k = 1, \ldots, d-1$ there is a polynomial mapping of degree $d-k+1$ from the cone of d-tensions of Δ to the cone of k-tensions of Δ. An all non-zero tension is always mapped to an all non-zero tension.*

By construction, in the case of embedding a d-tension is mapped to a 2-tension on the 1-skeleton of the manifold.

Corollary 3.5.3 *Let G be the 1-skeleton of a cell-decomposition Δ of $\mathbb{R}^d$ by convex polyhedra. If there is a convex surface which projects onto Δ, then G supports a positive equilibrium stress at all edges, and therefore is an infinite spider web.*

It turns out that the mappings from Proposition 3.5.2 can be extended from the cone of tensions to all the space of d-stresses, and the above construction can be carried out for arbitrary piecewise-linear realizations of orientable d-manifolds (not necessary embeddings). In order to formally establish this, we will need the concept of generalized volume introduced in Section 3.3.

The Minkowski theorem can be formulated for simplicial spheres arbitrarily realized in $\mathbb{R}^d$, and as we will see in Section 3.6 even for a large class of non-simplicial spheres realized in $\mathbb{R}^d$ with self-intersections. Let Δ be an oriented simplicial manifold realized in $\mathbb{R}^d$. For each oriented $(d-1)$-simplex F pick a unit normal vector $\mathbf{n}(aff(F))$, and let $Vol_{d-1}(F, \mathbf{n}(aff(F)))$ be the generalized volume of F computed in $aff(F)$ equipped with an orientation induced by $\mathbf{n}(F)$. We need the following lemma.

Lemma 3.5.4

$$\sum_{F \subset \Delta} Vol_{d-1}(F, \mathbf{n}(aff(F))\, \mathbf{n}(aff(F) = \mathbf{0}.$$

Proof. The orientation of Δ induces an orientation on a cone with Δ as base. Thus if F_1 and F_2 are two adjacent $(d-1)$-faces of Δ, the orientations of the cone over their common facet are opposite. Therefore the above formula can be rewritten as

$$\sum_{F \subset \Delta} \sum_{s^{d-1} \subset \partial\, 0 \cdot F} Vol_{d-1}(s^{d-1}, \mathbf{n}(aff(s^{d-1}))\, \mathbf{n}(aff(s^{d-1})) = \mathbf{0}.$$

where s^{d-1} stands for a facet of the cone $\mathbf{0} \cdot F$, and $\mathbf{n}(aff(s^{d-1}))$ is an arbitrary unit normal to hyperplane $aff(s^{d-1})$. Applying Minkowski theorem to each d-simplex $\mathbf{0} \cdot F$ we get the required formula. $\square$

The interplay between stresses and volumes in the case of convex polytope was also described by McMullen [59] and Lee [51].

3.6 A natural trace of d-stresses in skeletons of lower dimensions.

Remark 3.6.1 *Let Δ be an orientable homology $(d-1)$-manifold in Euclidean space of dimension d. An orientation of Δ induces the orientation of normals to Δ at the cells of maximal dimension by the following rule. Let $(v_1(S), \ldots, v_d(S))$ be an oriented simplex of Δ. If frame $[\mathbf{v}_1(S), \ldots, \mathbf{v}_d(S)]$ is positively oriented, then the corresponding normal to Δ at S has positive scalar product with all these vectors. Conversly, a consistent choice of the field of normals to Δ at their simplexes of maximal dimension determines an orientation of Δ (e.g. outer normals for a convex polytope; see Figure 3.6).*

In the case of an orientable d-manifold it is possible to fix the orientation of cells so that they form a d-cycle. By the above remark such orientation of cells induces the orientation of frames of normals corresponding to flags of cells. Thus, if Δ is an orientable d-manifold in $\mathbb{R}^d$ and the orientations of d-cells are picked up in such a way that it turns Δ into a d-cycle, any two flags of equal length having d-cells as maximal elements and distinct at only one position have corresponding frames of opposite orientation. A face-to-face partition of $\mathbb{R}^d$ provides a transparent example. Each of the two possible orientations of the partition correspond to either flags of inner normals or to the flags of outer normals.

Theorem 3.6.2 *Let Δ be an orientable homology d-manifold realized in $\mathbb{R}^d$. Then for $k = 1, \ldots, d-1$ there is a polynomial mapping $\mathfrak{p}_k$ of degree $d-k+1$ from the $Stress_d(\Delta)$ to $Stress_k(\Delta)$.*

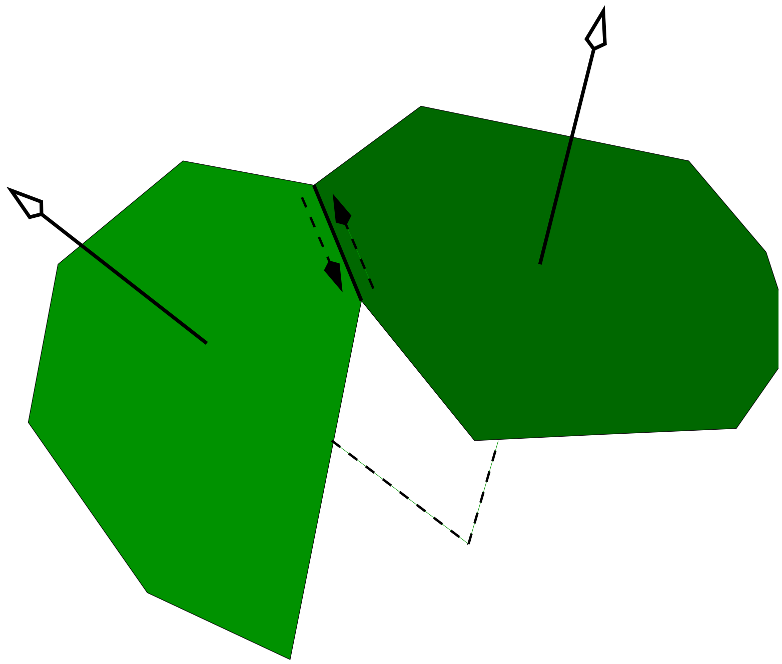

Figure 3.6: Orientation

Proof. For a cell-decomposition of a homology manifold there is so-called dual cell-decomposition (also called dual block decomposition). Consider the baricentric triangulation $T(\Delta)$ of the original cell-decomposition. Each cell of the original decomposition is a simplicial star in the baricentric triangulation. All $(d-k)$-simplexes of this triangulation which share the baricenter of a k-cell c form the dual cell (also called block) for c. This dual cell is a homology $(d-k)$-disk. The boundary of the dual cell is a homology sphere (for more details on the geometrical duality in homology manifolds see [61, 69]).

Let v be a vertex of Δ, and let D_v be the d-dimensional cell (block) corresponding to v in the dual decomposition of Δ. Obviously, the boundary of D_v is the link $Lk(v)$ of v. Each k-simplex of the baricentric triangulation of D_v, $k = 0, \ldots, d-1$, can be regarded as the result of the $(k-1)$-fold iterative coning starting from vertex v.

Each cell of Δ or Δ^* is itself an orientable homology manifold, namely a homology disk. Thus, an orientation of triangulation $T(\Delta)$ induces in a natural way orientation on cell-complexes Δ and Δ^*.

Let R be a (Euclidean) reciprocal for $St(v) \subset \Delta$ (see Section 3.4). By Theorem 3.4.1, the linear space of d-stresses on $St(v)$ is naturally isomorphic to the space of reciprocals with one fixed vertex. It turns out that one can introduce the notion of generalized "k-volume" ($k = 0, \ldots, d$) for the sub-reciprocals of R, corresponding to the stars of cells of $St(v)$ (we refer to them as "faces" of R). It is natural to call this function k-volume, because when a reciprocal can be regarded as the vertex set of a convex k-polytope, the absolute value of this function is equal to the k-volume of the polytope. We keep the same notation for the k-volume of a reciprocal that we used for generalized volumes, i.e. Vol_k.

Let C^{d-k} be a $d-k$-cell from the (open) star of v. Obviously $St(C^{d-k}) \subset St(v)$. The subset $R(C^{d-k})$ of R corresponding to this star spans an affine k-plane perpendicular to C^{d-k}. $R(C^{d-k})$ can be regarded as a realization of the vertex set of a cell of Δ^* dual to C^{d-k}. Thus it makes sense to talk about the (combinatorial) orientation of $R(C^{d-k})$. Recall that a k-cell of the dual decomposition corresponds to a $(d-k)$-cell of Δ. Choose a flag of full length in C^{d-k}. It corresponds to some simplex s of $T(\Delta)$ whose vertex set is the "baricenters" of the flag cells. The iterative coning of C^k with vertices of s is a cell from an amalgamation of the triangulation of the star of v in Δ^* into (non-simplicial, in general) blocks of form $v_0 \cdots (v_{d-k} \cdot Dual(C^k))$ constructed by successful coning of C^k. An orientation of T induces an orientation on $v_0 \cdots (v_{d-k} \cdot C^k)$. Therefore the choice of flag determines an orientation for C^k.

A flag of faces of C^{d-k} corresponds to an ordered $(d-k)$-tuple of normals to the faces of C^{d-k}. Denote it by $[N]$. This $(d-k)$-tuple induces an orientation of affine subspace spanned by $R(C^{d-k})$ by the following rule. A frame F in $aff(R(C^{d-k}))$ is said to be cooriented with the frame $[N]$ if $[N, F]$ is cooriented with the coordinate frame of $\mathbb{R}^d$. Therefore $Vol_{d-k}(R(C^{d-k})$ is well defined provided a flag of cells in C^{d-k}

(see Section 3.3) has been fixed. We have to show that $Vol_{d-k}(R(C^{d-k}))$ does not depend on the choice of flag in C^{d-k}. It is enough to show that for two flags in C^{d-k} that differ in one position the $Vol_{d-k}(R(C^{d-k}))$ is the same, since any two flags in C^{d-k} can be connected by a sequence of alterations. Obviously, two flags that differ in one position induce opposite combinatorial orientations on $R(C^{d-k})$. But on the other hand it means that the $(d-k)$-tuples of vectors corresponding to these flags have opposite orientations. Thus the generalized k-volume of $R(C^{d-k})$ is well defined and does not depend on the choice of a flag of faces in C^{d-k}.

Let s be a d-stress on Δ. Since the star of a $(d-k)$-cell of Δ is a homology d-disk, a d-stress restricted to the star of a vertex generates a k-dimensional reciprocal for this star (see Section 3.4). The distance between two vertices of the reciprocal corresponding to two adjacent d-cells equals the absolute value of stress on their common facet. Let $R(C)$ be the reciprocal of the star of a $(d-k)$-cell C corresponding to the stress s. Let us interpret $Vol_k(R(C))$ as the value of $(d-k+1)$-stress on C (recall that $(d-k)$-cells bear $(d-k+1)$-stresses). We have to check the equilibrium condition at every $(d-k-1)$-cell of Δ. Let F be a $(d-k-1)$-cell of Δ. Construct the reciprocal $R(F)$ for $St(F)$ corresponding to the d-stress s. Notice that if $F \subset C$, then the sub-reciprocal or $R(F)$ corresponding to the star of C coincides with $R(C)$ (up to translation). Let $\mathbf{n}(F, C)$ denotes the fixed unit normal to C at F whose orientation is induced by the orientation of Δ as it was explained in the beginning of this section. In the case where Δ is embedded into $\mathbb{R}^d$ we can think of $\mathbf{n}(F, C)$ as of inward unit normal.

$$\sum_{\{C \mid F \subset C\}} Vol_k\left(R(C), \mathbf{n}(F, C)\right) \mathbf{n}(F, C) = \sum_{\{R(C) \mid F \subset C\}} \sum_{S \subset C} Vol_k(S, \mathbf{n}(F, C)) \mathbf{n}(F, C)$$

where S is an oriented $(d-k)$-simplex from a baricentric triangulation of $R(C)$ arbitrarily realized in $aff(R(C))$. By Minkowsi theorem the last quantity is always zero. $\square$

Remark 3.6.3 *Recall our assumption that each cell has an underlying structure of a simplicial star. The above theorem still holds if the cells are not embedded, but realized as simplicial stars with self-intersections in such a way that the triangulation of each cell lies in the affine plane spanned by this cell.*

One way to show this is as follows. *Proof.* One can extend a d-stress s on Δ to a stress on the PL-realization of its baricentric triangulation $D(\Delta)$: set $s(S^{d-1}) = 0$ for any $(d-1)$-simplex S^{d-1} which does not belong to the triangulation of a $(d-1)$-cell of Δ, and set $s(S^{d-1}) = s(C^{d-1})$ if S is a $(d-1)$-simplex of a $(d-1)$-cell C^{d-1}. All simplicial cells of the baricentric triangulation are, indeed, embedded. Reorient (if necessary) all $(d-1)$-simplexes in the baricentric triangulation so that the positive direction of normal is always inwards. The space of d-stresses of the reoriented complex is isomorphic to the original space of d-stresses. This reorientation is required, because we want to use the definition of stress for complexes with embedded cells. By Theorem 3.4.1 there is a corresponding reciprocal $R(s)$ for $D(\Delta)$. Now, we can define the polynomial mappings $\mathfrak{p}_k$ for $D(\Delta)$. By construction of the reciprocal, (geometric) cycles of $R(s)$ corresponding to simplexes that belong to the same cell are congruent. Consider a $(k-1)$-cell C of Δ. The constructed k-stress $\mathfrak{p}_k(s)$ takes on the same values on any two $(k-1)$-simplexes of C that can be connected by a cell-facet path of $(k-1)$-simplexes of C, such that any two adjacent simplexes don't overlap; $\mathfrak{p}_k(s)$ takes on opposite values otherwise. Therefore $\mathfrak{p}_k(s)$ can be regarded as a k-stress on the original cell-partition of Δ. □

Another way to prove the theorem for the case of self-intersecting cells is to directly adopt the proof of Theorem 3.6.2. The only use of the notion of inner/outer normal in the proof of Theorem 3.6.2 was where we geometrically defined the orientation of the cells of the dual partition Δ^*. In the general case we just have to pick some combinatorial orientation of for Δ^*. The rest of the proof goes virtually unchanged.

As explained in the proof of Theorem 3.6.2, a Euclidean reciprocal R can be naturally regarded as the 1-skeleton of a PL-realization of the dual partition of Δ. To define the realization completely we just have to specify the positions of the baricenters of the cells of the dual partition. Now, we can ask the same questions about liftings, reciprocals, and stresses about the PL-realization of the dual partition. Notice, that in studies of liftings, stresses, and reciprocals the positions of the baricenters are not important. The above generalization is natural, since the class of PL-realization where each cell is realized as a simplicial star is closed under duality, whereas a dual complex for a PL-realization with embedded cells can have cells with self-intersections.

One should notice that the orientability of Δ is essential for our construction. Only in the case of orientable manifold the edges of a reciprocal can be separated into properly oriented and improperly oriented.

Since the generalized $(d-k+1)$-volume of R can be expressed (non-uniquely) as a homogeneous polynomial of degree $d-k+1$ in the (oriented) lengths of the edges of R, and the absolute values of the edges of R equal to the absolute values of corresponding d-stresses (see Section 3.3), the constructed mappings $\mathfrak{p}_k$ from $Stress_d(\Delta)$ to $Stress_k(\Delta)$, $k = 1, \dots, d-1$ are polynomial of degree $d-k+1$. The coefficients of these polynomials depend on geometry of Δ. According to Connelly, Sabitov, and Waltz [17] the 3-volume of an orientable simplicial 2-surface in $\mathbb{R}^3$ is an algebraic integer over the ring generated by the squared lengths of the surface edges. It means that if s is a d-stress on a $(d-2)$-primitive oriented Δ, the value of $(d-2)$-stress $\mathfrak{p}_{d-2}(s)$ on each $(d-3)$-cell of Δ is an algebraic integer over the ring generated by the squared values of s on the $(d-1)$-cells of the star of this cell. It would be interesting to know if there are any implications of this fact for the algebraic geometry of our mappings $\mathfrak{p}_k$.

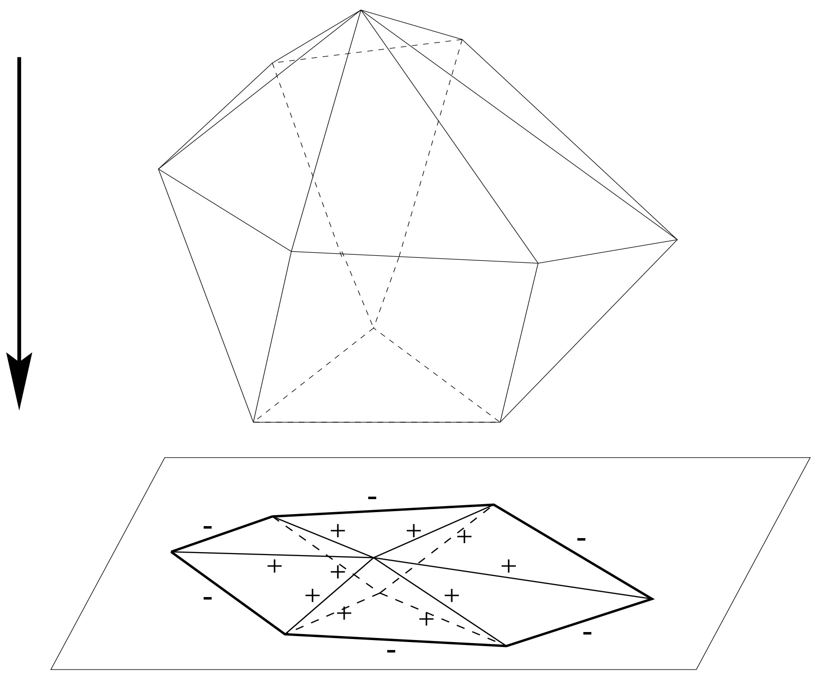

Figure 3.7: Maxwell convex stress

By construction, in the case of an embedding a d-tension is mapped to a 2-tension on the 1-skeleton of the manifold.

Corollary 3.6.4 *Let G be the 1-skeleton of a decomposition Δ of $\mathbb{R}^d$ by convex polyhedra. If there is a convex surface which projects onto Δ, then G is a spider web.*

Maxwell [53, 54] discovered the "convex self-stress" induced by projection of a convex polytope on the plane (see Figure 3.7).

Theorem 3.6.5 *The vertical projection of a strictly convex polyhedron, with no faces vertical, produces a plane framework with a self-stress that is negative on the boundary edges and positive on all edges interior to this boundary polygon.*

Now we can formulate a partial analog of Maxwell theorem on convex self-stresses and projections of spatial polyhedra. It immediately follows from our main theorem.

Theorem 3.6.6 *Let P^4 be a strictly convex polytope in $\mathbb{R}^4$ without vertical faces, and let G be the projection of $Sk^1(P^4)$ onto $\mathbb{R}^3 \subset \mathbb{R}^4$. Then G supports a self-stress s*

which is positive on all edges of G that belong to the interior of the projection. If all the edges of P^4 that project on the boundary of the projection are incident to exactly three 3-cells of P^4, then in addition s is negative on all edges of G that belong to the boundary of the projection.

Proof. Using our main theorem, let us construct the mapping $\mathfrak{p}_2 : Stress_3 \to Stress_2$ for the realization of our polytope P^4 in $\mathbb{R}^3$ induced by the vertical projection. Obviously, since the upper and the lower lids are convex, the reciprocals for the "interior" edges are convex (1-skeletons of convex polytopes) and have volumes of the same sign. The reciprocals of the "boundary" edges need not be convex; however if a boundary edge has a simplicial reciprocal, its volume ought to have the sign opposite to signs of the volumes of the reciprocals of the interior edges. □

Recall, that Maxwell correspondence states also that any equilibrium stress can be interpreted as one induced by the projection of a spatial polytope. On the CMS winter meeting of 1998 R. Connelly asked if the following conjecture is true for our correspondence.

Conjecture 3.6.7 *Let M^3 be a homology sphere realized in $\mathbb{R}^3$ and let s_2 be a self-stress (2-stress) on the 1-skeleton of M^3. There is a 3-stress s_3 on M^3 such that $\mathfrak{p}_2(s_3) = s_2$.*

As it was mentioned in the introduction, the generic realization of the boundary of the 4-dimensional cross-polytope O_4 provides a counterexample. According to Lee [51] $dim(Stress_3(O_4)) = 4$, but $dim(Stress_2(O_4)) = 6$ (self-stresses on a framework are 2-stresses). Since the mappings are algebraic the image of the space of 3-stresses cannot cover the space of 2-stresses. It would be interesting to give a geometric interpretation of those 2-stresses that can be interpreted as images of 3-stresses under the above mapping.

A cell-decomposition of a closed d-manifold is called k-primitive if the star of each k-cell has $d-k+1$ d-cells (some authors call 0-primitive decompositions *simple*; our terminology goes back to Voronoi and Delaunay). The meaning of this definition is that in a decomposition of $\mathbb{R}^d$ by convex polyhedra, $d-k+1$ is the minimal possible number of d-cells making contact in a k-cell. When a k-primitive cell-decomposition of $\mathcal{M}^d$ is assumed to be fixed, we will refer to this k-primitive decomposition of $\mathcal{M}^d$ as *k-primitive manifold* $\mathcal{M}^d$. If a PL-realization of a sphere $\mathbb{S}^d$ in $\mathbb{R}^d$ can be lifted to a convex polytope in $\mathbb{R}^{d+1}$, then 0-primitive vertices of $\mathbb{S}^d$ correspond to simple vertices of this convex polytope. The notion of k-primitive decomposition naturally arises in studies of space-fillers, lattice polytopes and stereohedra. For example, the affine equivalence between space-fillers and Dirichlet domains of lattices was proved by Voronoi only for 0-primitive (simple) tilings. The existence of a lattice Dirichlet domain which is affinely isomorphic to a space-filler Π is equivalent to the existence of a d-stress with some special symmetries on the lattice tiling $T(\Pi)$ by Π (Voronoi)[64, 67, 82]. Since any $(d-3)$-primitive decomposition of $\mathbb{R}^d$ is the projection of a convex surface [64, 67], we have the following corollaries.

Corollary 3.6.8 *The 1-skeleton of a $(d-3)$-primitive decomposition of $\mathbb{R}^d$ by convex polyhedra is always a spider web.*

A cell-decomposition of a d-manifold is referred to as *k-primitive* if the star of each internal k-dimensional cell has $d-k+1$ d-cells (some authors call 0-primitive decompositions *simple*; our terminology goes back to Voronoi [82]). For decompositions of $\mathbb{R}^d$ by convex polyhedra $d-k+1$ is the minimum possible number of tiles in the star of a k-face.

Corollary 3.6.9 *Let M be a realization in $\mathbb{R}^d$ of a $(d-3)$-primitive manifold Δ with trivial $H_1(\Delta, \mathbb{Z}_2)$. Suppose the body $|M|$ of this realization is convex and M is a double cover of int $|M|$. Then the 1-skeleton of M admits a convex self-stress.*

Conjecture 3.6.10 *Let Δ be a simplicial homology d-manifold with $H_1(\Delta, \mathbb{Z}_2) = 0$. Then for any realization of Δ in a general position in $\mathbb{R}^d$ mappings $\mathfrak{p}_k$, $k = 1, \dots, d$ have Jacobians of maximal possible rank at almost all points $s \in Stress_d(\Delta)$.*

One can ask about generic properties of the mappings $\mathfrak{p}_k$, only when any realization of Δ admits small perturbations not changing its combinatorial structure. For instance, this is the case when $d = 2$, when Δ is simplicial or when Δ admits a sharp lifting (for details on sufficient conditions for the existence of a sharp lifting see [64]). It is plausible that in these cases, for generic realizations the constructed mappings also have Jacobian of maximal possible rank. It is possible that the above conjecture holds for arbitrary orientable manifolds.

A necessary condition for our theorem is that $dim(Stress_d) \leq dim(Stress_k)$, $k > d$. Below we give a count that demonstrates that this condition holds for $k = 2$ (mapping of a d-stress to a self-stresses on the 1-skeleton). The dimension of the space of d-stresses on a simplicial d-pseudomanifold in $\mathbb{R}^d$ is at least $f_0 - d - 1$ [12] and is equal to $f_0 - d - 1$ if Δ is a manifold with $H_1(\Delta, \mathbb{Z}_2) = 0$ [64]. By the result of Fogelsanger [40] the 1-skeleton of a generic realization of a d-pseudomanifold in $\mathbb{R}^{d+1}$ is statically rigid. It means that $Sk^1(\Delta)$ can resolve any external load in $\mathbb{R}^{d+1}$ (see Introduction). Thus $dim\ Stress_2(\Delta, d+1) = f_1 - (d+1)f_0 + \binom{d+2}{2} = g_2(\Delta, d+1) \geq 0$ (the lower bound theorem for general simplicial pseudomanifolds).

For Conjecture 3.6.10 to be true, it is necessary that

$$dim\ Stress_2(\Delta, d) \geq dim\ Stress_d(\Delta, d) = f_0 - d - 1.$$

By the lower bound theorem $dim\ Stress_2(\Delta, d) - (f_0 - d - 1) = f_1 - (d+1)f_0 + \binom{d+2}{2} = dim\ Stress_2(\Delta, d+1) = g_2(\Delta, d+1) \geq 0.$

Chapter 4

Holey percolation

4.1 Introduction

Let M be a planar membrane in space clamped on its boundary. The membrane is made of some inextendable material so that it supports tension. If one makes a small convex hole in this membrane, the tension redistributes over the rest of the membrane. However the situation is more complicated when there are several hole in the membrane. It is rather clear that if you have a non-convex hole or a couple of convex overlapping holes, tension ought to vanish on the convex hull of this set (see **Fig. 4.1**).

However, it is less intuitive that tension may vanish at some subset of the complement of a collection of convex non-overlapping holes. For example, the convex

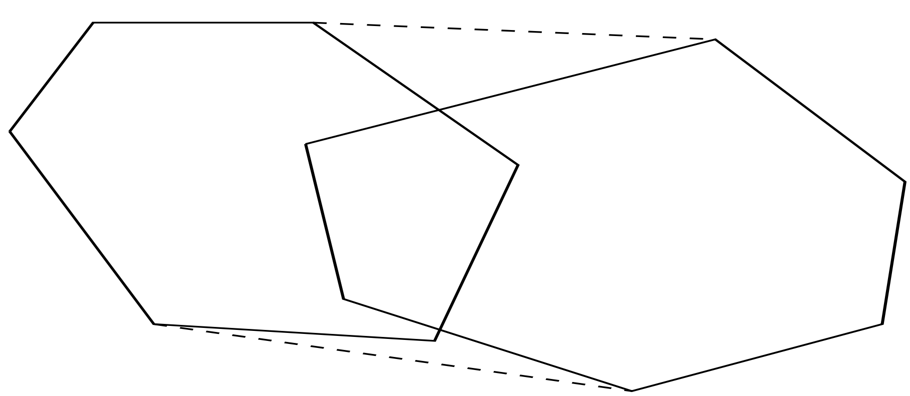

Figure 4.1: Two overlapping holes

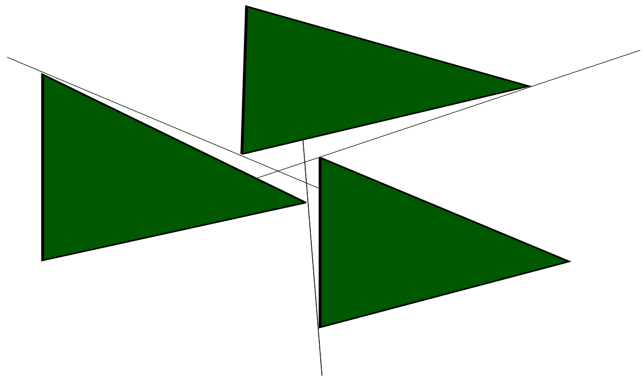

Figure 4.2:

hull of three holes shown on **Fig. 4.2** cannot support tension, which can be verified in practice using a sheet of some elastic material and scissors. Therefore, if the area where tension ought to vanish is interpreted as defective, all three polygons on **Fig. 4.1** ought to coalesce into one big defect. Such *coalescence effect of a "pinwheel configuration"* was first noticed by Connelly (1998). A mathematical explanation of this fact will be given in Subsection 4.1.2.

In the following subsection we give a formal treatment of tension in the language of graph rigidity. Our mathematical treatment of a tensed medium is discrete in nature and based on the ideas from rigidity theory, as opposed to classical methods of mechanics of continuous media that use smooth functions, tensor fields, and differential equations for describing elastic properties of materials. Notice that, in the planar case, holes with smooth boundary can be well approximated by polygons and that the theory based on graph rigidity seem is better suited for computer simulation.

The existence of a "discrete" tension in the complement of the holes has some important implications for rigidity properties of large planar graphs realized in the complement of the holes. This graph theoretic treatment of tension was introduced by Connelly (1998) (also Connelly, Mitchell, Rybnikov (1999)). In this paper we are primarily interested in the behavior of the system with very large or infinite number of holes. More specifically, we look at the question of existence of a subset supporting tension in the complement of an infinite set of holes in $\mathbb{R}^2$. It follows directly from definitions (see Subsection 4.1.1) that if such a set exists, it is infinite and almost surely (we abbreviate it as a.s. throughout the text) unique, which apparently is in good agreement with intuition. We consider a few models for the distribution of holes in an infinite membrane (Section 4.2), and completely resolve the question of the existence of a tensed subset for a "naive" Poisson model suggested by Connelly. In this model, holes are distributed according to a Poisson law, and their shapes are independent and identically distributed (we abbreviate it as IID). In fact we prove a somewhat stronger statement, which we hope has some reasonable physical interpretation (see Subsection 4.1.1). The implications of our main theorem for polyhedral stochastic geometry are outlined in Subsection 4.1.2. The percolation models that we describe in this paper are somewhat related to so-called "bootstrap percolation" introduced on trees by Chalupa, Leath and Reich (1979) and later on d-dimensional lattices by Kogut and Leath (1981). In these models, points are independently occupied with a low density and the resulting configuration is taken as the initial state for dynamics based on some collection of local rules, in which the occupation status of a point is updated according to the configuration of its neighbors. A rigorous analysis of this model have been conducted by Aizenman and Lebowitz (1988). Another work, showing that the critical probability $p_c = 1$ for some bootstrap percolation models was by van Enter (1987). Since then, there has been many publications devoted to this topic. For the latest results on bootstrap percolation see Dehghanpour and Schonmann (1997) and Gravner and McDonald (1997). Gravner and Schonmann (1999, personal communication) had some ideas how to extend their results for continuous case, but as far as the authors know, there has been no published results on the model which we describe and analyze below.

4.1.1 Rigidity and its Applications

Over the past two decades 2-dimensional and 3-dimensional random central-force networks have been used by physicists for modeling the elastic behavior of glasses within the framework of effective medium theory (see Duxbury, Thorpe, Jacobs, Moukarzel (1983, 1995, 1996, 1998, 1999)). It turns out that real glasses are well represented by generic random networks. The success of these methods resulted in good characterization of elastic properties of glasses like $Ge_xAs_ySe_{1-y}$ (Thorpe (1983)). The rigidity analysis of central-force networks has been also used for characterization of physical properties of other substances such has proteins and semiconductors (see Thorpe and Duxbury (1999)).

While discussing the property of a membrane to be able to support tension, we keep in mind (large) networks in the complement of the holes. It turns out that the existence of tension in the complement of the holes implies some important properties for a network realized in the complement. At this point we need to introduce some mathematical terminology. Notice that in the mathematics of rigidity there is a tendency to use term *framework* instead of *network* preferred by physicists.

A bar-and-joint framework is a realization of a graph in $\mathbb{R}^d$. Denote by $F(E;V,V_0)$ a framework in $\mathbb{R}^d$ with the edge set E, and the vertex set V, with pinned (fixed in $\mathbb{R}^d$) subset of vertices $V_0 \subset V$. Denote by $\mathbf{v}_i$ the vector of coordinates of vertex $v_i \in V$.

Definition 4.1.1 *An equilibrium stress (or self-stress) is an assignment of real numbers $s_{ij} = s_{ji}$ to the edges, a tension if the sign is positive, or a compression if the sign is negative, so that the equilibrium conditions*

$$\sum_{(ij)\in E} s_{ij}(\mathbf{v}_j - \mathbf{v}_i) = \mathbf{0}$$

hold at each vertex $\mathbf{v}_i \in V\backslash V_0$ (see **Fig. 4.3***).*

A finite framework $F(E;V)$ is called *rigid* in $\mathbb{R}^d$ if and only if there is a neighbor-

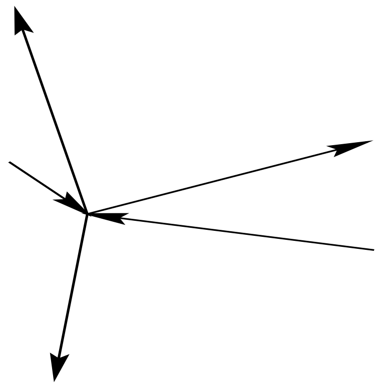

Figure 4.3:

hood $N(F)$ of F in the space of parameters $\mathbb{R}^{dV}$ such that any other $\mathbb{R}^d$ realization of graph (E, V) from $N(F)$ with the same lengths of all edges is congruent to $F(E;V)$. Sometimes it is interesting to study rigidity of graphs some of whose vertices are pinned. A framework $F(E;V,V_0)$ is called rigid with pinned vertices V_0 if F has a neighborhood $N(F)$ in $\mathbb{R}^{dV}$ such that any other realization of $(E;V)$ from $N(F)$ with the same lengths of all edges and the set of pinned vertices inherited from the set of pinned vertices of F is congruent to F. If the "neighborhood condition" is dropped the framework is called *globally rigid*. A framework that is not rigid is called *flexible*. A framework $F(E;V,V_0)$ that supports an all non-zero equilibrium tension is referred to as a *spider web*.

Definition 4.1.2 *An infinite framework $F(E;V,V_0)$ is referred to as rigid if any finite subgraph of $F(E;V,V_0)$ is contained in some rigid finite subgraph of $F(E;V,V_0)$.*

By polygonal partition of a planar set with piecewise-linear or no boundary we mean an edge-to-edge partition of this set into convex bounded polygons. The $1-$skeleton of a partition is a framework whose vertex set is the vertex set of the partition, and whose edge set is the partition edge set. A polygonal partition which consists of triangles is called a *triangulation*.

Definition 4.1.3 *Let M be a set with polygonal boundary in $\mathbb{R}^2$ (M might be all* *of $\mathbb{R}^2$), and let $\mathcal{H}$ be a collection of open polygons in M, such that the number of polygons intersecting any compact subset of $\mathbb{R}^2$ is finite. We call the elements of $\mathcal{H}$* holes *and denote by H the pointwise union of the holes. We say that $M\backslash H$ supports tension if $M\backslash H$ admits a partition with the set of edges E and vertices V, such that the framework $(E, V, V \cap \partial M)$ is a spider web with $V \cap \partial M$ pinned.*

Evidently, in this definition a general polygonal partition supporting tension can be replaced by a triangulation supporting tension without any loss of generality. A direct generalization of this definition to the case of general dimension is possible, but not quite natural, since not all spider webs in dimensions higher than 2 can be interpreted as 1-skeletons of polyhedral partitions (see Connelly and Whiteley (1993, 1996)). In the planar case the situation is simplified by the fact that any spider web with self-intersections can be turned into the 1-skeleton of a polygonal partition by adding points of self-intersections to the vertex set of the framework, and modifying the edge set accordingly (this observation is due to Bow, 1873). The cone of tension of the 1-skeleton of the new partition contains the cone of tension of the original skeleton. A more natural definition would one in which the existence of a nontrivial spider web in the complement of the holes is required. Let us now make some observations about holes. First, if a hole is non-convex, then there is no triangulation of the complement such that its 1-skeleton minus the boundary of M supports equilibrium tension. For instance, the equilibrium of forces at vertex $\mathbf{v}$ on **Fig. 4.4** is impossible if all edges incident to this vertex are under tension. Therefore, if two holes overlap, and their union is not convex, tension vanishes on all of their convex hull. This is called the *coalescence effect of overlapping holes.*

Notice that the above observation about the convexity of connected components of the complement of a set supporting tension is not valid for dimensions greater than three, whatever definition of tension we adopt. For instance, a three-dimensional polytonal

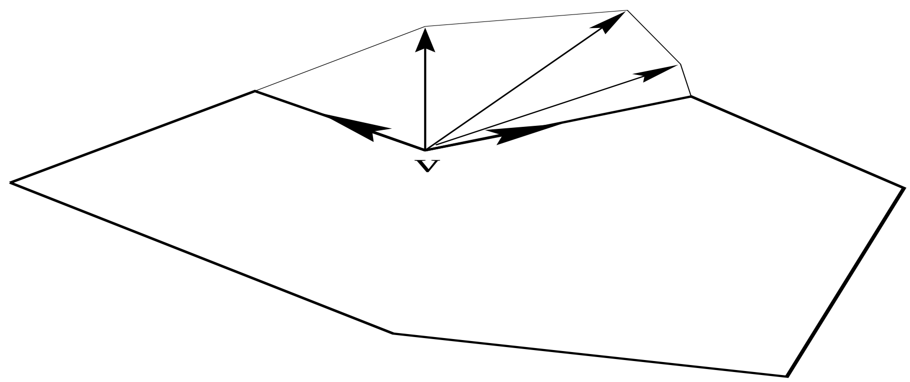

Figure 4.4: Non-convex hole

hole can have a saddle point, to which a number of edges lying in the complement of the holes can be attached so that this point will be in static equilibrium. However, a set supporting tension in $\mathbb{R}^d$ cannot have points of strict convexity.

It is natural to call subsets of M that cannot support tension *defects*. The primary defects are, obviously, holes themselves. The following definitions iteratively define the notion of a defect of k^{th} generation. Note that in these definitions the polygonality of the holes is not important.

Definition 4.1.4 *Let $\mathcal{H}$ be a set of holes. Elements of $\mathcal{H}$ are called defects of 0^{th} generation.*

Definition 4.1.5 *A connectivity component (understood topologically) of defects of k^{th} generation is referred to as a $k-$cluster.*

Definition 4.1.6 *A defect of $(k+1)^{th}$ generation is the convex hull of a $k-$cluster.*

Fig. 4.5 provides an illustration to our definitions. Here we used the solid line for the boundary of holes ($\equiv$ defects of 0^{th} generation), the dash line for the boundary of a defect of the first generation, and dotted line for the boundary of a defect of the second generation. Note that our definition of defect does not cover all subsets that cannot support tension, since an area that fails to support tension need not

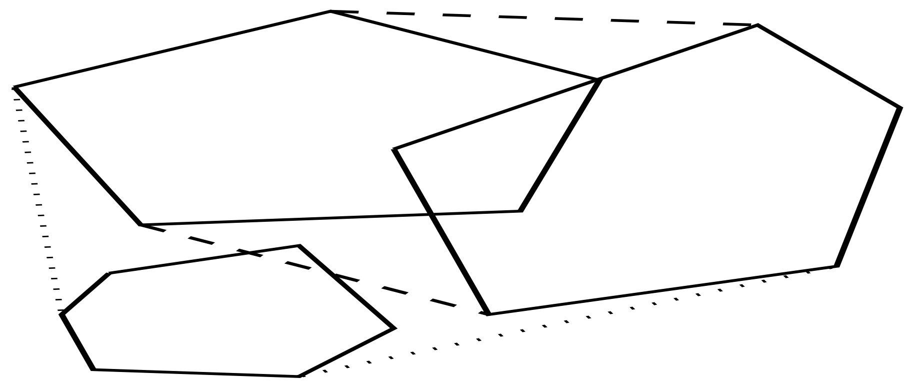

Figure 4.5: Defects of different generations

belong to the convex hull of a connectivity component of the set of holes as it was noticed above (see **Fig. 4.2** and explanations in Section 4.1.2). However, the effect of a "pinwheel configuration" is insignificant for the "naive" Poisson percolation model that we analyze in this paper. Even without taking this effect into account, we show that tension disappears on all of $\mathbb{R}^2$.

The existence of a triangulation which supports tension may have some interesting implications for the modeling of physical properties of materials with networks of Hooke springs and geometry of convex surfaces (see Subsection 4.1.2). Among other things it implies that every triangulation of $M \backslash H$ is *globally* rigid not only in $\mathbb{R}^2$, but also in $\mathbb{R}^3$, provided all its vertices lying on the boundary of M are pinned (Connelly (1993)).

The elasticity (rigidity) properties of a glass are related to how amenable the glass is to continuous deformations which require little energy. From a physical point of view, it is not enough to declare that the distance constraints force the structure to have only one configuration, since the bonds in a physical network do not behave as ideal bars in a framework. There should be a way of describing the behavior of the system as it is perturbed. That is why physicists often consider the energy function defined on the edges of a network of Hooke springs where each spring has some optimal length at which its energy is minimal, stretching or shortening a spring increases the energy of this connection

A tensegrity framework is a generalization of this model where, besides Hooke springs, there are members whose energy never decreases with the distance, and members whose energy never increases with the distance. In context of energy considerations it is often useful to work with the notion of tensegrity framework (Connelly, Whiteley (1996)). In a tensegrity framework all edges are partitioned into three types, cables E_+, struts E_-, and bars E_0, i.e. $E = E_0 \cup E_+ \cup E_-$. Together, struts, cables and bars are called members. Let $F(E, E_0, E_+, E_-; V, V_0)$ be some tensegrity framework in $\mathbb{R}^d$, and denote by l^0_{ij} the length of a member (ij). If a cable is stretched, the energy in the cable increases. If a strut is shortened, the energy in it decreases. Any change in the length of a bar forces the energy to increase. Therefore networks of Hooke springs are bar tensegrities from the mathematical point of view. Thus the energy $\mathfrak{H}_{ij}$ of member (ij) considered as the function of its squared length l_{ij}

is monotone increasing if (ij) is a cable,

is monotone decreasing if (ij) is a strut,

has a strict local minimum at l^0_{ij} called the equilibrium length.

In the spirit of the definition of equilibrium stress we assume that a strut can support only compression, a cable can support only tension, and a bar can be under both types of stress. For more detailed information on tensegrities see the works of Connelly and Whiteley (1996), and Connelly (1993).

It is natural to define an energy function of a finite bar tensegrity framework (finite network of Hooke springs) as the sum of the energy functions of its members. Thus

$$\mathfrak{H} = \frac{1}{2} \sum_{(ij)\in E} \mathfrak{H}_{ij}(|\mathbf{v}_j - \mathbf{v}_i|^2) = \frac{1}{2} \sum_{(ij)\in E} \mathfrak{H}_{ij}(l^2_{ij}). \tag{4.1}$$

The simplest way to define an energy function when all members are bars is as follows

$$\mathfrak{H} = \frac{1}{2} \sum_{(ij)\in E} a_{ij}(l_{ij} - l^0_{ij})^2, \tag{4.2}$$

where the sum is over all ordered pairs of vertices of the framework, l_{ij} is the length of the bond between i and j, l_{ij}^0 is the equilibrium bond length, and $a_{ij} > 0$ is the spring constant of the bond (ij).

In the spirit of the definition of equilibrium stress we assume that a strut can support only compression, a cable can support only tension, and a bar can be under either type of stress, depending on whether its length is larger or smaller than l_{ij}. For more detailed information on tensegrities see the works of Roth and Whiteley (1981), Connelly and Whiteley (1995), and Connelly (1993).

Consider now the energy $\mathfrak{H}$ as a function of the *coordinates of the vertices* of the framework. A finite tensegrity framework F in $\mathbb{R}^d$ with pinned vertices $V_0 \subset V$ is called *prestress stable* if

1) the differential of $\mathfrak{H}$ (considered as a function on the space of parameters $\mathbb{R}^{dV}$) at the point corresponding to F is zero,

2) the second differential of $\mathfrak{H}$ is a positive semidefinite quadratic form whose kernel restricted to infinitesimal motions leaving V_0 unmoved consists of trivial infinitesimal motions of the framework.

Definition 4.1.7 *An infinite tensegrity framework is called prestress stable if every its finite subgraph G is contained in a prestress stable subgraph whose un-pinned vertices contain the vertices of G.*

The concept of prestress stability is due to Connelly and Whiteley (1993, 1996) and basically accounts for local minima of the energy function. It turns out that if the complement of H in M supports tension, then for any triangulation T of $M \backslash H$ there is a an interpretation of the edges of T as struts and cables such that the resulting tensegrity is prestress stable in 3-space with ∂M pinned. In particular it implies that T is prestress stable as a bar framework in 3-space with ∂M pinned.

So far we discussed only statics of framework. There is an interesting implication of the non-existence of tension in the complement of a collection of holes for mechanics. It was conjectured by Connelly (1998) that in this case the complement has a triangulation which is a flexible in 3-space with boundary pinned. There also are interesting connections between our problem and convex geometry that in its original form are due to Maxwell (1864, 1869-1872) and Cremona (1872). They are outlined in Section 4.1.2.

Let us summarize the implications of the existence of tension in the complement of the holes (Connelly, Mitchell, and Rybnikov (1998, 1999)).

Proposition 4.1.8 *(Connelly) Let M be a convex set of $\mathbb{R}^2$ with polygonal or no boundary, and let $\mathcal{H}$ be a collection of convex open polygons in M, possibly overlapping. Suppose $M \backslash (H \cup \partial M)$ supports tension. Let P be a convex polygonal partition of $M \backslash H$ with the set of vertices V and the set of edges E, and let $F(E; V, V \cap \partial M)$ be the corresponding framework with pinned vertices $V \cap \partial M$. Then*

1) framework $F(E; V, V \cap \partial M)$ is globally rigid in $\mathbb{R}^3$ as a bar framework;

2) there is a partition of E into two sets E_+ and E_-, such that if members of E_+ are cable and members of E_- are struts, the resulting tensegrity is prestress stable in $\mathbb{R}^3$.

Conjecture 4.1.9 *(Connelly) Let M be a convex subset of $\mathbb{R}^2$ with polygonal or no boundary, and let $\mathcal{H}$ be a collection of convex open polygons in M. Suppose $M \backslash (H \cup \partial M)$ does not support tension. Then there is a triangulation of $M \backslash H$ which is flexible in 3-space with the boundary of M pinned.*

We say that tension vanishes at point $p \in M$ if there is no subset of $M \backslash H$ supporting tension that covers this point. In the following subsection we show that for a finite system $\mathcal{H}$ any subset of $M \backslash H$ which supports tension is, in fact, contained in the maximum subset $S_{max} \subseteq M \backslash H$ supporting tension. Thus, when the number of holes is finite, $M \backslash H$ can be partitioned into two polygonal subsets, the maximum subset supporting tension and its complement where tension vanishes. We conjecture that this is also true for infinite case (see Section 4.1.2). The above proposition suggests an interpretation of the area that supports tension as stable one, and of the area that cannot support tension as amenable to deformations requiring little energy.

One of the applications we have in mind is percolation of defects in presence of crystallization. Here $\mathcal{H}$ is not a set of holes but rather a dissemination of some alien substance that have poor contact with our material M. Crystallization cannot take place in a region where the tension has been lost, but we assume that the system has some memory, and therefore tension is vanishing gradually. The reason why we assume that crystallization cannot occur when tension has been lost is that the loss of tension means the loss of a rigid structure as it is suggested by Proposition 4.1.8 and Conjecture 4.1.9. We do not claim that the rigidity or flexibility of a triangulation of the complement of the defects directly explains the behavior of an actual physical system. However, if Connelly's Conjecture 4.1.9 is correct, the existence of tension is a *necessary and sufficient condition of the global rigidity of all triangulations of a planar set with polygonal boundary.*

We assume that tension cannot be lost instantaneously because there is some contact between the material and the alien substance. Before tension vanishes on defects of k^{th} generation it disappears on defects of $(k-1)^{th}$ generation. In Section 4.2 we show that there is $N \in \mathbb{Z}_+$ such that with probability 1 every point of $\mathbb{R}^2$ is covered by a defect of N^{th} generation. It can be interpreted as that there is time $t_c(\mathcal{H})$ such that if the system has not crystallized by time $t_c(\mathcal{H})$ it will never crystallize.

4.1.2 Tension and convexity

J. C. Maxwell (1864, 1869-1872), W. J. M. Rankine (1864), and later L. Cremona (1872) observed that a framework in the plane interpretable as the 1-skeleton of a polyhedral 2-sphere flatly realized in $\mathbb{R}^2$ admits an all non-zero equilibrium stress if and only if this framework is the vertical projection of a polyhedral sphere. This is also true for a partition of $\mathbb{R}^2$ into convex polygons, but in this case an infinite polyhedral surface plays the role of a 2-sphere. These theorems were later rigorously proved by Crapo and Whiteley (1982, 1984, 1993). A general theory of the relationships between stresses, liftings, and Voronoi diagrams can be found in Rybnikov (1999). In fact, this correspondence between stresses and polyhedral surfaces provides much insight into the problem of understanding the redistribution of stress caused by polygonal holes. It turns out that in the Maxwell construction a *tension* on the 1-skeleton of a polygonal partition corresponds to a *convex lifting* of the partition. Therefore all of the complement of the holes supports tension if and only if the holes can be lifted to *different* faces of some convex surface. Projecting such surface back on the plane we get a partition of the complement of the holes whose skeleton support an all non-zero tension.

Proposition 4.1.10 *Let $\mathcal{H}$ be a discrete system of holes in $\mathbb{R}^2$. The complement of H supports tension if and only if there is a continuous function defined on $\mathbb{R}^2$ whose graph is a convex polyhedral surface in $\mathbb{R}^3$ such that all connected components of H are vertical projections of different facets of this surface.*

For example, on **Fig. 4.2** we see that no convex piecewise-linear function can lift the triangles to three different facets, since it would imply that these triangles can be mutually separated in $\mathbb{R}^2$ by three lines meeting at one point. This is clearly impossible for the configuration depicted on **Fig. 4.2**. Similarly, on **Fig. 4.6** we see that the complement of the original system of holes (thin solid line) does not support a tension. However, the system of enlarged holes (dash line) supports tension. Holes A, B, and C form a "pinwheel configuration" like one on **Fig. 4.2**.

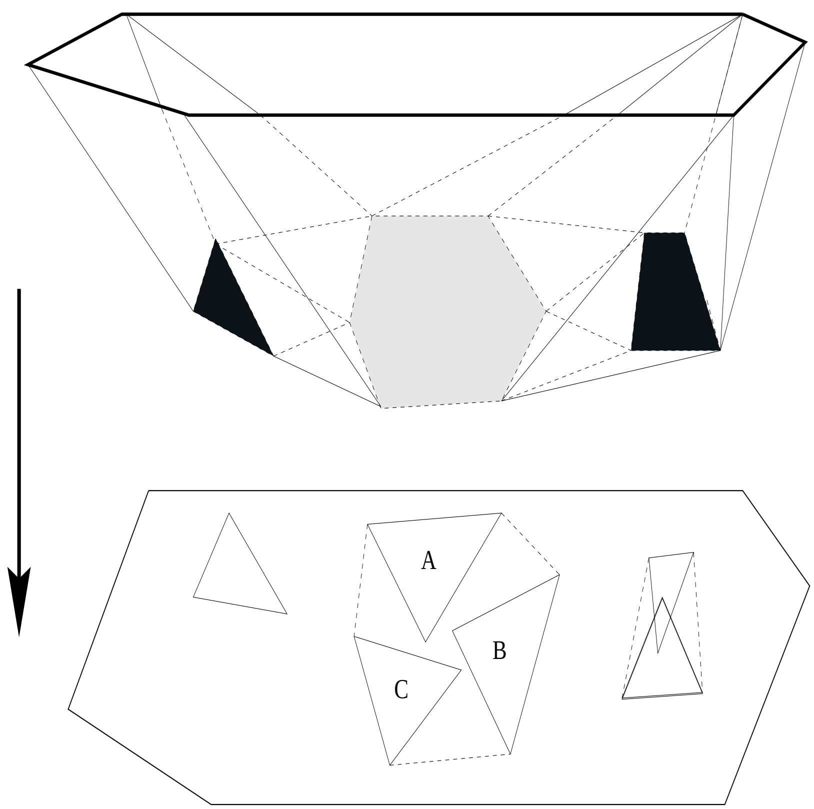

Figure 4.6: Lifting holes: three typical cases

As we have seen the Maxwell correspondence helps to understand the redistribution of tension in the case of a finite number of holes. In fact it also suggest an algorithm for computing the maximum subset of the complement of a finite system of holes that supports tension (Connelly, Mitchell, Rybnikov (1998, 1999)). The following theorem shows that one can exactly define the maximum subset of the complement that supports a tension, and illustrates the use of Maxwell correspondence.

Lemma 4.1.11 *Let M be a convex polygon in $\mathbb{R}^2$ (M might be all of $\mathbb{R}^2$), and let $\mathcal{H}$ be a finite collection of convex polygons in M. There is a set H_{min} such that*

1) $H \subset H_{min}$,

2) $M \backslash H_{min}$ supports tension,

3) There is no proper subset of H_{min} satisfying the first two conditions.

Furthermore, H_{min} is unique.

Proof. Let H' be a set of holes that satisfies the first two conditions, namely, $H \subset H'$ and $M \backslash H'$ supports tension. Recall, that any system of holes satisfying the first two conditions is actually a collection of edge-disjoint convex polygons, i.e. no pair of polygons in the collection share an edge.

Suppose not all of the vertices of H' come from the vertices of H. We show that in this case H' is not minimal, which means that there is a subset of H' that satisfies Conditions 1 and 2 and is contained in H'. Let D be a polygonal hole of H' with a vertex v which does not belong to the vertex set of H. Vertex v can be separated from all other vertices of D by a line passing though some two vertices of H. The resulting triangle Δ can be cut off D without violating the property of $M \backslash (H' \backslash \Delta)$ to support tension. Let us illustrate it with the use of Maxwell correspondence. $M \backslash H'$ has a triangulation that lifts to a convex surface. Denote the lifting map by L. Let us modify the surface by cutting off the corner of the vertex $L(v)$ by a plane passing through the other two vertices of Δ and cutting through all other edges incident to $L(v)$ at

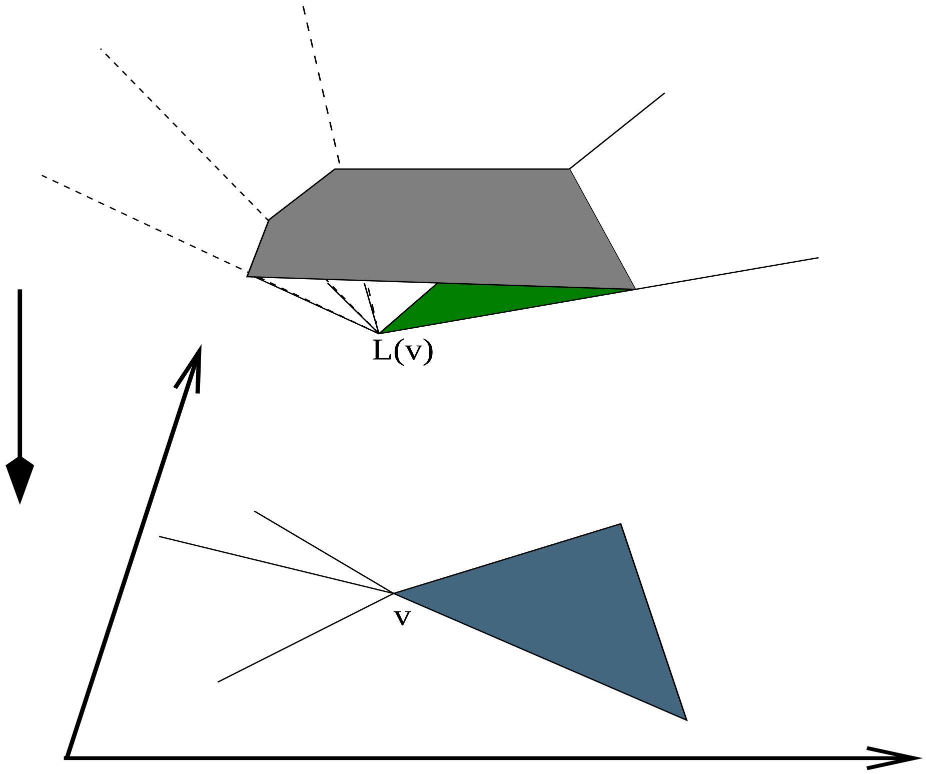

Figure 4.7: Cutting a corner

a sufficiently small distance from $L(v)$ (see **Fig. 4.7**). By Maxwell correspondence the projection of the resulting surface back on the plane gives a triangulation that supports tension in its 1-skeleton. Thus, if a minimum subset exists then it consists of holes all whose vertices come from the vertices of H.

There is a finite number of way to construct a system of holes in M such that

1) it covers H,

2) its vertices come from vertices of H and M,

3) its complement supports tension.

Pick among such systems of holes a system with minimal area. Denote it by H_1. We claim that it is, indeed, H_{min}. Assume that there is a proper subset H_2 of H_1 that satisfies the first two conditions. The procedure described above can be repeatedly applied to H_2 until the vertex set of what is left of H_2 is a subset of the vertex set of H. But then the resulting collection of holes satisfies the first two conditions, and has smaller area than H_1. It contradicts to the choice of H_1. Thus H_1 is minimal with respect to inclusion. Now, let's show that H_1 is actually contained in any system of holes satisfying the first two conditions. Suppose H' is another system of holes which does not contain H_1, but satisfies the first two conditions. Consider a system of holes whose holes are intersections of the holes of H' and H_1. Denote it by H_{mesh}. Since both H' and H_1 cover H, H_{mesh} covers H too.

By definition $M \backslash H_1$ and $M \backslash H'$ both support tension, i.e. have partitions whose 1-skeletons support tension. Denote these skeletons G_1 and G'. Consider a new graph, whose vertex set consists of all the vertices of the old graphs, G_1 and G', and all edge intersections of the old graphs (this graph is what we see when G_1 and G' are drawn on the same picture of M). The edges of the new graph are defined accordingly. Denote it by G_{mesh}. Tensions on G_1 and G' give rise to a tension on G_{mesh}. Thus, G_{mesh} gives a partition of $M \backslash H_{mesh}$ whose 1-skeleton supports an all non-zero tension. Therefore we have a system of holes on M that satisfies the first two conditions, and is contained in H_1. But we have already shown that this is impossible. Thus $H_1 = H_{min}$. $\square$

If $\mathcal{H}$ is infinite we can still conclude that a maximal (with respect to inclusion) set supporting tension is the complement of a set of edge-disjoint polygons all whose vertices come from the vertex set of H, since the argument about cutting a corner can be also applied to an infinite surface.

As a consequence of the above theorem, the problem of finding the maximal area supporting tension can be rephrased as the following optimization problem on convex polyhedra. Find a convex surface in $\mathbb{R}^3$ such that each hole of $\mathcal{H}$ is covered by the vertical projection of a facet of the part of this surface visible from the plane, and the total area of the projections of the facets that cover holes is minimal possible. Another important implication of this theorem is that for finite systems of holes the complement of the set where tension *vanishes* must *support* tension, i.e. it has a triangulation which is a spider web. The case of infinite system of holes is more complicated. Even under additional restrictions on the system of holes, like that the vertices of the holes form an (r, R)-system, or that the sizes of the holes are uniformly bounded from both above and below it is not obvious that the union of all subsets of holes supporting tension can be represented as the complement of a discrete set of edge-disjoint polygons.

Conjecture 4.1.12 *Let $\mathcal{H}$ be an infinite discrete system of polygons in $\mathbb{R}^2$. Then the union of all subsets of $\mathbb{R}^2 \setminus H$ supporting tension can be represented as the complement of a discrete set of edge-disjoint convex polygons.*

In fact, we conjecture that the complement of the set where tension vanishes is not only polygonal, but also supports tension. Notice that our main result is about the *vanishing* of tension and the validity of this conjecture are irrelevant to our proof.

4.2 Tension Percolation Models

When not all of the complement of H supports tension, it is natural to ask what is the maximum subset of $S \setminus H$ supporting tension.

Connelly and coauthors (1998, 1999) showed that the problem of finding the subset of maximum area supporting tension can be reduced to a linear programming problem, and as a consequence there is a polynomial (in the number of vertices of the holes) algorithm finding the maximal tensed subset.

This polynomial algorithm can be effectively implemented and used for computer simulation of the membrane model for different distributions of holes and their parameters. However, this empirical evaluation for finite systems does not allow us to understand the limiting behavior of the system when the number of holes is infinite. In this section we consider a number of probability models for the distribution of holes and their sizes.

In the previous sections we looked at the behavior of tension on a finite membrane with polygonal defects. A natural question is how the tension is redistributed on an infinite membrane (film) with polygonal holes. One can assume different distributions of the holes and their shapes. The naïve model would be that all holes are associated with nodes distributed according to Poisson law, and their shapes are independent of the Poisson distribution. We show that assuming that the holes are distributed according to Poisson law with parameter λ, and the shapes of the holes are IID and independent of the Poisson distribution, the tension vanishes on all of $\mathbb{R}^2$ for any positive value of Poisson parameter λ.

A more realistic approach would be as follows. Consider a Voronoi partition of the plane whose set of sites is a realization of a Poisson distribution with parameter λ. Declare each Voronoi domain normal with probability p and defective with probability $1-p$ (for percolation on Voronoi partitions see Vahidi-Asl and Wierman (1990)). For each defective domain define a convex hole centered at the Voronoi center of this domain. We suggest a family of model in which each convex polygonal hole is situated at a small neighborhood of its Voronoi domain. We associate with each hole three distributions: the distribution of the number of vertices of the hole, the distribution of the angular coordinate of a vertex of the hole, and that of the radial coordinate of the vertex. Let ν be any discrete distribution for the number of vertices of a defect. Let ρ be a distribution for the length of the radius-vector of vertex. A reasonable choice for ρ would be some random variable taking on all values between the radius of the inscribed circle and the radius of circumscribed circle of the domain. A natural choice for the distribution of the angular coordinate is a uniform distribution on the circle. The basic question is whether there is a critical value of parameter p such that there is an infinite connected component of $\mathbb{R}^2$ supporting tension. A special case of such model would be one where the holes coincide with the closures of their Voronoi domains. Notice that, the problem is meaningful, only if we consider the removal of the closures of the domains, since the 1-skeleton of any Voronoi partition supports tension (see Rybnikov (1999)). We conjecture that in this case for any value of $\lambda > 0$ and for any value of $p < 1$ there is no infinite connected component supporting tension. The constructions involving Voronoi partitions are motivated by a natural desire to have a model in which the effect of *pinwheel configuration* is significant. In the "naïve" Poisson model the vanishing of tension is caused by the overlapping of holes, which is in turn is caused by the independence of the distribution of the hole sizes and the underlining Poisson process.

A lattice analog of continuous tension percolation models was introduced by Connelly and Rybnikov (1998, 1999). Let $\mathbb{T}$ be a triangular lattice on the plane. Remove each edge with probability $1-p$. The question is whether there is a critical value of probability p_c^t such when $p > p_c^t$ there is an infinite subgraph of $\mathbb{T}$ supporting tension. This looks reasonable for modeling tension in a membrane, but the assumption that all atoms lie on three families of parallel lines does not seem to be realistic for describing properties of molecular systems like glasses. Thus, a generic version of this problem should have instead of $\mathbb{T}$ a generic triangular lattice. A paper by Connelly, Rybnikov, and Volkov in which the critical probability p_c^t is found for both the regular and generic triangular lattices is to appear. Note that the problem of tension percolation on a lattice is related to rigidity percolation on lattices. See Holroyd (1998) for rigorous estimates of the critical probability of rigidity percolation and Duxbury, Jacobs, Thorpe, Moukarzel (1995, 1996, 1997, 1998, 1999) for simulation results and their physical interpretation.

4.3 Percolation on iterative defects

In this section we show that if the centers of holes are distributed in $\mathbb{R}^2$ according to Poisson law and their shapes are IID, tension disappears on all of $\mathbb{R}^2$. Moreover, the following results on iterative convex hulls are true not only for dimension two, but also for $\mathbb{R}^d$. The proofs can be adopted straightforwardly. While discussing iterative convex hulls it is convenient to adopt a slightly more general definition of hole, without the polygonality restriction.

Definition 4.3.1 *A hole (f-hole) centered at $p \in \mathbb{R}^d$ is a region*

$$H(p, f) = \{p + f(\frac{\mathbf{x}}{\|\mathbf{x}\|})\mathbf{x} \mid \|\mathbf{x}\| \leq 1\}$$

where f is a continuous positive function defined on a unit $(d-1)$-sphere.

Definition 4.3.2 *We say that a collection of holes $\mathcal{H} = \{H_1(p_1, f_1), H_2(p_2, f_2), \dots\}$ is uniformly bounded from below, if there exist $r > 0$ such that $f_i(\mathbf{x}) \geq r$ for all f_i and $\|\mathbf{x}\| = 1$. Similarly, we say that $\mathcal{H}$ is bounded from above, if there exist $R > 0$ such that $f_i(\mathbf{x}) \leq R$ for all f_i and $\|\mathbf{x}\| = 1$.*

Consider a $d-$dimensional Poisson point process with rate λ. Let $Y = Y(\omega)$ be the collection of nodes of some realization ω of the process. Each node $y \in Y(\omega)$ is the center of a hole $H(y, f_y)$, where function f_y is positive and continuous.

Let μ be a probability measure on some subspace of $C(\mathbb{S}^{d-1}, \mathbb{R}_+)$. Suppose that for each y the function f_y is chosen from a distribution μ independently of the other functions and the configuration ω. Therefore, the holes $H(y, f_y)$ are IID. Under this condition, it is known that if $\mathcal{H}$ is bounded both from above and below, there exists some $\lambda_c = \lambda_c(\mathcal{H}, \mu)$ such that for $\lambda < \lambda_c$ there exists no infinite cluster, and for $\lambda > \lambda_c$ such cluster exists a.s (see Menshikov and Sidorenko (1987)).

The next statement is our main percolation result.

Theorem 4.3.1 *For any distribution μ and any $\lambda > 0$ there exists a nonnegative integer $N = N(\mu, \lambda)$ such that a.s. there is an infinite N-cluster whose convex hull coincides with $\mathbb{R}^d$.*

To prove this theorem, we reduce it in two steps (see Lemmas 4.3.3 and 4.3.4) to a simpler proposition (see Proposition 4.3.6).

Lemma 4.3.3 *It is sufficient to prove Theorem 4.3.1 only for the case when the set of defects is bounded from below by some $r > 0$.*

Proof. According to definition, a hole corresponds to a continuous and positive function $f(u)$ on a unit sphere $\mathbb{S}^{d-1}$. Since $\mathbb{S}^{d-1}$ is compact and $f(u)$ is continuous, $f(u)$ achieves its minimum on $\mathbb{S}^{d-1}$, and this minimum is strictly positive. Let $\rho(f) := \min_{u \in U} f(u) > 0$. The distribution μ on functions f induces a distribution on a real-valued *positive* random variable ρ. Therefore, there must exist $r > 0$ such that $P(\rho \geq r) = p_0 > 0$.

Now we couple the original process with rate λ and the functions f_y distributed according to μ, with a process which culls all the defects for which $\rho(f) < r$. It is straightforward that this process is also a Poisson process, but with rate $\tilde{\lambda} = \lambda p_0$. The distribution of defects $\tilde{\mu}$ for the thinned process is such that the radius of the inscribed circle for *any* defect is at least r. Consequently, the set of holes is now uniformly bounded below. If we show that the statement of Theorem 4.3.1 holds for $\tilde{\mu}$ and any $\tilde{\lambda} > 0$, it will imply Theorem 4.3.1 for a generic μ, as the set of holes in "tilde" model is stochastically smaller than that of the original one. □

Hence, we can assume that each hole ($\equiv$ defect of 0^{th} generation) corresponding to a node $y \in Y$ contains a ball of radius r with the center at y. Since taking a convex hull is a monotonous operation (namely, if $A \subseteq B$ then $conv(A) \subseteq conv(B)$), it suffices to prove Theorem 4.3.1 for the case when all holes are the balls of radius

r. Moreover, without loss of generality we can assume that $r = 1$ as we can always re-scale the space. Consequently, Theorem 4.3.1 follows from the following lemma.

Lemma 4.3.4 *Let $\mathcal{H}$ be a collection of balls of unit radius whose centers are distributed in $\mathbb{R}^d$ according to Poisson law with parameter $\lambda > 0$. Then there exists a nonnegative integer $N = N(\lambda)$ such that a.s. there is an infinite N-cluster whose convex hull is $\mathbb{R}^d$.*

4.3.1 Proof of Lemma 4.3.4

We will present the proof for the planar case, i. e. $d = 2$. The arguments can be generalized for $d \geq 3$ rather easily. Pick a hole (which is a unit circle) from $\mathcal{H}$, and denote it by $G(0)$. Examine all circles from $\mathcal{H}$ which have non-empty intersection with $G(0)$, take the convex hull of them and $G(0)$, and denote the resulting set by $G(1)$. Then consider all circles of $\mathcal{H}$ which intersect $G(1)$, and call the convex hull of $G(1)$ and their union $G(2)$. Iterating this procedure we construct the sets $G(2), G(3), \dots$. If for some k no element of $\mathcal{H}$ intersects with $G(k)$, we have $G(k + l) \equiv G(k)$ for all $l \geq 0$.

Proposition 4.3.5 *$G(k)$ is contained in some defect of k^{th} generation.*

This statement is obvious, because the operation of taking convex hull is commutative.

For any point $a \in \mathbb{R}^2$ let (ρ_a, φ_a) be its polar and (x_a, y_a) be its Cartesian coordinates. Without loss of generality assume that $G(0)$ is centered at the origin. Denote by $\mathcal{C}(k)$ the circle of radius k centered at the origin.

Proposition 4.3.6 *With a positive probability depending on λ only,*

$$\mathcal{C}(k+1) \subseteq G(k) \text{ for all } k = 1, 2, 3, \dots \tag{4.3}$$

that is, $G(k)$'s eventually cover all $\mathbb{R}^2$ as $k \to \infty$.

Proof. Here we actually prove a stronger result: we show that (4.3) holds even if $G(k+1)$ were constructed using $G(k)$ and *only* those circles of $\mathcal{H}$ whose centers lie inside the ring

$$R_{k+1} = \{a \in \mathbb{R}^2 : \ k + \frac{3}{2} \le \rho_a \le k+2\}.$$

$G(\cdot)$ is an increasing process taking values in the subsets of $\mathbb{R}^2$. Observe that $G(k-1) \subseteq \mathcal{C}(k+1))$, and let E_k be the event "$\mathcal{C}(k+1) \subseteq G(k)$". Observe that the "smallest" value $G(k)$ can take on E_k is $\mathcal{C}(k+1)$. Therefore, if we can demonstrate that

$$\begin{aligned} &\mathsf{P}(E_{k+1} \,|\, G(k) = \mathcal{C}(k+1), G(k-1), G(k-2), \ldots, G(1)) = \\ &\qquad = \mathsf{P}\left(E_{k+1} \,|\, G(k) = \mathcal{C}(k+1)\right) \ge 1 - \gamma_k, \end{aligned} \tag{4.4}$$

it will imply that $\mathsf{P}(E_{k+1} \,|\, E_k, E_{k-1}, \ldots, E_1) \ge 1 - \gamma_k$ and as a result

$$\mathsf{P}(\bigcap_{k=1}^{\infty} E_k) = \prod_{k=1}^{\infty} (1 - \gamma_k) > 0 \tag{4.5}$$

as soon as $\sum_k \gamma_k < \infty$.

It is straightforward that $\mathsf{P}(E_{k+1} \,|\, G(k) = \mathcal{C}(k+1)) > 0$ for all k, so it suffices to estimate γ_k from (4.4) only for large values of k. Suppose that $G(k) = \mathcal{C}(k+1)$ and break the ring R_k into $M = M(k) := 2\pi\sqrt{k+3}$ congruent pieces

$$R_{k+1}^{(i)} \;=\; \{a \in R_{k+1} : \ \varphi_a \in (\alpha i, \alpha(i+1)]\}, \quad i = 0, 1, \ldots, M-1$$

$$\text{where } \alpha \;=\; \alpha(k) = \frac{2\pi}{M} = \frac{1}{\sqrt{k+3}}$$

Let the event $E_{k+1}^{(i)}$ be "$R_{k+1}^{(i)}$ contains at least one center of a circle from $\mathcal{H}$". Then these events are independent for different values of i and k, and consequently the probability of the event $\tilde{E}_{k+1} = \cap_{i=0}^{M-1} E_{k+1}^{(i)}$ "each $R_{k+1}^{(i)}$ contains such center" is

$$\mathsf{P}(\tilde{E}_{k+1}) = \left(1 - e^{-\lambda\alpha(k+7/4)/2}\right)^M \ge 1 - Me^{-\lambda\alpha(k+7/4)/2} = 1 - C_1 k^{1/2} e^{-C_2 k^{1/2}}$$

for some positive constants C_1 and C_2 and large k.

Our next step is to show that $\tilde{E}_{k+1}$ implies E_{k+1} whenever $\mathcal{C}(k+1) \subseteq G(k)$. Once this is established, we have

$$\gamma_k \leq C_1 k^{1/2} e^{-C_2 k^{1/2}},$$

and, as a result, $\sum \gamma_k < \infty$, which yields (4.5) and proves the proposition.

Assume that $\tilde{E}_{k+1}$ occurs and consider an arbitrary point

$$a \in \mathcal{C}(k+2). \tag{4.6}$$

Since each of $R_{k+1}^{(i)}$ contains a center of a circle of $\mathcal{H}$ and every such circle obviously intersects $\mathcal{C}(k+1)$, there exist two points b and c in $G(k+1)$ such that

$$\rho_b \geq k + 5/2, \qquad \rho_c \geq k + 5/2,$$

$$\varphi_b \leq \varphi_a \leq \varphi_c \mod 2\pi\ ,$$

$$\varphi_a - \varphi_b \leq \alpha \mod 2\pi\ , \qquad \varphi_c - \varphi_a \leq \alpha \mod 2\pi\ . \tag{4.7}$$

Since $G(k+1)$ is convex, it must contain all points inside the triangle formed by the origin 0 and points b and c. For any two points e and f of the plane, let $[e, f]$ denote the segment with these points being its endpoints. There exists a point h lying on the segment $[b, c]$ such that $[0, h]$ is orthogonal to $[b, c]$. Notice that this point has the smallest distance from the origin among all points of the segment $[b, c]$. Let us get the lower bound on ρ_h. From (4.7) it follows that at least one of the angles between $[0, h]$ and $[0, b]$ or $[0, c]$ does not exceed α. Suppose it is the first one. The length of $[h, b]$ in this case is smaller or equal to

$$\rho_b \sin\alpha \leq \rho_b \alpha = \frac{\rho_b}{\sqrt{k+3}}$$

whence

$$\rho_h \geq \sqrt{\rho_b^2 - \frac{\rho_b^2}{k+3}} \geq \sqrt{\frac{(k+5/2)^2(k+2)}{k+3}} \geq k+2$$

as $(k+5/2)^2 > (k+2)(k+3)$. Consequently, equations (4.7) and 4.6 yield that the point a lies inside the triangle $\Delta(b0c)$ and hence belongs to $G(k+1)$. $\square$

Let us make some observations which rely on obvious generalizations of Proposition 4.3.6. In our discussion we will rely on the proof presented above.

First, color the plane as an infinite chess-board with the cell size of $1/4$, so that a point with coordinates (x,y) is green if $\lfloor 4x \rfloor + \lfloor 4y \rfloor$ is even, and blue if this sum is odd. The arguments in the proof will remain valid if instead of $\mathcal{H}$ we consider only those circles of it which have "green" centers (these centers form a Poisson point process with the same rate λ on the "green part" of the plane). The similar statement about the "blue" process is valid as well.

Next, note that if while constructing $G(k+1)$ from $G(k)$ we do not use any circle with center a such that

$$\begin{aligned} &-k^{-2/3} \leq \varphi_a \leq k^{-2/3} \bmod 2\pi \quad \text{or} \\ &\pi - k^{-2/3} \leq \varphi_a \leq \pi + k^{-2/3} \end{aligned} \tag{4.8}$$

(and use only the green part of the plane), the arguments above still hold for large k, as $k^{-2/3} = o(\alpha(k))$. Denote the resulting process $G'(k)$, and the probabilities corresponding to (4.4) as $1-\gamma'_k$. Since γ'_k are asymptotically of the same form as γ_k, we have $\sum \gamma'_k < \infty$.

The third observation is that if the circle $\mathcal{C}(k_0)$ $(k_0 > 1)$ is completely covered by some elements of $\mathcal{H}$ with all their centers lying inside $\mathcal{C}(k_0)$, then the probability $p(k_0)$ that the process $G'(k)$ described above (using only green part of the plane and "avoiding" points close to the horizontal axis), can be continued indefinitely, in a way described by (4.3), has the property

$$\lim_{k_0 \to \infty} p(k_0) = 1.$$

This follows from the fact that $p(k_0) = \prod_{k=k_0}^{\infty}(1-\gamma'_k) \geq 1 - \sum_{k \geq k_0} \gamma'_k$ and the sum on the right hand side converges to zero as $k_0 \to \infty$.

Consequently, there exist $k_0 > 1$ such that $p(k_0) \geq 0.9$. Fix k_0 and let ν denote the probability that indeed $\mathcal{C}(k_0)$ is completely covered by circles of $\mathcal{H}$ as described in the previous paragraph. The value ν may be small, however it is positive and therefore there exist a positive integer T such that

$$(4.9) \qquad (1-\nu)^{T+1} < 0.1$$

Consider $T+1$ circles of radius k_0 on the plane with their centers located at the points

$$(4.10) \qquad (0,0),\ (k_0^3,0),\ (2k_0^3,0),\ \ldots,\ (Tk_0^3,0).$$

According to (4.9), the probability that at least one of them is completely covered by circles of $\mathcal{H}$ is larger than 0.9. Arbitrarily pick one such circle (let it be, say, the i^{th} one). The event "the process $G'(k)$ starting from $(ik_0^3,0)$ can be continued indefinitely" is independent of the configuration inside the other circles, since when the process $G'(\cdot)$ reaches the j^{th} circle $(j \neq i)$ $k \approx |i-j|k_0^3 =: mk_0^3$, $m \geq 1$, and therefore the j^{th} circle lies entirely inside the angle

$$-\frac{1}{mk_0^2} < \varphi < \frac{1}{mk_0^2} \text{ or } \pi - \frac{1}{mk_0^2} < \varphi < \pi + \frac{1}{mk_0^2}$$

and at the same time

$$\frac{1}{mk_0^2} \leq \frac{1}{m^{2/3}k_0^2} \approx \frac{1}{k^{2/3}}.$$

Hence, with probability exceeding $(1-(1-\nu)^{T+1}) \times p(k_0) \geq 0.81$ one of the k_0-radius circles with a center at one of the points of (4.10) is completely covered by elements of $\mathcal{H}$ *and* the process $G'(k)$ which starts from i^{th} circle can grow indefinitely.

Now let us tile the plane with the boxes $L \times L$ where $L = 5Tk_0^3$, so that for any $(i,j) \in \mathbb{Z}^2$, the box $B(i,j)$ consists of the points

$$\{\, (iL+x', jL+y') \text{ where } (x',y') \in (0,L]^2 \,\}$$

Suppose that $i+j$ is an even number. We say that $B(i,j)$ is *open* (see **Fig. 4.8**), if

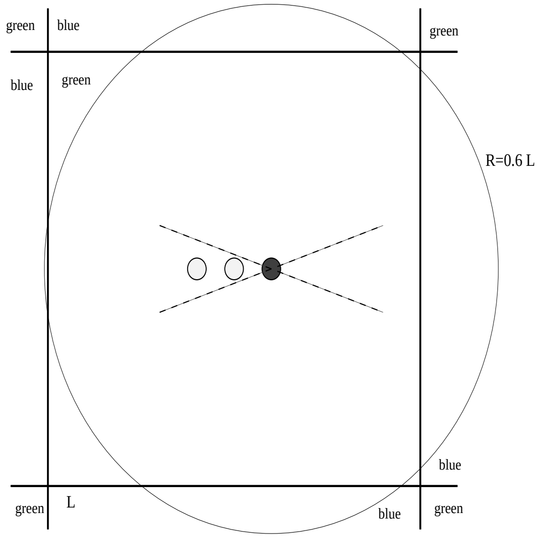

Figure 4.8: Local explosion and coupling

(A) one of $T+1$ circles of radius k_0 with the centers in the set

$$(X_i, Y_j),\ (X_i + k_0^3, Y_j),\ (X_i + 2k_0^3, Y_j),\ \ldots, \qquad (X_i + Tk_0^3, Y_j)$$
$$\equiv ((i+0.6)L, Y_j)$$
$$\text{where } X_i = (i+0.4)L,\ Y_j = (j+0.5)L$$

is completely covered by the elements of $\mathcal{H}$ *and*

(B) the process $G'(\cdot)$ starting with the center a_{ij} of a circle satisfying (A) and using only green points for $k > k_0$ reaches the circumference of radius of $0.6L$ (i.e. $G'(k_1 - 1)$, where $k_1 = \lceil 0.6L \rceil$, contains the entire circle $a_{ij} + \mathcal{C}(k_1)$).

In the case when $i+j$ is odd, we call $B(i,j)$ open if the same is true except $G'(k)$ uses *blue* points.

From the translation invariance and the previous arguments it follows that

$$\mathsf{P}(B(i,j) \text{ is open}) > 0.8$$

Moreover, the boxes $B(i,j)$ are open independently of each other, since the neighboring (by an edge) boxes use points of different colors and $G'(k_1 - 1)$ does not reach any of the corners of the box (since $0.6L < \sqrt{(0.5L)^2 + (0.4L)^2}$). Also observe that when two boxes $B(i,j)$ and $B(i',j')$ sharing a common edge are open, the circles $\mathcal{C}(k_1) + a_{ij}$ and $\mathcal{C}(k_1) + a_{i'j'}$ have a non-empty intersection.

We say that two boxes are connected, if there exists a sequence of open boxes starting with the first one and ending with the second one such that each pair of two consecutive boxes in this sequence share an edge. The "box"-cluster containing the box $B(i,j)$ is the set of all boxes connected to it.

Hence, we can couple the boxes $\{B(i,j)\}$ with the site percolation process on $\mathbb{Z}^2$ where each site is open independently of the others with probability exceeding 0.8. However, as follows from Zuev (1987), the critical probability p_c for the site percolation on $\mathbb{Z}^2$ is smaller than $0.68 < 0.8$ and therefore there exists a.s. an infinite open cluster of boxes $B(i,j)$. As a result, Proposition 4.3.5 yields that there exist a.s. an infinite $(k_1 - 1)$-cluster.

Let us prove the following

Proposition 4.3.7 *Consider a site percolation model on $\mathbb{Z}^2$ in the supercritical regime (i.e. $p > p_c$). Then the convex hull of a (unique) infinite cluster is $\mathbb{R}^2$.*

Proof of the proposition. First, we remark that the fact that the infinite cluster is unique whenever it exists follows from Aizenman, Kesten and Newman (1987). Let $C(\infty)$ be such infinite cluster. On the plane with coordinates (x,y) consider the angle

$$A = \{(x,y) \in \mathbb{Z}^2 : \; x \geq 0, \; 0 \leq y \leq x\}.$$

Notice that the plane can be partitioned into 8 cones congruent to A. Let E_A be the event $|A \bigcap C(\infty)| = \infty$, i.e. there are infinitely many points of the infinite cluster lying inside of A. Since this is a tail event, $\mathsf{P}(E_A)$ is either 0 or 1. If it were 0, it would imply that the probability that there are infinitely many points of $C(\infty)$ in each of the other seven congruent to A cones is also 0. This contradicts to the fact that $|C(\infty)| = \infty$. Hence, $P(E_A) = 1$ and the similar is true for the others cones. However, it is easy to see from geometrical observations that if each of this eight cones contains infinitely many points of $C(\infty)$ then $conv(C(\infty)) = \mathbb{R}^2$. $\square$

Hence, the convex hull of the $(k_1 - 1)$-cluster is at least as large as that of an infinite open cluster for the site percolation on $\mathbb{Z}^2$ (rescaled L times), and Lemma 4.3.4 has been proven. $\square$

Chapter 5

Tension percolation on a lattice

5.1 Introduction

We consider regular and generic triangular lattices on the plane where each bond is removed with probability $1-p$. We show that in both cases the lattice cannot support tension for any positive value of $1-p$. Moreover, the complete relaxation of tension (as defined in Section 5.4) occurs in finite time a.s. The problem is solved by reducing it to a bootstrap percolation model on a triangular lattice where the set of rules follows from the geometry of stresses. Those rules are of geometric nature as opposed to simple counts used in standard bootstrap percolation models. Over the past two decades 2-dimensional and 3-dimensional random central-force networks have been used by physicists for modeling the elastic behavior of glasses within the framework of effective medium theory (see Duxbury, Thorpe, Jacobs, Moukarzel (1983, 1995, 1996, 1998, 1999)). It turns out that real glasses are well represented by generic random networks. The success of these methods resulted in good characterization of elastic properties of glasses like $Ge_xAs_ySe_{1-y}$ (Thorpe (1983), Thorpe and Duxbury (1999)). The rigidity analysis of central-force networks has been also used for characterization of physical properties of other substances such has proteins and semiconductors (see Thorpe and Duxbury (1999)). Background information on rigidity theory and its applications to physics can be found in Sections 4.1 and 4.1.2.

Our problem is somewhat similar to the problem of rigidity percolation. Rigidity percolation was intensively studied by physicists and mathematicians over the last 20 years [44, 46, 48]. Obviously, every finite subgraph of a triangular (regular or generic) lattice is contained in a finite rigid subgraph. One can inquire if the graph maintains this property after we remove each edge independently with probability $1-p$. Notice, that the focus of this problem is the combinatorial rigidity in the plane, whereas the focus of our investigation is the behavior of the planar system in the space, since the spider-web property implies a sort of stability in three-space (see Section 4.1). Denote the critical probability of rigidity percolation by p_c^r. For rigidity percolation on a generic triangular lattice Holroyd has shown that $0 < p_c^r < 1$. However he did not give any estimates besides $0 < p_c^r < 1$. Using a combinatorial algorithm of Hendrickson and numerical curve fitting technics Jacobs and coauthors conducted large scale Monte-Carlo simulation experiment [46] and estimated p_c^r for a generic triangular lattice as 0.6602 ± 0.0003. For a regular triangular lattice it is estimated to be lower, but greater than 0.6. Of course their estimations are based on certain assumptions about the behavior of the system as $size \to \infty$. There are no rigorous estimates of the critical probability of rigidity percolation. Menshikov and Rybnikov found a constructive argument allowing to estimate it from above, but their estimate is very poor (worse than 1-10^{-3}) and is of no interest.

5.2 Rigidity and Spider Webs

5.2.1 Self-Stresses on Frameworks

A bar-and-joint framework is a realization of a graph in $\mathbb{R}^d$. Denote by $F(E; V, V_0)$ a framework in $\mathbb{R}^d$ with the edge set E, and the vertex set V with pinned (fixed in $\mathbb{R}^d$) subset of vertices $V_0 \subset V$. Notice, that in mathematics of rigidity there is a tendency to use term *framework* instead of *network* preferred by physicists. Denote by $\mathbf{v}_i$ the

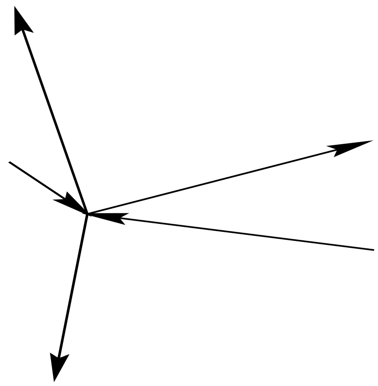

Figure 5.1: Equilibrium stress

vector of coordinates of vertex $v_i \in V$.

Definition 5.2.1 *An equilibrium stress (or self-stress) is an assignment of real numbers $s_{ij} = s_{ji}$ to the edges, a tension if the sign is positive, or a compression if the sign is negative, so that the equilibrium conditions*

$$\sum_{\{j \mid (ij) \in E\}} s_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0$$

hold at each vertex $\mathbf{v}_i \in V \backslash V_0$ (see **Fig. 5.1***).*

A finite framework $F(E;V)$ is called *rigid* in $\mathbb{R}^d$ if and only if there is a neighborhood $N(F)$ of F in the space of parameters $\mathbb{R}^{dV}$ such that any other $\mathbb{R}^d$ realization of graph (E,V) from $N(F)$ with the same lengths of all edges is congruent to $F(E;V)$. Sometimes it is interesting to study rigidity of graphs some of whose vertices are pinned. Framework $F(E;V,V_0)$ is called *rigid with pinned vertices* $V_0 \subset V$ if F has a neighborhood $N(F)$ in $\mathbb{R}^{dV}$ such that any other realization of $(E;V)$ from $N(F)$ with the same lengths of all edges and the set of pinned vertices inherited from the set of pinned vertices of F is congruent to F. If the "neighborhood condition" is dropped the framework is called globally rigid. A framework that is not rigid is called *flexible*.

Definition 5.2.2 *An infinite framework $F(E; V, V_0)$ is referred to as rigid if any finite subgraph of $F(E; V, V_0)$ is contained in some rigid finite subgraph of $F(E; V, V_0)$.*

Some authors call such rigidity *finite rigidity*, because it is defined through finite subgraphs of an infinite framework. There are alternatives to the above definition which employ infinitely dimensional loads or flexes, but it is pretty hard to get sharp results along the lines of "truly infinite" rigidity.

Definition 5.2.3 *We say that a graph $F = (E, V)$ with vertex set V and edge set E realized in $\mathbb{R}^d$ supports tension if there is a non-negative self-stress t on F positive on some edges.*

A framework that has a self-stress strictly positive on *all* edges is called a *spider web*. The following obvious observations immediately follow from the definition of stress.

Proposition 5.2.4 *If $F = (E, V)$ is a spider web, then F has infinite number of edges and vertices and their convex hull is an affine subspace of $\mathbb{R}^d$.*

Proposition 5.2.5 *If $F = (E, V)$ is a spider web, then for each vertex $\mathbf{v}$ of F the convex hull of the vertices adjacent to $\mathbf{v}$ contains $\mathbf{v}$.*

Denote the set of vertices adjacent to $\mathbf{v}$ by $A(\mathbf{v})$.

Proposition 5.2.6 *Let $\mathbf{v}$ be a vertex of a spider web $F = (E, V)$ in $\mathbb{R}^d$. Suppose there is a subset of the vertices of $A(\mathbf{v})$ such that its convex hull affinely spans a hyperplane in $\mathbb{R}^d$. Then, if $A(\mathbf{v})$ affinely spans $\mathbb{R}^d$, the convex hull of $A(\mathbf{v})$ intersects both open half-spaces determined by this hyperplane.*

5.2.2 Tension

The existence of a tension (a strictly positive self-stress) on a framework in the plane implies some important rigidity properties for this network considered as a three-dimensional object. This fact may have some interesting implications for modeling physical properties of materials with networks of Hooke springs and geometry of convex surfaces. Among other things it implies that F is *globally* rigid not only in $\mathbb{R}^2$, but also in $\mathbb{R}^3$ (if F is finite some vertices have to be fixed).

The elasticity and rigidity properties of a glass are related to how amenable the glass is to continuous deformations requiring little energy. From a physical point of view, it is not enough to declare that the distance constraints force the structure to have only one configuration, since the bonds in a physical network do not behave as ideal edges in a framework. There should be a way of describing the behavior of the system as it perturbed. That is why physicists often consider the energy function defined on the edges of a network of Hooke springs; i.e. each spring has some optimal length at which its energy is minimal, stretching or shortening a spring increases the energy of this connection. A tensegrity framework is a generalization of this model where besides Hooke springs there are members whose energy increases with the distance, and members whose energy decreases with the distance. the notion of tensegrity framework is often useful in context of energy considerations (Connelly, Whiteley (1996)).

In a tensegrity framework all edges are partitioned into three types, cables E_+, struts E_-, and bars E_0, i.e. $E = E_0 \cup E_+ \cup E_-$. Together, struts, cables, and bars are called members. If a cable is stretched, the energy in the cable increases. If a strut is shortened, the energy in it decreases. Any change in the length of a bar forces the energy to increase. Therefore, networks of Hooke springs are bar tensegrities from a mathematical point of view.

Let $F(E, E_0, E_+, E_-; V, V_0)$ be some tensegrity framework in $\mathbb{R}^d$. Denote by l_{ij}^0 the length of a member (ij). Note, that the energy $\mathfrak{H}_{ij}$ of member (ij) considered as the function of its squared length l_{ij}

– is monotone increasing if (ij) is a cable,

– is monotone decreasing if (ij) is a strut,

– has a strict local minimum at l_{ij}^0 called the equilibrium length.

It is natural to define an energy function $\mathfrak{H}$ of a finite tensegrity framework as the sum of the energy functions of its members. Thus

$$(5.1) \qquad \mathfrak{H} = \frac{1}{2} \sum_{(ij)\in E} \mathfrak{H}_{ij}(|\mathbf{v}_j - \mathbf{v}_i|^2) = \frac{1}{2} \sum_{(ij)\in E} \mathfrak{H}_{ij}(l_{ij}^2).$$

when all members are bars the simplest way to define the energy function is as follows

$$(5.2) \qquad \mathfrak{H} = \frac{1}{2} \sum_{(ij)\in E} a_{ij}(l_{ij} - l_{ij}^0)^2,$$

where the sum is over all ordered pairs of vertices of the framework, l_{ij} is the length of the bond between i and j, l_{ij}^0 is the equilibrium bond length, and $a_{ij} > 0$ is the spring constant of the bond between i and j.

In the spirit of the definition of equilibrium stress we assume that a strut can support only compression, a cable can support only tension, and a bar can be under either type of stress, depending on whether its length is larger or smaller than l_{ij}^0. For more detailed information on tensegrities see the works of Roth and Whiteley (1981), Connelly and Whiteley (1996), and Connelly (1993).

Consider now the energy $\mathfrak{H}$ as a function of the *coordinates of the vertices* of the framework. A finite tensegrity framework F in $\mathbb{R}^d$ with pinned vertices $V_0 \subset V$ is called *prestress stable* if

1) the differential of $\mathfrak{H}$ – considered as a function on the space of parameters $\mathbb{R}^{dV}$ – at the point corresponding to F is zero, and

2) the second differential of $\mathfrak{H}$ is a positive semidefinite quadratic form; its kernel restricted to infinitesimal motions leaving V_0 unmoved consists of trivial infinitesimal motions of the framework.

Definition 5.2.7 *An infinite tensegrity framework is called prestress stable if every its finite subgraph G is contained in a prestress stable subgraph whose non-pinned vertices contain the vertex set of G.*

The concept of prestress stability is due to Connelly and Whiteley (1993, 1996) and, basically, accounts for local minima of the energy function.

The spider web property implies some interesting properties for planar frameworks which can be interpreted as a sort of strong stability. This is well illustrated the following theorem of Connelly (part 2 is unpublished).

Proposition 5.2.8 *Let $F(E; V, V_0)$ be a (possibly infinite) framework in $\mathbb{R}^2$ with pinned vertices $V_0 \subset V$. Then*

1) framework $F(E; V, V_0)$ is globally rigid in $\mathbb{R}^3$ as a bar framework;

2) if F has no edge intersections, there is a partition of E into two sets E_+ and E_-, such that if members of E_+ are cables and members of E_- are struts, the resulting tensegrity is prestress stable in $\mathbb{R}^3$.

5.3 Triangular lattice models

We consider a regular or slightly perturbed triangular lattice $\mathbb{T}$ on the plane where each edge is removed independently with probability $1 - p$, $p > 0$. Is there a critical value $p_c < 1$, such that for $p > p_c$ there is an infinite spider web subgraph a.s.? We show that for any $p < 1$ there no spider web subgraph a.s.. Thus, no non-trivial p_c exists. Our percolation model is somewhat related to so-called "bootstrap percolation" introduced on trees by Chalupa, Leath, and Reich (1979) and, later, on d-dimensional lattices by Kogut and Leath (1981). In these models, points are independently occupied with a low density and the resulting configuration is taken as the initial state for dynamics based on some collection of local rules, in which the

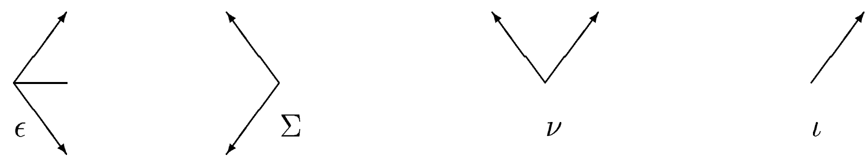

Figure 5.2: Local removal rules

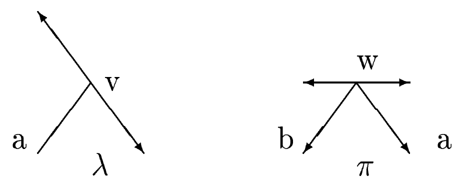

Figure 5.3: Local partial removal rules

occupation status of a point is updated according to the configuration of its neighbors. Aizenman and Lebowitz (1988) were first to conduct a rigorous analysis of this model.

Consider the affine plane $\mathbb{R}^2$ and two vectors $\vec{\mathbf{e}}_{\mathbf{1}}$ and $\vec{\mathbf{e}}_{\mathbf{2}}$ with coordinates $(1,0)$ and $(\frac{\sqrt{3}}{2},\frac{1}{2})$ respectfully . Also, set $\vec{\mathbf{e}}_{\mathbf{3}} = \vec{\mathbf{e}}_{\mathbf{2}} - \vec{\mathbf{e}}_{\mathbf{1}}$. The regular triangular lattice $\mathbb{T}$ is a graph whose vertex set is the collection of all points with coordinates $V(\mathbb{T}) = \{i\vec{\mathbf{e}}_{\mathbf{1}}+j\vec{\mathbf{e}}_{\mathbf{2}},\ (i,j)\in\mathbb{Z}^2\}$, and whose edge set $E(\mathbb{T})$ consists of all edges between vertices $\mathbf{a},\mathbf{b}\in V(\mathbb{T})$ such that $\mathbf{a}-\mathbf{b}=\vec{\mathbf{e}}_{\mathbf{k}}$ or $\mathbf{a}-\mathbf{b}=-\vec{\mathbf{e}}_{\mathbf{k}}$ for $k=1$, 2 or 3. Let us denote an edge between a and b by $(\mathbf{a},\mathbf{b})$.

Suppose that some edges were removed from $\mathbb{T}$. Denote the resulting lattice by $\mathbb{T}'$. By Proposition 5.2.5 edges in configurations congruent to those depicted on Figure 5.2 cannot support tension. We call configurations congruent to those depicted on Figure 5.2 ϵ-, Σ-, ν-, and ι -configurations respectfully, and refer to any such configuration as *relaxed*. By Proposition 5.2.6 edge $(\mathbf{v}a)$ and edges $(\mathbf{w}a)$ and $(\mathbf{w}b)$ on Figure 5.3 cannot support tension. We call such edges *legs* in λ- and π-configurations. We refer to λ- and π-configurations as *partially relaxed*. Therefore, if $\mathbb{T}'$ contains a spider web, this spider web does not contain edges in configuration depicted on Figure 5.2 and edges that are legs in λ- or π-configurations.

Assume we have an infinite parallel Turing machine that can operate on the stars of the vertices of an infinite (but locally finite) grid; the machine works on all stars simultaneously. Once the machine sees a star where edges form one of the configurations congruent to those depicted on Figure 5.3 or Figure 5.2 (ϵ, Σ, ν, ι, λ, π), it removes all edges that cannot support tension. The machine proceeds for as long as there are edges that can be removed using the local rules given by Figure 5.3 and Figure 5.2. In Section 5.4 we show that if the initial lattice $\mathbb{T}^p$ was obtained as the result of the independent edge removal with probability p, the parallel machine operating on the grid requires only a finite number of steps to turn $\mathbb{T}^p$ into a graph with no infinite components.

Now, suppose that each edge is removed from $E(\mathbb{T})$ with probability $p > 0$ independently of the others. We will refer to the resulting lattice with randomly removed edges as $\mathbb{T}^p$. Our main result is

Theorem 5.3.1 *For any $p > 0$ the lattice $\mathbb{T}^p$ cannot support tension a.s.*

However, first we want to prove

Lemma 5.3.1 *With a* positive *probability $\mathbb{T}^p$ cannot support tension.*

Proof of the Lemma. As follows from Proposition 5.2.5, any configuration of edges congruent to those depicted on Figure 5.2 cannot support tension. Therefore, the lattice $\mathbb{T}^p$ can support tension if and only if the lattice $\mathbb{T}^p(1)$ obtained from $\mathbb{T}^p$ by removing all edges in such configurations can support tension. We call these edges *implicitly* removed, as opposed to *initially* removed edges, that is, $\mathbb{T}\backslash\mathbb{T}^p$. Similarly, we construct the lattice $\mathbb{T}^p(2)$ by removing all edges from $\mathbb{T}^p(1)$ in configurations congruent to the ones on Figure 5.2. In the same manner we define lattices $\mathbb{T}^p(3)$,

$\mathbb{T}^p(4), \ldots$, etc. Notice, that if $\mathbb{T}^p(n+1) \equiv \mathbb{T}^p(n)$ for some n, then $\mathbb{T}^p(n+k) \equiv \mathbb{T}^p(n)$ for any positive integer k.

Let $\mathcal{H}(k)$ be a regular hexagon centered at the origin with a side of length k, i.e., the hexagon with the vertices $k\vec{\mathbf{e}}_1$, $k\vec{\mathbf{e}}_2$, $k\vec{\mathbf{e}}_3$, $(-k)\vec{\mathbf{e}}_1$, $(-k)\vec{\mathbf{e}}_2$ and $(-k)\vec{\mathbf{e}}_3$. Let $F(k)$ denote the event "all interior edges of $\mathcal{H}(k)$ have been (possibly implicitly) removed from $\mathbb{T}^p(k')$ for some k'". It is obvious that

$$\mathsf{P}\left(F(k+1) \mid \bigcap_{i=k_0}^{k} F(i)\right) = \mathsf{P}(F(k+1) \mid F(k)).$$

Let us show that

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i) \mid F(k_0)\right) = \prod_{k=k_0}^{\infty} P(F(k+1) \mid F(k)),$$

Indeed,

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i) \mid F(k_0)\right) = \mathsf{P}\left(\bigcap_{i=k_0+2}^{\infty} F(i) \mid F(k_0+1)F(k_0)\right) \mathsf{P}\left(F(k_0+1) \mid F(k_0)\right)$$

$$= \mathsf{P}\left(\bigcap_{i=k_0+2}^{\infty} F(i) \mid F(k_0+1)\right) \mathsf{P}\left(F(k_0+1) \mid F(k_0)\right) = \ldots$$

$$= \mathsf{P}\left(\bigcap_{i=N+1}^{\infty} F(i) \mid F(N)\right) \mathsf{P}\left(F(N) \mid F(N-1)\right) \ldots \mathsf{P}\left(F(k_0+1) \mid F(k_0)\right) =$$

$$\prod_{k=k_0}^{\infty} P(F(k+1) \mid F(k)).$$

We claim that the probability of the event $\{F(k+1) \mid F(k)\}$ for large k is greater than $1-\gamma_k$, for some sequence $\{\gamma_k\}$, such that $\sum_{k=k_0}^{\infty} \gamma_k < \infty$. This would yield

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i) \mid F(k_0)\right) \geq \prod_{k=k_0}^{\infty} (1-\gamma_k) > 0,$$

which, in turn, would prove the Lemma, since $P(F(k_0)) > 0$ for any fixed k_0 and positive p. Indeed, the probability that on each of the six sides of $\mathcal{H}(k)$ at least one edge has been initially removed is

$$\left(1-(1-p)^k\right)^6 = \left(1-e^{-\alpha k}\right)^6 > 1-6e^{-\alpha k}$$

where $\alpha = -\log(1-p) > 0$. Now, pick k_0 so large that $1-6e^{-\alpha k}$ is positive as soon as $k \geq k_0$. Set $\gamma_k = e^{-\alpha k}$. Then $\sum \gamma_k$ is, indeed, finite. Meanwhile, as one can conclude upon studying Fig. 5.3, whenever there are no edges inside $\mathcal{H}(k)$, and at least one

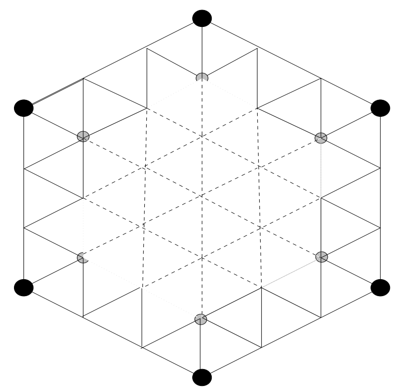

Figure 5.4: Typical propagation of a regular hexagon. Solid lines are remaining edges, dotted lines are removed ones. Dark circles are vertices of $\mathcal{H}(k)$, and grey circles are vertices of $\mathcal{H}(k-1)$

edge is removed on each side of it, an incremental application of the removal rules will eventually, (in a number of steps not exceeding k), delete all edges inside $\mathcal{H}(k+1)$. Therefore, with a positive probability $F(k)$ implies that all the edges of our lattice are eventually removed. □

Below, we will refer to the process described in the above proof as "hexagon propagation". We will make use of the following definition.

Definition 5.3.2 *We say that the sequence of planar lattices $L(n)$* eventually disappears *and write $L(n) \to \emptyset$, if for any fixed bounded subset A of the plane there exists $N > 0$ such that $L(n) \bigcap A = \emptyset$ for all $n \geq N$.*

Therefore, the above Lemma immediately implies

Corollary 5.3.3 *With a positive probability, $\mathbb{T}^p(n) \to \emptyset$. Moreover, conditioned on the event "all edges are initially removed in $\mathcal{H}(k)$",*

$$\mathsf{P}(\mathbb{T}^p(n) \to \emptyset) \to 1$$

as $k \to \infty$.

We would like to make another observation about the proof of Lemma 5.3.1. Suppose the interior of $\mathcal{H}(k)$ is empty. Evidently, to remove all edges from $\mathcal{H}(k+1)$ with the use of the local removal rules described above we need that at least one edge is absent (initially removed) on each side of $\mathcal{H}(k)$. Suppose we are not allowed to look for such initially removed edges in the planar cones (angles) defined by inequalities $|\varphi| \leq 30^o$ and $|\varphi - 180^o| \leq 30^o$, in the polar coordinate system (ρ, φ) (see Fig. 5.3). Evidently, the arguments of the proof can be carried through virtually unchanged.

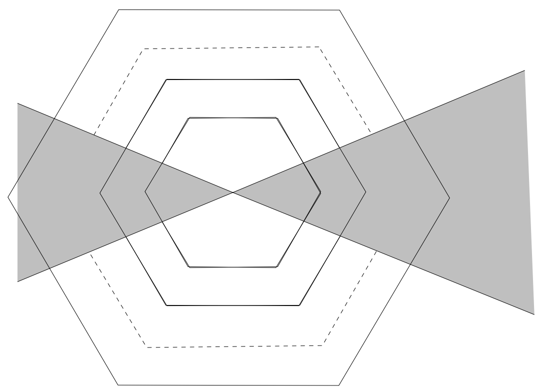

Figure 5.5: Hexagon propagation avoiding two angles

Thus we have

Corollary 5.3.4 *Independently of the initial configuration inside the above mentioned angles*

$$\mathsf{P}(\mathbb{T}^p(n) \to \emptyset | \text{“all edges are initially removed in”} \mathcal{H}(k)) \to 1$$

as $k \to \infty$

Proof of Theorem 5.3.1. Fix $\epsilon > 0$. By Corollary 5.3.4 there is N such that if each edge in $\mathcal{H}(N)$ has been removed, the probability that $\mathbb{T}^p(n) \to \emptyset$ is greater than $1 - \epsilon/2$, regardless of the configuration inside the two angles. Let $q = q(N)$ be the probability that all edges inside $\mathcal{H}(N)$ have been initially removed. Obviously, $q > 0$ for any positive p. There is a positive integer M such that

$$1 - (1-q)^M > 1 - \epsilon/2.$$

Consider M non-overlapping hexagons of size N along the horizontal axis with the centers at $\mathbf{0}$, $N\vec{\mathbf{e}}_\mathbf{1}$, $2N\vec{\mathbf{e}}_\mathbf{1}, \ldots,$ $(M-1)N\vec{\mathbf{e}}_\mathbf{1}$ (see Fig. 5.3). Notice, that each of the

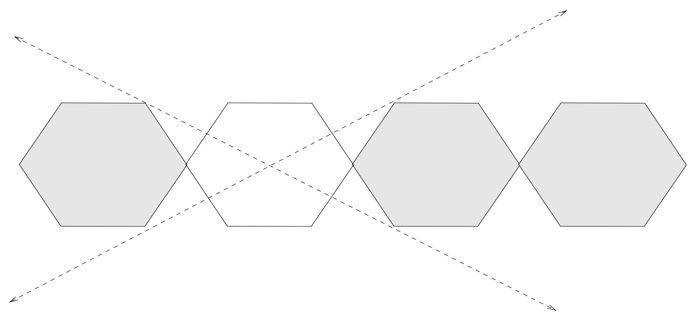

Figure 5.6: M hexagons; with probability $1 - \epsilon/2$ all edges are removed in at least one of them

hexagons lies fully inside $\pm 30^\circ$ angles for all the others; therefore, due to symmetry and space homogeneity, each of them propagates to infinity in the way described in the proof of Lemma 5.3.1, with an angular restriction of Corollary 5.3.4, with probability at least $1 - \epsilon/2$ *independently* of the initial configuration inside the others. Thus, the probability that inside at least one of the M hexagons all edges have been initially removed *and* it will propagate to infinity is larger than

$$(1 - \epsilon/2)^2 \geq 1 - \epsilon.$$

Now recall the definition of tension. $\mathbb{T}$ supports tension only if there is a subgraph that can bear an all non-zero equilibrium tension. The removal of an edge means that this edge can not belong to a subgraph supporting an all non-zero equilibrium tension. However, the arguments above show that eventually *all* edges are removed with probability $1 - \epsilon$. Since $\epsilon > 0$ is arbitrary, Theorem 5.3.1 holds. $\square$

Studies of rigidity percolation (Jacobs et. al. (1995,1996,1997), Holroyd 1998) show that the behavior of a regular triangular lattice may differ from the behavior of a generic triangular lattice. A generic lattice in a strong sense is a realization of a graph in $\mathbb{R}^d$where the matrix of stresses of any finite subgraph of the lattice has maximal possible rank. All theorems and lemmas in this section hold not only for a regular triangular lattice, but also for any generic triangular lattice obtained from $\mathbb{T}$ by a sufficiently small perturbation, for we essentially need only four removal rules: the ν-rule, the Σ-rule, the ϵ-rule, and the ι-rule (see Lemma 5.3.1 and Figure 5.3). Of course, our tension percolation problem for a perturbed triangular lattice makes sense only if there are perturbations of the regular lattice preserving the property of the lattice to support an all non-zero tension. It follows from the results of Bezdeks and Connelly (1996) on the uniform stability of sphere packings that there is $\epsilon > 0$ such that any ϵ-perturbation of the regular triangular lattice supports an all non-zero tension. In fact, we suspect that all our results hold for a larger class of generic triangular lattices. Our method cannot be applied straightforwardly to the case of an arbitrary generic triangular lattice because a perturbation can turn a (partially) relaxed configuration into a non-relaxed configuration.

A general tension percolation problem can be stated as follows. Let F be an infinite framework in $\mathbb{R}^d$ with discrete vertex set. Suppose we independently remove each edge with probability $1-p$. Denote the resulting graph by F^p. What is the infimum of p's such that F^p supports tension a.s.? We have a general conjecture about tension percolation on planar graphs. To formulate this conjecture we need to introduce the notion of directional specter of a framework. By the direction of a line on the plane we understand the angle it form with a horizontal axis. Let F be a framework on the plane. We call the set of directions defined by the edges of F the *directional specter* of F.

Conjecture 5.3.5 *Let $F = (E; V, V_0)$ be an infinite framework on the plane realized without self-intersections. Suppose the directional specter of F is finite, and the set of edge lengths of F is bounded from below by $l > 0$, and from above by $L < \infty$. Then the critical probability of tension percolation is 1.*

The notion of an (r, R) point set is widely used in discrete geometry and mathematical crystallography. A point set V is called an (r, R)-system, or a Delaunay system, if

1) for any point $\mathbf{v} \in V$ the ball of radius r centered at $\mathbf{v}$ does not contain any other vertices of V, and

2) any ball of radius R contains at least one point of V.

Notice that, for a graph with a finite directional specter the (l, L)-property of the edge set is equivalent to the (r, R)-property of the vertex set.

A worthy goal would be to connect the problem of tension percolation to the general theory of local rules for quasicrystalls developed by Danzer and Dolbilin.

5.4 Finite Time of Relaxation

Assume that it takes one unit of time for an infinite parallel Turing machine to remove all the configurations of edges congruent to those depicted on Figure 5.2. Thus, the lattice $\mathbb{T}^p$ is transformed to $\mathbb{T}^p(n)$ by time n. Let us call this process the *relaxation of tension on* $\mathbb{T}^p$ and say that tension has been completely lost if there is no infinite connected component of non-removed edges on the lattice. We want to show now that the complete relaxation of tension occurs in a finite time a.s.

Theorem 5.4.1 *With probability* 1 *there exists $\bar{n}$ such that $\mathbb{T}^p(\bar{n})$ is a union of finite disjoint graphs.*

Proof. The idea of the proof is based on Theorem 5.3.1. Let N and M are the same as in the proof of Theorem 5.3.1. Pick $\epsilon < \frac{1}{10}$ and N and M corresponding to this ϵ. Consider a partition of the plane into the boxes with the side length $S = 13(M-1)N > (4\sqrt{3}+6)(M-1)N$. Assume one of the boxes – call it B_0 – is centered at $\mathbf{0}$. In this box consider hexagons with the side N centered at $\mathbf{0}$, $N\vec{\mathbf{e}}_\mathbf{1}$, $2N\vec{\mathbf{e}}_\mathbf{1}, \ldots, (M-1)N\vec{\mathbf{e}}_\mathbf{1}$.

We call box B_0 *open* if (1) one of these hexagons has all the edges removed *and* (2) using the procedure of implied edge removal as described by Lemma 5.3.1, it will grow till its upper and lower sides coincide with those of the box B_0 *and* (3) one of the edges on its upper side with the X-coordinate between 0 and $(M-1)N$ has been initially removed (see Fig. 5.4 and 5.3).

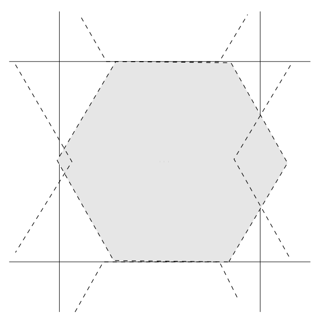

Figure 5.7: Hexagon propagation inside a box

Following the line of arguments in Theorem 5.3.1, we can conclude that the probability that B_0 is open can be made greater than 0.9 (however, we might need to have N quite large). The same is true about the other boxes of the tiling $\{B_0 + i\vec{\mathbf{e}}_\mathbf{1} + \mathbf{j} \times (\vec{\mathbf{1}, \mathbf{0}}),\ (\mathbf{i}, \mathbf{j}) \in \mathbb{Z}^\mathbf{2}\}$. Moreover, both vertical and horizontal neighbors are open independently, since they "look for" different initially removed edges; again, we ignore the interior of the cones described between Corollaries 3.4 and 3.5. Therefore, all the boxes are open independently of each other. Besides, if two neighboring (at a side) boxes are open, their inside areas where edges are removed are connected.

Now, let us couple the boxes with the vertices in the site percolation model on $\mathbb{Z}^2$ where each site is open with probability 0.9 and closed otherwise. There is a unique open cluster of open sites and no infinite cluster of closed sites (e.g. see Grimmett

(1989)). Therefore, each cluster of closed sites is surrounded by a finite contour of open site. Geometrically, for our triangular lattice, it implies that each connected component of non-removed edges is surrounded by a contour of removed ones, and therefore each such component is finite. □

5.5 Tension on Finite Subgraphs of a Triangular Lattice

While discussing tension on finite graphs, we assume that some of the vertices are pinned. For example, if a finite graph can be regarded as the 1-skeleton of a tiling of a convex polygon, we normally assume that all the boundary vertices of the polygon are pinned. There are a few reasons to study tension percolation on finite subgraphs of a regular infinite graph. First, a finite spider web has nice stability properties (see Section 5.2.2). Second, it is reasonable to suggest that a large finite piece of a triangular lattice describes the behavior of a physical system better than an infinite triangular lattice. Third, to study tension percolation on 3-dimensional lattices it is important to understand quantitatively the effect of edge removal on the ability of a finite subset of a triangular lattice to support tension. While the method developed in the previous sections explain how a triangular lattice loses the ability to support tension as a result of any non-neglectable edge removal, it barely helps to estimate the probability of the existence of tension on a subset of a triangular lattice where each edge has been removed with probability $1 - p$ as a function of the size of the subset.

Let us sketch the connections between tension percolation on a 3-dimensional triangular lattice and spider web properties of finite subgraphs of a 2-dimensional triangular lattice. To introduce a three-dimensional analog of $\mathbb{T}$ we need to enlist the notion of point lattice. Recall, that a point lattice is the set of all points in $\mathbb{R}^d$ that can be represented as integer linear combinations of the vectors of a fixed coordinate frame. A face-centered cubic point lattice – *fcc* lattice – is a lattice of points spanned by the vertex set of a regular 3-simplex. This lattice is a natural generalization of a triangular point lattice. Denote by $\mathbb{T}_3$ the graph whose vertex set is the *fcc* lattice, and whose edge set is all the shortest vectors of the *fcc* lattice. Remove each edge independently with probability $1-p$ and denote the resulting lattice by $\mathbb{T}_3^p$. For what values of p does the modified lattice support tension with a positive probability? Suppose we want to approach this problem in the same way we approached the 2-dimensional problem. Here, instead of a propagating hexagon we have a propagating 3-polytope (see Lemma 5.3.1). Notice, that regular triangular and square lattice are the only types of 2-sublattices of $\mathbb{T}_3$. We call a polytope a lattice polytope if all its faces lie on periodic subgraphs of $\mathbb{T}_3$; note, that our definition of a lattice polytope differs from the standard definition of a lattice polytope used in the theory of lattice points. Thus, the facets of a lattice 3-polytope can be of only two sorts: lying on a square sublattice and lying on a triangular sublattice. The geometry of a facet as well as the geometry of a lattice polytope is not important, since there is only a finite number of lattice polytope in $\mathbb{T}_3$ up to homothety. From this remark it becomes clear that, in principal, tension percolation on $\mathbb{T}_3$ is no different from tension percolation on any periodic graph in $\mathbb{R}^3$ with triangular planar subgraphs. Now, let us compare the hexagon propagation and the polytope propagation. If a side of the propagating hexagon misses an edge, all the side has to go; however, one missing edge on a facet of a triangular type is not enough to conclude that the rest (with fixed boundary) is not able to bear tension. Let n be the length of the longest edge of a facet of $\mathbb{T}_3$. Denote by $P_p(n)$ the probability that after the independent edge removal from a facet of size n with probability $1-p$ the resulting graph with fixed boundary can support tension in internal edges. If $\sum_n P_p(n)$ converges, the arguments of Lemma 5.3.1 and

Theorem 5.3.1 work, and $\mathbb{T}^3_p$, $p < 1$ cannot support tension a.s.

Let us denote by $\mathcal{H}_n$ a hexagonal chunk of a regular triangular lattice with each side of length n. We define the distribution function $\mathbb{F}_n(p)$ (where $0 \leq p \leq 1$, $pn \in \mathbb{Z}$) as the ratio of the number of supporting tension subgraphs of $\mathcal{H}_n \backslash \partial\mathcal{H}_n$ on pn edges (with $\partial\mathcal{H}_n$ pinned) and the total number of subgraphs of $\mathcal{H}_n \backslash \partial\mathcal{H}_n$ on pn edges. $\mathbb{F}_n(p)$ can be, indeed, interpreted as the probability that after the independent deletion of pn edges $\mathcal{H}_n$ still supports tension (with the boundary pinned). Obviously, for each n it is a decreasing function of p. Numerical experiments also show that for each p $\mathbb{F}_n(p)$ decreases as $n \to \infty$. For large n function $\mathbb{F}_n(p)$ should look like a non-decreasing function of p with one inflexion point, although it is very hard to formally prove that $\mathbb{F}_n(p)$ converges to such a function. It is known that for the connectivity and rigidity percolation problems the analogous distribution function has such shape. In connectivity percolation it is the proportion of the subgraphs on pn edges having a component connecting two opposite sides of a rectangle (hexagon) of size n. In the case of rigidity percolation it is the proportion of the subgraphs on pn edges having a rigid component connecting two opposite sides of a rectangle (hexagon)of size n. Jacobs, Thorpe, and Douxbury's simulation results suggest that in the case of rigidity percolation the distribution function converges to an increasing function with one inflexion point. We believe that the limiting behavior of $\mathbb{F}_n(p)$, as $n \to \infty$, is described by the distribution function for the probability model described in Section 5. Let $\mathsf{P}(p)$ be the probability that $\mathbb{T}^p$ supports tension, or in other words, that it has a subgraph which is a spider web. We conjecture that $\mathbb{F}_n(p) \to \mathsf{P}(p)$, as $n \to \infty$.

Before having proved that the critical probability for the 2-dimensional problem is one, we had conducted numerical experiments for the hexagonal fragments of the *regular* triangular lattice of sizes $n = 10$–75. The purpose of the experiments was not only to see the behavior of the value of the critical threshold, but also to produce pictures for subsequent visual analysis. We also compared two algorithms finding the maximum spider web in a graph. One of these algorithms is an integer LP algorithm which solves an optimization problem for the stress matrix of the graph. The other one is a combinatorial approximation algorithm that removes edges from the graph according to the local rules (see above). When it cannot find a removable configuration of edges it declares the remaining subgraph a "spider web", which may not be true. The advantage of this algorithm is its linear running time.

The threshold value of p was estimated through Monte-Carlo trials of the following kind. Remove independently a fraction of edges, and check if the remaining subgraph $\mathbb{T}'$ supports tension. If not, then remove a smaller fraction of edges and start from the beginning. Otherwise remove an edge at random from $\mathbb{T}'$, and check if the resulting graph still has a spider web subgraph; do it until it does not have a spider web component. The concentration of the remaining edges is the threshold value of the conducted trial. For each concentration the average of the threshold values of different trials (we used 50-100 trials) was taken as its threshold value. In fact, only for small values of n we verified the existence of a spider web subgraph. For larger sizes (> 10) we confined ourself to applications of local rules (see above). Therefore, we got the estimates of the threshold value from below; however, for sizes 10–12 we did not encounter any situations were local rules were not able to prove the absence of a spider web subgraph.

The exact algorithm needs to use an integer linear programming (LP) routine over integers, since the matrix of stresses for a piece of a regular triangular grid has only 0, 1, and 2 entries, most of which are 0. Of course, at the implementation stage it is possible to replace an integer LP feasibility routine by a floating number LP routine, but it did not work well for LP implementations that we used.

In Table 5.1 we give the Monte-Carlo estimates of the value of the critical threshold for finite $\mathcal{H}_k$. The first column gives the inverse of the linear size of the hexagonal grid. The second column is the Monte-Carlo estimate of the concentration of retained edges at which the lattice stops to support tension.

$1/n$	Y
.10000000000	.7958333336
.08333333333	.8021727394
.07692307692	.8062717778
.07142857143	.8097171716
.06666666667	.8098670214
.06250000000	.8140784320
.05882352941	.8138015374
.05555555556	.8186090220
.05263157895	.8169435026
.05000000000	.8181771634
.04545454545	.8222804782
.04166666667	.8212864870
.03703703704	.8266064252
.03448275862	.8291585516
.03225806452	.8301513160
.02941176471	.8308974362
.02702702703	.8323039906
.02500000000	.8328988380
.02380952381	.8430081056
.02272727273	.8420009690
.02173913043	.8438519094
.02000000000	.8451033646
.01818181818	.8457304990
.01666666667	.8472701975
.01515151515	.8524133963
.01388888889	.8513971422

Table 5.1: The inverse of the linear size versus the Monte-Carlo estimate of the critical threshold

Physicists are convinced that in connectivity and rigidity percolation the value of the critical threshold for a finite system (which is defined differently for different problems) approximately follows the power law, namely $Y(n) \asymp P_c + B(\frac{1}{n})^C$; here n is the linear size of the system, $Y(n)$ is the value of the critical threshold for a piece of linear size n (we used a regular hexagonal piece–$\mathcal{H}_n$ in our notation), P_c is the value of critical probability for the infinite lattice, and B and C are constants (see Stauffer (1989), Jacobs (1995)). If this hypothesis is also correct for our tension percolation model, the parameters B and C can be estimated as follows ($P_c = 1$ by Theorem 5.4.1).

	$Value$	$Std.Error$	$t - value$
P_c	1	0	0
C	0.1607495929	0.00405487	−72.3876
B	−0.293521855	0.00423650	37.9440

Table 5.2: Parameters of the model $Y \asymp 1 + B(\frac{1}{n})^C$

The residual standard error is 0.0020877 on 26 degrees of freedom. Thus, C is estimated to be 0.1607495929, etc. The error bounds on the estimates are $estimate + / - 1.96 * (Std.Error)$. Although our numerical statistics is not sufficient for making strong conjectures, we suspect that the critical concentration approaches the critical probability according to a power law with a small exponent. If it is so, the power series $\sum_n P_p(n)$ introduced above diverges, and Lemma 5.3.1 should not work for a 3-lattice. In other words, in this case $\mathbb{T}^3$ does not have the property that once the tear starts (see Lemma 5.3.1), it propagates to infinity with a positive probability. Of course, even if Lemma 5.3.1 does not work in dimension 3, the critical probability of tension percolation for $\mathbb{T}^3$ may well be 1. We end up with the following problem.

Problem 5.5.1 *The critical probability of tension percolation for $\mathbb{T}^3$ is less than 1.*

Chapter 6

Summary and Conclusions

The thesis offers a number of new results on the geometry and combinatorics of piecewise-linear structures in Euclidean spaces. It especially emphasizes various connections between different areas of discrete geometry. The notion of stress plays a key role throughout the work. Our main objective in Chapters 2 and 3 was to understand connections between stresses and liftings (projections) of PL-manifolds. This program was realized in two ways. First, we obtained interesting results characterizing geometrical, combinatorial, and complexity properties of piecewise-linear partitions of manifolds having certain regularity properties. The existence of a convex piecewise-linear surface that projects on the partition is a good example of such property. Second, we linked geometric and homological characteristics of PL-manifolds to rigidity theory, obtaining some results on geometric statics of planar and spatial graphs. Our second major objective was to consider the geometry of liftings and stresses from a probabilistic point of view. In Chapters 4 and 5 we considered two classes of tension percolation models: the continuous membrane models, and the lattice models. The work in this direction was partially motivated by applications of rigidity theory to molecular chemistry. We succeeded in finding the critical probability threshold for a continuous model, and a discrete 2-dimensional tension percolation model. The results obtained for continuous and discrete tension percolation have many similarities, such as a finite time of destruction, a positive probability of the propagation to infinity of a relaxed region, etc. This thesis as well as other papers by K. R. and his co-authors introduces a new class of bootstrap percolation models, where the local removal rules are of geometric nature, as opposed to simple counts.

Concerning possible directions for further work in areas explored in Chapters 2 and 3, we would like to remark that any further progress in problems, such as determining the space of liftings for a PL-manifold, finding combinatorial relations similar to the g-theorem, and studying properties of stresses (see Chapters 2 and 3) will probably require an intensive use of advanced algebraic geometry and homological algebra guided by combinatorial and geometric intuition. For interesting new prospectives see Whiteley (1996).

Speaking of further research in the area of geometric bootstrap percolation, the following problems seem to be of a considerable interest.

Conjecture 6.0.2 *Let F be an infinite framework on the plane with a finite directional specter and edge lengths bounded from below and above. Then the critical probability of tension percolation is 1.*

Conjecture 6.0.3 *Let F be a spatial framework whose vertex set is the face-centered cubic lattice D_3, and whose edge set is the set of the shortest vectors of D_3. Then the critical probability of tension percolation is less than 1.*

In the continuous case, it is interesting to know, if for the model where the Dirichlet-Voronoi region of a node of a Poisson point process is a hole with probability p, the critical probability of tension percolation is 1 (see Chapter 4).

Simulation results of Jacobs et. al. show that for rigidity percolation the value of the critical threshold for a finite system approaches the value of the critical probability at infinity according to a power law. It would be interesting to investigate the same problem for continuous and lattice tension percolation models.

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Curriculum Vitae

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Awards:

Name of the Scholarship	Eligibility	Period
Distinct Stipend for maintaining A average in all courses taken in the previous term	institutional (MSU)	Sept. 1988 - June 1992 Jan. 1993 - June 1993
R. Samuel McLaughlin Fellowship	institutional (Queen's)	Sept. 1995 - Aug. 1996
E. G. Bauman Fellowship	institutional (Queen's)	Sept. 1996 - Aug. 1997
Queen's Graduate Fellowship	institutional (Queen's)	Sept. 1997 - Aug. 1998
Fields Institute Graduate Scholarship	national (Canada)	Jan. 1998 - Mar. 1998
Ontario Graduate Scholarship	provincial (Ontario)	Sept. 1998 - Aug. 1999
Fields Institute Graduate Scholarship	national (Canada)	Jan. 1999 - Mar. 1999
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Publications in refereed journals

K. A. Rybnikov Jr., On the density of three-dimensional compacta. *Doklady Mathematics*, vol. **48**, No. 1, (1994), 110-113.

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Conference contributions and other publications

K. A. Rybnikov, On the density of three-dimensional normal spaces, M.Sc. paper, 34 pp., Dept. of Discrete Math., Faculty of Mathematics and Mechanics, Moscow State University, June 1993 (the results of this M.Sc. work were presented on the *II National conference on Discrete Mathematics*).

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In preparation

R. M. Erdahl, K. Rybnikov, On the relationship between perfect and Delaunay type partitions of the cone of positive quadratic forms, 1998-1999.

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