Stresses and liftings of cell-complexes

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Abstract

This paper introduces a general notion of stress on cell-complexes and reports on connections between stresses and liftings (generalization of C_1^0-splines) of d-dimensional cell-complexes in $\mathbb{R}^d$. New sufficient conditions for the existence of a sharp lifting for a "flat" piecewise-linear realization of a manifold are given. Our approach also gives some new results on the equivalence between spherical complexes and convex and star polytopes. As an application, two algorithms are given that determine whether a piecewise-linear realization of a d-manifold in $\mathbb{R}^d$ admits a lifting to $\mathbb{R}^{d+1}$ which satisfies given constraints. We also demonstrate connections between stresses and Voronoi-Dirichlet diagrams and show that any weighted Voronoi-Dirichlet diagram without non-compact cells can be represented as a weighted Delaunay decomposition and vice versa.

1 Introduction

In this paper we develop a variety of geometrical and algorithmic methods that are useful for studying piecewise-linear surfaces, weighted Voronoi and Delaunay diagrams, self-stresses in frameworks and geometrical cell-complexes and parallel drawings of polyhedral pictures. The space of d-stresses on a cell-complex introduced in this paper plays an important role in investigations of many affine and projective properties of a broad class of geometrical cell-complexes. Our notion of stress generalizes the notion of affine stress introduced by Lee [28].

In sections 3-7 we prove that for a piecewise-linear realization in $\mathbb{R}^d$ of a homology manifold $\mathcal{M}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, and for an arbitrary decomposition of $\mathbb{R}^d$ by convex polyhedra the linear space of d-stresses is isomorphic to:

1. the linear space of liftings (with one fixed d-cell);
2. the linear space of reciprocals (with one fixed vertex).

These results can be considered as generalizations of similar results by Crapo and Whiteley for $d = 2, 3$ [11, 12, 13, 47]. For the first time this equivalence is proved under the general condition $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ for $d > 3$. Thus, there are non-spherical closed compact manifolds for which the equivalence between stresses and liftings holds. If Δ is a decomposition of $\mathbb{R}^d$ by convex polyhedra, then cone of d-tensions is equivalent to: 1) the cone of *additively weighted* Voronoi-Dirichlet diagrams, representing Δ; 2) the cone of convex liftings; 3) the cone of convex reciprocals. The equivalence between the last three objects was proved earlier by Aurenhammer [4] and McMullen [34]. In section 8 we consider a general notion of duality for decompositions of $\mathbb{R}^d$ by convex polyhedra, and show that the classes of weighted Voronoi and Delaunay diagrams (decompositions) coincide.

New sufficient conditions for the existence of a sharp lifting of a piecewise-linear realization of a cell-decomposition of a manifold in $\mathbb{R}^d$ are given in Sections 8 and 9. For instance, we show that any closed $(d-3)$-primitive manifold $\mathcal{M}^d$ (where the star of each $(d-3)$-cell has only four d-cells) with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ has a sharp lifting, but the existence of a sharp lifting for a $(d-2)$-primitive manifold $\mathcal{M}^d$ (where the star of each internal $(d-3)$-cell has only three d-cells) with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ requires the existence of a non-trivial d-stress on the star of each $(d-3)$-cell. In the same sections the problems of convexity and uniqueness of a lifting are analyzed. These results are improvements upon the well-known theorem of Davis [14] on the existence and uniqueness of a sharp convex lifting for a simple cell-decomposition of $\mathbb{R}^d$, $d > 2$, and upon similar results of Crapo and Whiteley on liftings of simple piecewise-linear spheres [13, 49]. As an application of the developed geometrical approach, two algorithms are given that determine whether a piecewise-linear realization of a d-manifold in $\mathbb{R}^d$ admits a lifting to $\mathbb{R}^{d+1}$ which satisfies given constraints, and find the dimension of the space of liftings. For decompositions of $\mathbb{R}^d$ by convex polyhedra, often referred to as tilings, these algorithms also recognize whether a decomposition is a weighted Voronoi-Dirichlet diagram and determine whether there is a convex surface which projects onto this decomposition. The first algorithm (Section 11) applies only to $(d-2)$-primitive manifolds, and has linear running time in the number of $(d-1)$-faces, which is optimal. This algorithm is similar to an algorithm of Aurenhammer [3], but is more general and can be applied to a broader class of complexes. The second algorithm (Section 12) applies to general cell-decompositions of homology manifolds (including arbitrary cell-decompositions of $\mathbb{R}^d$), but has worse time complexity, although it is polynomial. All theorems and algorithms are also interpreted for spherical cell-complexes (Section 13). These interpretations give new criteria and algorithms for recognizing whether a spherical complex is the radial projection of a convex polytope.

2 Polyhedral cell-complexes

All complexes which we shall consider are simplicial complexes from the topological point of view. However, all theorems and algorithms in this paper are stated for fixed decompositions of simplicial complexes into polyhedral *cells* (also called blocks or simplicial stars in combinatorial topology, see [29, 36]) which are not necessarily simplexes. We assume that all complexes have at most countable number of cells and are locally finite. A homology k-sphere (k-disk) is a polyhedron with the homology groups of a standard k-sphere (k-disk). A compact k-cell is understood to be a polyhedron (simplicial complex) which is a cone with a homology k-sphere as base. Cells of co-dimension 1 are referred to as *facets*. To include into consideration general cell-decompositions of Euclidean spaces into convex polyhedra we allow non-compact cells in a realization of a cell-complex in $\mathbb{R}^d$; i.e. a polyhedral k-dimensional subset of the realization may be considered as a whole cell from the geometrical point of view, if it lies in a k-dimensional affine subspace, and is homeomorphic to a linear half-space of $\mathbb{R}^k$. We also assume that each non-compact cell has finitely many faces.

We denote the star of a k-dimensional cell C^k by $St(C^k)$, and the k-dimensional skeleton of a complex $\mathcal{K}^d$ by $Sk^k(\mathcal{K}^d)$. Denote the linked complex, or simply the link (see [29, 36]), of a cell C^k by $Lk(C^k)$ (we understand a link as a combinatorial polyhedron and are not interested in its realizations). For us the relative boundary $\partial\mathcal{K}^d$ of a complex $\mathcal{K}^d$ is a sub-complex of $\mathcal{K}^d$ which consists of the closures of all $(d-1)$-cells which are not shared by at least two d-cells. We shall refer to cells that belong to the relative boundary as *boundary* cells, and to the other cells as *internal*. We denote the number of k-cells of $\mathcal{K}^d$ by $f_k(\mathcal{K}^d)$ and the number of internal k-cells by $f_k^\circ(\mathcal{K}^d)$. A *(combinatorial) path* in a complex $\mathcal{K}^d$ is a finite ordered sequence $\mathfrak{p} = [C_1, \dots, C_k]$ of d-cells, where every two consecutive d-cells share a common $(d-1)$-cell. A *circuit* is a path where the first and last cells coincide.

We shall consider a somewhat more general construction than an embedding or an immersion of a cell-complex into Euclidean space, such as a *piecewise-linear (PL-) realization* of a cell-complex in Euclidean space. In all geometric discussions cell-complexes will be considered as fixed piecewise-linear realizations, rather than abstract combinatorial objects.

Such general construction can be helpful, for example, for studying frameworks with bar intersections, polyhedral scenes, Schlegel diagrams, splines over triangulations (in the planar case this point of view was adopted in [11, 48, 51, 52]; in the three-dimensional case such PL-realizations were considered by Crapo and Whiteley in [11, 47]). For example, a Schlegel d-diagram (see Ziegler's book [25] for the theory of Schlegal diagrams) is a PL-realization of a $(d+1)$-polytope P^d in $\mathbb{R}^d$ obtained by radial projection of P^d onto one of its facets (a Schlegel diagram of a 4-cube is drawn in *figure 1*).

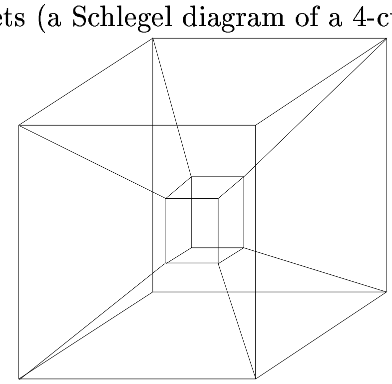

figure 1

One can identify an abstract combinatorial cell-complex $\mathcal{K}^d$ with its embedding into $\mathbb{R}^{2d+1}$ (since it can be triangulated). A PL-realization of a combinatorial simplicial complex $\mathcal{K}^d \subset \mathbb{R}^{2d+1}$ with a fixed decomposition into polyhedral cells is a continuous PL-mapping r of $\mathcal{K}^d$ in $\mathbb{R}^N$ $(N \geq d)$ such that *the closure of each k-cell, $k = 0, \dots, d$ is embedded by r into $\mathbb{R}^N$ as a "flat" (lying in a k-subspace) k-polyhedron.* For a PL-realization $(\mathcal{K}^d, r)$, we reserve an upper case Roman font $K^d = (\mathcal{K}^d, r)$, and for an abstract combinatorial structure an upper case script $\mathcal{K}^d$. The body of $r(\mathcal{K}^d)$ is denoted by $|K^d|$.

If we refer to the metric, projective, or affine properties of a cell-complex, these should be understood as the properties of its fixed PL-realization. However, when we consider the combinatorial or homological properties of a cell-complex, we are referring to its abstract combinatorial structure. A PL-realization of the star of a k-cell in $\mathbb{R}^d$ is called generic (for us) if all its $(d-1)$-cells lie on different $(d-1)$-planes.

We shall consider only strongly connected, pure dimensional complexes. A *homology d-manifold* (with boundary) is a cell-complex such that the link of each k-cell, $k = 0, \dots, d-1$ is either a homology $(d-k-1)$-sphere or a homology $(d-k-1)$-disk. A manifold is closed if each facet is adjacent to exactly two d-cells. All statements in the paper are formulated for both closed manifolds and for manifolds with a boundary, unless stated otherwise. Since we consider manifolds only from the combinatorial point of view, a manifold is always understood to be a *homology* manifold.

The star of an internal k-cell in a d-manifold is called $(d-k)$-primitive if it has $d-k+1$ d-cells. A cell-decomposition of a d-manifold is referred to as *k-primitive* if the star of each internal k-dimensional cell has $d-k+1$ d-cells (some authors call 0-primitive decompositions *simple*; our terminology goes back to Voronoi [46]). For decompositions of $\mathbb{R}^d$ by convex polyhedra $d-k+1$ is the minimum possible number of tiles in the star of a k-face. If a PL-realization of a homology sphere $\mathcal{S}^d$ can be lifted onto a convex polytope in $\mathbb{R}^{d+1}$, then 0-primitive vertices of $\mathcal{S}^d$ correspond to simple vertices of this convex polytope. When a k-primitive cell-decomposition of $\mathcal{M}^d$ is assumed to be fixed, we will refer to this k-primitive decomposition of $\mathcal{M}^d$ as *k-primitive manifold* $\mathcal{M}^d$. The notion of k-primitive decomposition naturally arises in studies of space-fillers, lattice polytopes and stereohedra [46, 26, 18, 19]. For example, the affine equivalence between space-fillers and Dirichlet domains of lattices was proved by Voronoi only for 0-primitive (simple) tilings. Later Zhitomirski [26] proved that this equivalence holds for $(d-2)$-primitive tilings, and Erdahl [24] for zonotopes, but it still remains unknown whether there are more general sufficient combinatorial conditions on a space filler to be the Dirichlet domain of a lattice (for details see [18, 19]). The existence of a lattice Dirichlet domain which is affinely isomorphic to a space-filler Π is equivalent to the existence of a convex lifting with some special simmetries for the lattice tiling $T(\Pi)$ by Π (Voronoi [46]).

3 Stresses on cell-complexes

If (V, E) are the vertices and edges of a framework in $\mathbb{R}^d$, then a self-stress (or simply stress) is an assignment of real numbers $s_{ij} = s_{ji}$ to the edges, a tension if the sign is positive or a compression if the sign is negative, so that the equilibrium conditions $\sum_j s_{ij}(\vec{v}_j - \vec{v}_i) = 0$ hold at each vertex $\vec{v}_i$. The notion of stress can be naturally generalized to k-stresses on cell-complexes. This generalization proves to be useful in the theory of space-fillers, the combinatorics and geometry of piecewise-linear manifolds, and the rigidity theory.

Consider a PL-realization K^d in $\mathbb{R}^N$ of a cell-complex $\mathcal{K}^d$, and let $\vec{n}(C^{k-2}, C^{k-1})$ be the inner unit normal to a $(k-1)$-cell C^{k-1} at its $(k-2)$-face C^{k-2}.

Definition 3.1 *A real-valued function $s(\cdot)$ on $(k-1)$-cells of K^d is a k-stress if for each internal $(k-2)$-cell C^{k-2} of K^d*

$$\sum_{C^{k-1}} s(C^{k-1})\, \vec{n}(C^{k-2}, C^{k-1}) = 0,$$

where C^{k-1} ranges over the $(k-1)$-cells such that $C^{k-2} \subset \partial\, C^{k-1}$. The quantities $s(C^{k-1})$ are the coefficients of k-stresses, a tension if the sign is positive and a compression if the sign is negative.

Notice that this definition works for cell-complexes in $\mathbb{S}^N$ too. If not all $s(C^{k-1})$ are zero, the k-stress s is called non-trivial. *Figure 2* illustrates the geometry of the equilibrium condition from the Definition 3.1 for a 3-stress on the star of an edge of a cell-complex in $\mathbb{R}^3$.

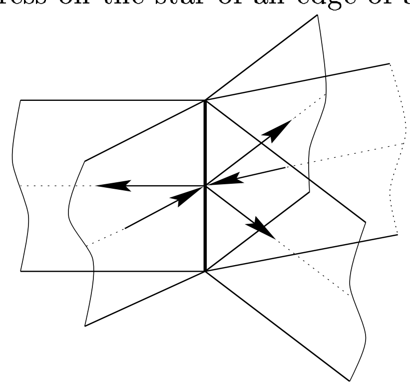

figure 2

By the relationship between self-stresses in a planar framework and liftings to $\mathbb{R}^3$ of the cell-decomposition induced by this framework (this was originally discovered by Maxwell: see [30, 31, 37, 38]), d-stresses in d-dimensional manifolds can be thought of as a generalization of self-stress in frameworks. To complete the analogy with stresses on frameworks the coefficients assigned to faces should be divided by their volumes, but for our purposes this would be inconvenient since our formulas would then become more cumbersome, and because we admit cells of infinite volume.

Our generalization of the notion of stress is geometrical and does not involve coordinates, whereas that of Lee [28] is algebraic and more restrictive (for finite simplicial complexes). If $b_a(C^{k-1})$ denotes Lee *affine* k-stress on a $(k-1)$-simplex C^{k-1}, then $s(C^{k-1}) = b_a(C^{k-1})\ vol\ (C^{k-1})$. It is more natural from the geometrical point of view to assign coefficients of k-stresses to k-cells, rather than to $(k-1)$-cells, as it is done in our definition, but this shifted notation was already used by several authors [28, 44, 13] and we will keep to this convention. Tay, White, and Whiteley remarked in [44] that the notion of linear stress can be extended to general cell-complexes. Note that our notion of stress corresponds to a certain subspace of the space of linear stresses mentioned by Tay and coauthors. McMullen also uses the language of outer normals in [35], where he defines weights (on simple polytopes), a notion dual to stresses.

It easy to see that k-stresses form a linear space, and that k-tensions (where all coefficients of k-stresses are positive) and k-compressions (where all coefficients of k-stresses are negative) form congruent polyhedral cones in this linear space. We will denote the space of all k-stresses on M^d by $Stress_k(M^d)$, and the cone of all k-tensions in this space by $Tension_k(M^d)$.

Theorem 3.2 *Let M^d be a PL-realization of an orientable manifold $\mathcal{M}^d$ in $\mathbb{R}^d$. For each $k = 1, \ldots, d-1$, there is a polynomial mapping of degree $d-k+1$ from the space $Stress_d(M^d)$ to the space $Stress_k(M^d)$, such that a d-tension (compression) is mapped to a k-tension (compression).*

The proof of this theorem is based on the construction of a local Euclidean reciprocal (see Section 7) for the star $St(v)$ of a vertex v of $\mathcal{M}^d$. Then define a function (polynomial in the linear and angular parameters of the reciprocal) on all subcomplexes of this reciprocal corresponding to the stars of faces, and show that for each $(k-2)$-face of $St(v)$ this function represents a k-stress which is well-defined on all the manifold. We conjecture that our mapping somehow generalizes a mapping constructed with algebraic methods by Lee in [28] (Theorem 6). We will explore the connection between stresses on skeletons of different dimensions in a subsequent paper [23].

4 Quality transfer

In this section, we introduce the notion of quality transfer which allows us to make a connection between stresses and the geometry of PL-manifolds. Let Q be a set of *qualities* and let $\mathfrak{G}$ be a group acting on Q. Here we consider the problem of assigning an element of Q, a quality, to each of the d-cells of M^d so that qualities assigned to adjacent cells are governed by rules associated with the common facet. If an arbitrary quality is assigned to some d-cell C_0, a quality can be assigned to any other d-cell C_k by translating qualities along a path connecting C_0 to C_k, using the rules associated with facets. Suppose that $\mathfrak{f}$ maps every ordered pair of adjacent cells into a group element, so that the reversed pair is mapped into the reciprocal group element. If $\mathfrak{p} = [C_0, \ldots, C_k]$ is a combinatorial path and q is the quality assigned to C_0, then the quality assigned to C_k via $\mathfrak{p}$ is given by the formula.

$$q \circ \mathfrak{f}(\mathfrak{p}) = q \circ \mathfrak{f}([C_0, C_1]) \cdot \mathfrak{f}([C_1, C_2]) \ldots \mathfrak{f}([C_{k-1}, C_k])$$

The qualities assigned to cells are well defined (i.e. independent of path) if and only if every circuit lies in the kernel of $\mathfrak{f}$.

Consider a 1-complex (graph) $\mathcal{G}$ where the vertices are d-cells, and the edges are the $(d-1)$-cells of $\mathcal{M}^d$ (therefore $\mathcal{G}$ may be infinite). Two vertices share an edge if, and only if, the corresponding d-cells are adjacent. The graph $\mathcal{G}$ is called the combinatorial dual of $\mathcal{M}^d$, and it is somewhat easier to consider quality transfer on $\mathcal{G}$ rather than on $\mathcal{M}^d$. In this model, the edges are associated with elements of $\mathfrak{G}$, and qualities are assigned to the vertices. Assigning qualities to the vertices of $\mathcal{G}$ is well defined if it is well defined over all cycles of $\mathcal{G}$. A more manageable criterion that the qualities assigned to d-cells are well defined is the following:

Lemma 4.1 *Let $\{c_i\}$ be a generating system for $H_1(\mathcal{G}, \mathbb{Z}_2)$. Then quality transfer is well defined on $\mathcal{G}$ if and only if it is well defined over all cycles from $\{c_i\}$.*

Proof. Fix a quality for a vertex v_0 of $\mathcal{G}$, and denote by $Q(v_i)$ the set of qualities that can be assigned to a vertex v_i by translating qualities from v_0. Consider the 1-complex $cov\ \mathcal{G}$, where the vertices are the pairs $(v_i, q(v_i))$ where v_i is a vertex of $\mathcal{G}$ and $q(v_i) \in Q(v_i)$. Vertices (v_i, q) and (v_j, q') of $cov\ \mathcal{G}$ share an edge in $cov\ \mathcal{G}$ if and only if v_i and v_j are adjacent in $\mathcal{G}$ and there is $g \in \mathfrak{G}$ (G is the group acting on Q) such that $g(q) = q'$. It is easy to see that $cov\ \mathcal{G}$ is a covering of $\mathcal{G}$. Since $\mathcal{G}$ is a 1-complex and $\{c_i\}$ is a generating system of $H_1(\mathcal{G}, \mathbb{Z}_2)$, then $\{c_i\}$ is also a generating system for the fundamental group $\pi(\mathcal{G})$. The covering map p from $cov\ \mathcal{G}$ onto $\mathcal{G}$ induces a monomorphism $p^* : \pi(cov\ \mathcal{G}) \to \pi(\mathcal{G})$. Every cycle from $\{c_i\}$ is lifted onto $cov\ \mathcal{G}$ in the trivial way (i.e. the lifting of c_i is a single cover of c_i). Since $\{c_i\}$ is a generating system for $\pi(\mathcal{G})$, p^* is an epimorphism. Therefore, p^* is an isomorphism, and the covering map p is one-to-one. As the covering is trivial, the quality transfer is well-defined on $\mathcal{G}$. $\square$

For the remaining portion of this paper we shift our attention from general cell-complexes to d-dimensional homology manifolds realized in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$. PL-manifolds with non-compact cells are permitted, if in the PL-realization non-compact cells can be retracted onto (new) compact cells (which are subsets of old non-compact cells) whose internal faces span the same affine subspaces as replaced non-compact cells. For instance, it allows the inclusion of decompositions of $\mathbb{R}^d$ by convex polyhedra (for example, weighted Voronoi diagrams) into the class of manifolds where the theory of d-stresses and liftings works. We will consider all homologies with coefficients in the group of two elements: $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.2 *Let $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. If quality transfer is well defined over the links of all $(d-2)$-cells, then it is well defined on $\mathcal{M}^d$.*

Proof. Since $\mathcal{M}^d$ is a homology manifold, one can consider a cell-decomposition $\mathcal{D}$ of $\mathcal{M}^d$ (in the combinatorial sense) which is dual to the original (for a description of this construction see [36, 29]). By the definition of a polyhedral cell-complex, the cells of the decomposition of $\mathcal{M}^d$ can be triangulated so, that all cells of the original decomposition become triangulated in a baricentrical fashion. A k-cell of the dual decomposition is defined as the union of the k-simplexes that share a common vertex in the baricentrical triangulation (which is the same for both the original, and the dual). If M^d has non-compact cells, we understand $\mathcal{D}$ to be a cell-decomposition of a manifold (with boundary) obtained from M^d by contraction of all its non-compact cells onto their compact subsets (in section 2 we assumed that it is always possible). This operation does not change the homotopical class and therefore preserves the homology too. Since $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, $Sk^2(\mathcal{D})$ has trivial $H_1(Sk^2(\mathcal{D}), \mathbb{Z}_2)$. In other words, a cycle on $Sk^1(\mathcal{D})$ can be represented as the sum of the boundaries of 2-cells of $\mathcal{D}$. It is easy to see that the dual graph $\mathcal{G}(\mathcal{M}^d)$ is a subgraph of $Sk^1(\mathcal{D})$ and coincides with $Sk^1(\mathcal{D})$ when $\mathcal{M}^d$ is a closed manifold. The d-cells having facets that belong to $\partial\mathcal{M}^d$ require a more detailed consideration. If $\mathcal{M}^d$ has a boundary, $Sk^2(\mathcal{D})$ has edges and vertices corresponding to boundary $(d-1)$-cells, and 2-cells corresponding to boundary $(d-2)$-cells. Indeed, any cycle of $\mathcal{G} \subset Sk^1(\mathcal{D})$ is the boundary of a chain whose carrier does not contain such 2-cells. Thus, the cycles of $\mathcal{G}$ that correspond to the stars of internal $(d-2)$-dimensional cells form a generating system for $H^1(\mathcal{G}, \mathbb{Z}_2)$. By Lemma 4.1, quality transfer is well defined on $\mathcal{M}^d$. □

The technique of quality transfer on Euclidean tilings goes back to Voronoi [46]. However, his technique was essentially homotopical, not homological (see [18, 37]). The homological version of this procedure was later implicitly used by several authors, including Crapo, Whiteley [13, 47], and McMullen [33].

If M^d is a fixed realization in $\mathbb{R}^d$ of a manifold $\mathcal{M}^d$, one can attempt to assign $+1$ or -1 to each cell (recall that every d-cell is *embedded* into $\mathbb{R}^d$) so that two adjacent d-cells have the same orientation if and only if their outer normals at their common facet have opposite directions. By the definition of a cell-complex, $\mathcal{M}^d$ can be baricentrically triangulated. One can prove that $\mathcal{M}^d$ is orientable in the sense defined above, if and only if it is orientable as a simplicial complex (in the usual sense). Since we need our geometrical notion of orientation in the following section, we now show that any manifold $\mathcal{M}^d$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ is orientable in this sense.

Assuming that for a d-cell C_0 of M^d the orientation of the embedding is o, we define the orientation for an adjacent cell C_1 as follows. If outer normals to C_1 and C_0 at their common facet have opposite directions (when C_1 and C_0 are convex, it means that they point to different half-spaces determined by the facet), assign orientation $+o$ to C_1; if the normals are cooriented, assign orientation $-o$ to C_1. Since $\mathcal{M}^d$ is strongly connected, an orientation can be assigned to any cell of M^d by transferring the orientation along paths of adjacent cells in this fashion. We have to show that the orientation assigned to a cell in this manner is independent of a particular path used for the transfer, and is therefore well defined. By Lemma 4.2, this requires that the transfer of orientation be well-defined over the links of the internal $(d-2)$-cells which are all $\mathbb{S}^1$. Since the closure of each cell of $\mathcal{M}^d$ is *embedded* into $\mathbb{R}^d$, the orientation is correctly defined over the links of all $(d-2)$-cells, so, by Lemma 4.2, the orientation can be properly introduced on all of M^d. Throughout the paper we will denote the orientation of a cell C by $o(C)$.

5 Stresses, splines and liftings

Let K^d be a PL-realization of a combinatorial cell-complex $\mathcal{K}^d$ in $\mathbb{R}^d$. A family of affine functions $L(\mathbf{x}; C^d)$ on $\mathbb{R}^d$ corresponding to the d-cells of K^d is referred to as a lifting of K^d if $L(\mathbf{x}; C^d)$ determines a PL-realization L^d of $\mathcal{K}^d$ in $\mathbb{R}^{d+1}$. In other words, each d-cell of K^d is the vertical projection of the corresponding d-cell of the realization L^d determined by the affine functions $L(\mathbf{x}; C^d)$. Since L^d is a PL-realization of $\mathcal{K}^d$, each k-cell $(0 \le k \le d)$ of K^d is a vertical projection of the corresponding k-cell of L^d. A lifting is called *sharp* if each pair of adjacent d-cells of K^d is lifted onto distinct hyperplanes in $\mathbb{R}^{d+1}$. $L(\mathbf{x}; C^d)$ is called locally convex (concave) if on each sub-complex of K^d which is embedded into $\mathbb{R}^d$, $L(\mathbf{x}; C^d)$ is a convex (concave) PL-function. We refer to the angle between two adjacent d-cells of a lifting as a dihedral angle. Denote the linear space of liftings defined up to the choice of transferring a supporting plane by $Lift(K^d)$. Locally convex liftings form the polyhedral cone $CLift(K^d)$ in this space. Lifting is a natural generalization of the notion of continuous PL-function (C^0_1-spline) on a cell-decomposition of a polyhedral region. A locally convex lifting is a (locally) convex PL-function on the region. The conception of lifting is very convenient in studies of stresses in planar frameworks, and in the analysis of polyhedral scenes [2, 11, 13, 48, 52].

Let M^d be a PL-realization in $\mathbb{R}^d$ of a *manifold* $\mathcal{M}^d$. A real-valued function c^0_1 on the ordered pairs of adjacent d-cells of M^d is called a C^0_1-cofactor if
1) $c^0_1([C, C']) = -c^0_1([C', C])$
2) for each internal $(d-2)$-cell C^{d-2}

$$\sum_{i=1}^{i=n} c^0_1([C_i, C_{i+1}])\vec{n}([C_i, C_{i+1}]) = 0$$

where $[C_1, \dots, C_n, C_{n+1} = C_1]$ is a cyclical order of the d-cells making contact in C^{d-2}, and $\vec{n}([C_i, C_{i+1}])$ is the outer unit normal to C_i at its facet shared with C_{i+1}. We will also use the term C^0_1-*cofactor* for referring to the value of a function $c^0_1(\ )$ on a facet.

A lifting $L(\mathbf{x}; C) = \langle \vec{a}(C), \mathbf{x} \rangle + a_0(C)$ determines a C^0_1-cofactor c^0_1 by $c^0_1([C, C']) = \langle \vec{a}(C') - \vec{a}(C)), \vec{n}([C, C']) \rangle$. In theorem 5.1 we show that if $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, then a cofactor determines a lifting up to the choice of an affine function. The following theorem connects stresses and liftings for PL-realizations of a manifold with $H_1 = 0$ over $\mathbb{Z}_2$.

Theorem 5.1 *Let M^d be a PL-realization of a manifold $\mathcal{M}^d$ with trivial $H_1(\mathcal{M}^d, \mathbb{Z}_2)$. The linear space $Stress_d(M^d)$ is isomorphic to the linear space $Lift(M^d)$. All-non-zero stresses correspond to sharp liftings. If M^d is an embedding, then $Tension_d(M^d) \cong CLift(M^d)$.*

Proof. Take the set A of affine scalar valued functions on $\mathbb{R}^d$ to be the set of qualities. This set has a linear space structure and acts on itself by transfer. We will denote this transferal group by $\mathfrak{G}$. If each ordered pair of adjacent cells is associated with a group element from $\mathfrak{G}$, and an affine function is assigned to any particular d-cell, affine functions can be assigned to adjacent cells using the action of $\mathfrak{G}$ on A. Lemma 4.2 gives sufficient conditions that the quality assigned to a cell in this manner be independent of the path used for the transfer, and is therefore well defined. As an application of this lemma, we shall establish a linear correspondence between d-stresses and liftings.

Let s be a non-trivial d-stress on M^d, and fix the orientations of the d-cells of M^d so that the orientations agree for pairs of adjacent cells as it was described above. If d-cells C_1, C_2 are adjacent, denote by $E_{1,2}(\mathbf{x}) = \langle \vec{n}, \mathbf{x} \rangle - c = 0$ the equation of their common facet C^{d-1} which is adjusted so that the vector $\vec{n}$ is an outer unit normal to C_1 at C^{d-1} (if C_1 is convex, this is the same as saying that $E_{1,2}(\mathrm{x}) \leq 0$ on C_1). If $s(C_1, C_2) = s(C_2, C_1)$ is the stress on C^{d-1}, let $g(\mathbf{x}, [C_1, C_2]) = s(C_1, C_2) o(C_1)(\langle \vec{n}, \mathbf{x} \rangle - c)$ be the affine function associated with the ordered pair $[C_1, C_2]$ (as an element of $\mathfrak{G}$). If $q(\mathbf{x}, C_1)$ is the affine function for C_1, then the affine function for C_2 is given by $q(\mathbf{x}, C_1) + g(\mathbf{x}, [C_1, C_2])$. It is easy to see that this definition is symmetric and for the path $[C_1, C_2, C_1]$ we have $g(\mathbf{x}, [C_1, C_2]) + g(\mathbf{x}, [C_2, C_1]) = 0$. Since $g(\mathbf{x}, \mathfrak{p})$ is defined on two-element paths, it is also defined on arbitrary paths. Since $\mathcal{M}^d$ is a manifold, the link of each internal $(d-2)$-cell is $\mathbb{S}^1$. Consider an ordered circuit $[C_1, \dots, C_n, C_{n+1}]$, where C_i are the d-cells from the star of an internal $(d-2)$-cell C^{d-2} and $C_{n+1} = C_1$. It is easy to see that $g(\mathbf{x}, [C_1, \dots, C_n, C_1]) = 0$ if and only if

$$\vec{v} = \sum_{i=1}^{i=n} s([C_i, C_{i+1}]) o(C_i) \vec{n}([C_i, C_{i+1}]) = 0 \tag{1}$$

Rotate $\vec{v}$ by 90° in the orthogonal complement to *span* (C^{d-2}) in such way that the result of the rotation of $(o(C_1)\vec{n}([C_1, C_2]))$ becomes the inner unit normal to the common facet of C_1 and C_2 at C^{d-2}. Denote this rotation operator by r. Consider instead of $St(C^{d-2})$ the section of $St(C^{d-2})$ with a perpendicular 2-plane. By convention we will refer to the direction of rotation from C_1 to C_2 through their common facet as clockwise. For each C_i, $i = 1, \dots, n$, call the facet where C_i contacts C_{i-1} (C_n if $i = 1$) the first facet and the facet where C_i contacts C_{i+1} the second facet. If $o(C_i) = o(C_1)$, then the direction of rotation from the first facet of C_i to the second is clockwise. In this case, the rotation transforms vector $o(C_1)\vec{n}([C_i, C_{i+1}])$ into the inner unit normal to the common facet of C_i and C_{i+1} at C^{d-2}. If $o(C_i) = -o(C_1)$, then the direction of rotation from the first facet of C_i to the second is counter-clockwise. It is easy to see that, in this case, $o(C_1)\vec{n}([C_i, C_{i+1}])$ is also transformed into the inner unit normal to the common facet of C_i and C_{i+1} at C^{d-2}. Thus $r(\vec{v}) = 0$, for $r(\vec{v})$ is the vector sum of d-stresses at C^{d-2}. It therefore follows that formula (1) holds.

By Lemma 4.2, since the quality transfer is well defined over the links of all $(d-2)$-cells, it is well defined on all of $\mathcal{M}^d$. Our construction provides continuous gluing over facets of M^d. If an affine function is fixed for any d-cell (the "origin"), affine functions can be assigned to the other cells using the group $\mathfrak{G}$ and the quality transfer. The resulting family of affine functions is a lifting by construction and Lemma 4.2. It is easy to see that if the coefficients of d-stresses are all-non-zero, then the lifting is sharp.

Conversely, let $L^d = L(\mathbf{x}, C)$ be a lifting of M^d. If $L(C)$ is the lifting of a d-cell C of M^d, define $o(L(C)) := o(C)$ (recall that the orientations of embedding $o(C)$ have been adjusted for the d-cells of M^d). The d-stress on M^d corresponding to L^d can be found in the following way. Let $L(C_1)$ and $L(C_2)$ be adjacent d-dimensional cells of the lifting projecting onto cells C_1 and C_2 of the "flat" realization M^d, and defined by affine functions $L(\mathbf{x}, C_1) = \langle \vec{a}_1, \mathbf{x} \rangle + c_1$ and $L(\mathbf{x}, C_2) = \langle \vec{a}_2, \mathbf{x} \rangle + c_2$. Let $\vec{n}_1$ be the outer unit normal to C_1 at facet C^{d-1} shared with C_2. Now, we can assign the coefficient of stress $o(L(C_1))\langle \vec{a}_2 - \vec{a}_1, \vec{n}_1 \rangle$ to the facet C^{d-1}. Reversing the arguments in the proof of formula (1) from the first part of the theorem, one can see that introduced quantities $o(L(C_1))\langle \vec{a}_2 - \vec{a}_1, \vec{n}_1 \rangle$ are actually coefficients of d-stresses. Notice that these arguments have local character and do not employ the condition that $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. □

Since, in the second part of Theorem 5.1, the homological condition that $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ is not used, the above theorem constructs a monomorphism from $Lift(M^d)$ into $Stress_d(M^d)$ for an arbitrary orientable $\mathcal{M}^d$. This monomorphism maps a C^0_1-cofactor c^0_1 to a d-stress s via formula $s(C^{d-1}) = c^0_1(C^{d-1})$, where C^{d-1} is an internal facet of M^d. Notice that in the proof we did not assume the convexity of cells. Theorem 5.1 was independently proved by Whiteley for homology 3-spheres in preprint [13], although the topological part of the proof was only sketched and the condition $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ was not mentioned as sufficient for the existence of the isomorphism between $Stress_d(M^d)$ and $Lift(M^d)$.

Although the space of lifting is always a subspace of the space of stress, the converse is not necesseraly true. A realization of the torus depicted in *figure 3* demonstrates that in general one cannot drop the condition $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ (triangles (123), (456) and hexagon (7 8 9 10 11 12) are not loaded; edges (16), (24), (35) can have arbitrary tensions; all other edges are under compression; since lines 16, 24, 35 do not pass through a common point, the torus does not lift).

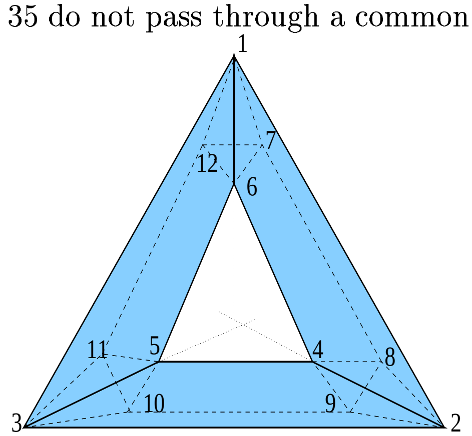

figure 3

If Δ is a cell-decomposition of a region in $\mathbb{R}^d$, then a lifting of Δ is a continuous PL-function on Δ, i.e. a C^0_1-spline. More generally, a C^r_m-spline over Δ is a C^r-smooth function which is represented by polynomials of degree at most m on each d-cell. Such functions form a vector space over $\mathbb{R}$, which is denoted by $S^r_m(\Delta)$. The theory of splines on manifolds embedded into $\mathbb{R}^d$ is similar to the theory of stresses and liftings. Recall that coefficients of d-stresses define gluing of affine functions over facets of a decomposition in the way shown in Section 5. Regarding smooth splines as generalizations of PL-functions, one can say that C^{r-1}_r-*cofactors* play the same role for C^{r-1}_r-splines as stresses for C^0_1-splines. They define the smooth gluing of polynomial patches over facets (the core of the theory of cofactors of smooth splines can be found in [49].) Let's fix orientations for the links of all internal $(d-2)$-cells. The orientation of the link of a $(d-2)$-cell induces the orientations of normals to supporting hyperplanes of facets making contact in this $(d-2)$-cell.

Definition 5.2 *A real-valued function $c_r^{r-1}(\ )$ on the $(d-1)$-cells of Δ is called a C_r^{r-1}-cofactor if for each internal $(d-2)$-cell C^{d-2} of Δ, and for each non-negative integer vector $(i_1, \ldots, i_n)$ such that $i_1 + \ldots + i_n = r$*

$$\sum_{C^{d-2} \subset \partial F} c_r^{r-1}(F)\ n_1^{i_1}(F) \cdots n_d^{i_d}(F) = 0$$

where F ranges over all facets making contact at C^{d-2} and $n_1(F), \ldots, n_d(F)$ are the coordinates of the unit normal to the hyperplane spanned by F whose orientation is induced by the fixed orientation of the link of C^{d-2}.

Clearly the C_r^{r-1}-cofactors form a linear space, which we denote by $COF_r^{r-1}(\Delta)$. The following lemma [49] explains the importance of the above definition.

Lemma 5.3 *Let c_r^{r-1} be a C_r^{r-1}-cofactor for Δ. Then, adopting the notation from definition 6.2, for every internal $(d-2)$-cell of Δ we have*

$$\sum_{C^{d-2} \subset \partial F} c_r^{r-1}(F)\ (n_1(F)x^1 + \cdots + n_d(F)x^d + n_{d+1}(F))^r = 0$$

where $(x_1, ..., x_d)$ are the usual Euclidean coordinates in $\mathbb{R}^d$ and $n_1(F)x_1 + \cdots + n_d(F)x_d + n_{d+1}(F) = 0$ is an equation for F.

This statement is known as the C_r^{r-1}-cofactor reduction lemma [49]. The following lemma explains why C_r^{r-1}-cofactors determine the smooth gluing of polynomial patches (for the proof see [8]).

Lemma 5.4 *Let function f be represented by polynomials of at most degree m on each d-cell of Δ. Then f is a C_m^r-spline if and only if for each pair of adjacent d-cells the difference between polynomials corresponding to these cells is divided by the $(r+1)$-power of an affine function vanishing on the common facet of these d-cells.*

The following theorem, which generalizes a Billera's theorem [9, 49] for $d > 2$ is a consequence of Lemma 4.2 on quality transfer. It underlines the analogies between stresses and cofactors of C_r^{r-1}-splines.

Theorem 5.5 *If $H_1(\Delta, \mathbb{Z}_2) = 0$, then*

$$dim\ S_r^{r-1}(\Delta) = \left(\begin{array}{c} r+d \\ d \end{array}\right) + \ dim\ COF_r^{r-1}(\Delta)$$

Proof. A proof of this theorem can be obtained by direct substitution of C_1^0-cofactors by C_r^{r-1}-cofactors in the proof of Theorem 5.1. □

6 Reciprocals, liftings and stresses

Consider a planar framework (V, E) that is in a state of static equilibrium, and assume that the framework determines a cell-decomposition $D(V, E)$ of $\mathbb{R}^2$ (assuming that the framework has vertices at infinity), or of a simply-connected region in $\mathbb{R}^2$. Consider a vertex of (V, E). The sum of vectors of stresses applied to this vertex is equal to zero. Therefore, when rotated on $90°$ clockwise they form a polygon (self-intersecting in general). It was noticed by Maxwell (for a proof see [48]) that the positions of rotated edges of (V, E) can be adjusted so that they form a reciprocal graph (or simply reciprocal). Each edge of this reciprocal corresponds to an edge of (V, E) and each vertex to a cell of $D(V, E)$ (one vertex corresponds to the complement of $D(V, E)$ if any). We introduce and explore a similar notion for d-manifolds in $\mathbb{R}^d$ (see also [47, 13].

The *combinatorial dual graph* $\mathcal{G}(\mathcal{M}^d)$ of a manifold $\mathcal{M}^d$ is a (multi)graph where the vertices are the d-cells of $\mathcal{M}^d$, and the edges are the internal $(d-1)$-cells of $\mathcal{M}^d$.

A *reciprocal* of a PL-realization M^d of a manifold $\mathcal{M}^d$ in $\mathbb{R}^d$ is a rectilinear realization R in $\mathbb{R}^d$ of the combinatorial dual graph $\mathcal{G}(\mathcal{M}^d)$ such that the edges of R are perpendicular to the corresponding facets. If none of the edges of a reciprocal collapses into a point, the reciprocal is called non-degenerate.

Let $v(C_1)$ and $v(C_2)$ be vertices of a reciprocal R corresponding to adjacent d-cells C_1 and C_2. If $\vec{v}(C_2) - \vec{v}(C_1)$ is cooriented with an outer normal to C_1 at the facet shared with C_2, then the edge $[v(C_2)v(C_1)]$ is called *properly oriented.* Otherwise it is called improperly oriented. If realization M^d is an embedding and all edges of R are properly oriented, R is called a convex reciprocal (since the cycles of R corresponding to the stars of the $(d-2)$-cells are convex in this case). Reciprocals with one fixed vertex form a linear space. Denote it by $Rec(M^d)$. Reciprocals were originally considered by Maxwell [30] in connection with stresses in plane frameworks. The linear space of planar reciprocals was studied in [11]. Convex reciprocals form a cone $CRec(M^d)$ in the linear space $Rec(M^d)$. Maxwell noticed that convex reciprocals corresponded to convex liftings of planar cell-complexes. The convex reciprocals were studied in [4, 11, 12, 48, 37, 38]. The following theorem is a simple consequence of the definitions given above and of affine properties of affine spaces. It holds regardless of the homological properties of a manifold.

Theorem 6.1 *Let M^d be a PL-realization of a manifold $\mathcal{M}^d$ in $\mathbb{R}^d$. There is an isomorphism between $Lift(M^d)$ and $Rec(M^d)$. Sharp liftings correspond to non-degenerate reciprocals. If M^d is an embedding, then convex dihedral angles correspond to properly oriented edges of the reciprocal, concave dihedral angles correspond to improperly oriented edges, and $CLift(M^d) \cong CRec(M^d)$.*

Proof. Let R be a reciprocal for M^d, let $v(C_1)$ and $v(C_2)$ be vertices of R corresponding to d-cells C_1 and C_2, and let $\vec{n}$ be the outer unit normal to C_1 at the facet F shared with C_2. Since $\vec{v}(C_2) - \vec{v}(C_1)$ is orthogonal to F, there is $c \in \mathbb{R}$ such that $\langle \vec{v}(C_2) - \vec{v}(C_1), \mathbf{x} \rangle + c = 0$ is the equation of F. A lifting corresponding to R can be constructed as follows. If $L(\mathbf{x}; C_1)$ is an affine function corresponding to C_1, then the affine function corresponding to C_2 is given by $L(\mathbf{x}; C_1) + \langle \vec{v}(C_2) - \vec{v}(C_1), \mathbf{x} \rangle + c$. Let's fix an arbitrary affine function for a d-cell C_0. Connect a d-cell C with C_0 by a combinatorial path. We determine the affine function $L(\mathbf{x}; C)$ corresponding to C via the use of the above formula. Since every cycle of $\mathcal{G}(\mathcal{M}^d)$ is realized as a rectilinear cycle of R in $\mathbb{R}^d$, $L(\mathbf{x}; C)$ is path independent and is therefore well defined. It is easy to see that $L(\mathbf{x}; C)$ determines a PL-realization of $\mathcal{M}^d$ in $\mathbb{R}^{d+1}$ where non-degenerate dihedral angles correspond to non-degenerate edges of R.

Let $L^d = L(\mathbf{x}; C)$ be a lifting of M^d, and let $L(\mathbf{x}; C_1) = \langle \vec{a}_1, \mathbf{x} \rangle + c$ and $L(\mathbf{x}; C_2) = \langle \vec{a}_2, \mathbf{x} \rangle + c$ be the affine functions determining adjacent d-cells of L^d which correspond to d-cells C_1 and C_2 of M^d.

If v_1 is a vertex of a reciprocal corresponding to C_1, then the vertex corresponding to C_2 is given by $v_1 + \vec{a}_2 - \vec{a}_1$. Fix a vertex corresponding to a d-cell C_0 at the origin and construct all other vertices and edges of the reciprocal using this formula. It is easy to see that the resulting rectilinear 1-complex is a reciprocal for M^d and non-degenerate dihedral angles of L^d correspond to non-degenerate edges of the reciprocal. $\square$

In the 2-dimensional case, the connection between liftings and reciprocals was first noticed by Maxwell. The isomorphism between spaces of liftings and reciprocals for manifolds in $\mathbb{R}^2$ was proved by Crapo and Whiteley [11, 12]. The relationship between convex liftings of cell-decompositions of $\mathbb{R}^d$ and convex reciprocals was also shown in [3].

7 Voronoi diagrams: duality, projections and stresses

The aim of this section is to show some important connections between stresses, liftings and weighted Voronoi and Delaunay diagrams. A point s in Euclidean space $\mathbb{R}^d$ is said to be additively weighted if it is associated with a real number $w(s)$ referred to as the weight of s. The weighted distance from s is

$$\mathbf{d}^2(s, x) - w(s)$$

where $\mathbf{d}(s, x)$ denotes the Euclidean distance between points s and x.

Let S be a discrete set of additively weighted points in $\mathbb{R}^d$, such that all weights are bounded in absolute value. Points of S are called sites. A point $x \in \mathbb{R}^d$ belongs to the *Voronoi domain* of a site $p \in S$ with weight $w(p)$ if and only if

$$\mathbf{d}^2(p, x) - w(p) \leq \mathbf{d}^2(s, x) - w(s)$$

for all $s \in S$.

It is easy to see that each non-empty domain of a weighted Voronoi diagram is a convex polyhedron and that these polyhedra form a face-to-face decomposition of $\mathbb{R}^d$. Such a decomposition is called an *(additively) weighted Voronoi diagram* (also often referred to as a weighted Dirichlet decomposition, or a power diagram). If all weights are equal, then the diagram is referred to as a Voronoi (Dirichlet) diagram. Let Δ be a weighted Voronoi diagram. Then Δ can be represented as a weighted Voronoi diagram in different ways. The location and the weight of at least one site can be chosen arbitrarily. (Notice that weighted sites can lie outside their domains.) Weighted Voronoi diagrams have many applications. (For more information on weighted Voronoi diagrams see [5]). Algorithms constructing weighted Voronoi diagrams and data structures related to these diagrams are described in [5, 20]. A discrete set of points S in $\mathbb{R}^d$ uniquely defines a special decomposition of the convex hull of S which is combinatorially and metrically dual to the Voronoi diagram of S [16, 17]. Such decomposition is called a Delaunay decomposition. Every d-cell of that decomposition is inscribed into a sphere which does not contain points of S in its interior.

Let S be a discrete set of points in $\mathbb{R}^d$. A convex polytope P in $\mathbb{R}^d$ is called a Delaunay cell of the system of points S if:
1) all vertices of P belong to S;
2) there is a sphere circumscribed around P;
3) no points of S except the vertices of P lie inside or on the sphere.

Delaunay cells form a face-to-face decomposition of *conv* S. This decomposition is defined uniquely by S. Delaunay decompositions have many applications in computational geometry, mesh generation, the theory of lattices, mathematical crystallography, *etc.* One of the generalizations of Delaunay decompositions, the weighted Delaunay decompositions, are intensively employed in computational geometry [21, 22]. For generically distributed sites and weights, the weighted Delaunay decomposition is a triangulation. Such triangulations are called *regular.* A sphere circumscribed around a cell of a Delaunay decomposition is called an empty sphere. Informally, a Delaunay decomposition consists of cells which sit in their own empty spheres. Generalizing Delaunay diagrams to weighted Delaunay diagrams we replace standard Euclidean spheres by virtual spheres defined through weighted distances.

Let S be a discrete set of points in $\mathbb{R}^d$. Let W be a set of real weights bounded in absolute value which are associated with these points. A polytope $P \subset \mathbb{R}^d$ is called a weighted Delaunay cell of the system of weighted points (S, W) if:
1) all vertices of P belong to S;
2*) there is a point $c(P) \in \mathbb{R}^d$ called the weighted center of P and $r(P) \in \mathbb{R}$ such that for every vertex v of P:

$$\mathbf{d}^2(c(P), v) - w(v) \;=\; r(P)$$

3*) for each $s \in S$ which is not a vertex of P:

$$r(P) < \mathbf{d}^2(c(P), s) - w(s)$$

Weighted Delaunay cells form a face-to-face decomposition of *conv* S.

The weighted vertices of a weighted Delaunay decomposition are the weighted sites for the dual weighted Voronoi diagram. The duality between weighted Delaunay and Voronoi decompositions can be proved exactly in the same way as the duality for ordinary Voronoi and Delaunay decompositions (see [16, 17]). The duality between Voronoi diagrams and Delaunay decompositions can be naturally generalized in the following way.

Let Δ be an arbitrary locally finite decomposition of $\mathbb{R}^d$ into compact convex polyhedra. A decomposition Δ^* of $\mathbb{R}^d$ is called dual of Δ if the following conditions hold:

1. *combinatorial duality:* there is a one-to-one correspondence between m-dimensional faces of Δ^* and $(d-m)$-dimensional faces of Δ, $0 \le m \le d$; this correspondence induces an isomorphism between the incidence graphs (infinite) of Δ and Δ^*;

2. *orthogonality and proper orientation:* 1-skeleton of Δ^* is a convex reciprocal for Δ.

It is easy to see that this relationship is reciprocal, i.e. if Δ^* is dual of Δ, then Δ is dual of Δ^*. The above definition captures all properties of the duality between ordinary (or weighted) Voronoi and Delaunay decompositions except for the relationship between Voronoi sites and vertices and Delaunay centers and vertices. The property that Delaunay vertices are Voronoi centers and vice-versa does not find its generalization in this construction. One can ask what does the existence of a dual decomposition for an arbitrary Δ imply in terms of the geometry of Δ? In this section we will show that two decomposition are dual if and only if it is possible to present one of them as a weighted Voronoi diagram (S, W) and the other as the dual Delaunay decomposition (later we show that these two properties are equivalent). The notion of dual decomposition can also be regarded as a d-dimensional analog of the notion of planar convex reciprocal.

If Δ^* is a dual of Δ, then any decomposition obtained from Δ^* by scaling and translating is also a dual of Δ. It is obvious that the vector sum of any two dual decompositions of Δ is again a dual decomposition for Δ. Thus all decompositions (considered up to translation) which are dual for decomposition Δ form a cone $Dual(\Delta)$.

Theorem 7.1 *A decomposition Δ of $\mathbb{R}^d$ has a dual decomposition if and only if Δ has a d-tension. The cone of dual decompositions $Dual(\Delta)$ is isomorphic to the cone of tensions $Tension_d(\Delta)$.*

Proof. Let Δ^* be a dual for Δ. By definition, $Sk^1(\Delta^*)$ is a non-degenerate convex reciprocal for Δ. Thus, by Theorem 6.1 we have a monomorphism from the cone of dual decompositions into the cone of tensions. In fact, these cones are isomorphic. The cone of tensions is isomorphic to the cone of convex liftings by the results of Section 5. Consider a convex lifting of Δ. Let $p \in \mathbb{R}^{d+1}$ be a point lying above this surface. Intersections of cones dual to the stars of vertices of the lifting with $\mathbb{R}^d$ are convex polyhedra which form a face-to-face decomposition of $\mathbb{R}^d$. This decomposition is a dual of Δ by construction. □

Aurenhammer [4] proved that a finite decomposition of $\mathbb{R}^d$ is the projection of a convex polyhedral surface if and only if it is a weighted Voronoi diagram. This theorem holds for infinite decompositions by finite polyhedra as well (adopting the proof is straightforward). The following theorem follows from Theorem 5.1, Theorem 6.1 and Aurenhammer's theorem.

Theorem 7.2 *A decomposition of $\mathbb{R}^d$ by finite convex polyhedra is a weighted Voronoi decomposition if and only if it has a d-tension.*

If (S_1, W_1) and (S_2, W_2) are two diagrams representing a decomposition Δ, then there is a collection of weights W, such that $(S_1 + S_2, W)$ represents Δ. If a weighted diagram (S, W) represents a decomposition Δ, and $h(S)$ is a homothetic image of S, then there is a set of weights W' such that Δ is a weighted diagram with site set $h(S)$ and weight set W'. The same is true about translating the set of sites, i.e. if $t(S)$ is a translated copy of S, then there is a set of weights W' such that Δ is a weighted diagram with site set $t(S)$ and weight set W'. (All these statements can be proved via the use of the arguments from the first part of the proof of the Theorem 6.5; essentially it is the idea of quality translation over the links of $(d-2)$-cells). Hence, diagrams representing Δ constitute a cone $WVor(\Delta)$ whose elements are sets of weighted sites defined up to translation such that their diagrams give partition Δ. A d-tension on Δ defines an element of $WVor(\Delta)$ uniquely. The preceding theorems imply the following proposition.

Proposition 7.3 *$WVor(\Delta)$ is isomorphic to the cone of dual decompositions $Dual(\Delta)$ (up to translation) and to the cone of d-tensions $Tension_d(\Delta)$.*

Ash and Bolker [1] proved that a plane decomposition by finite convex polyhedra is a weighted diagram if and only if it is a section of a higher-dimensional Voronoi decomposition. As a non-trivial application of this theorem one can mention that any rhombic Penrose tiling can be represented by a weighted diagram, since it is a section of the standard decomposition of $\mathbb{R}^5$ by cubes. In fact, the Ash-Bolker theorem is d-dimensional, as their proof does not depend on d. There is an easy way to establish a correspondence between weighted Voronoi diagrams and sectional Voronoi diagrams. Assume that all weights are negative. (Since all weights are bounded in absolute value, we can always add to all weights an appropriate constant and make them negative.) A weighted site s with coordinates $(x^1, ..., x^d)$ and weight $w(s)$ corresponds to a not-weighted site in $\mathbb{R}^{d+1}$ with coordinates $x^1, ..., x^d, \sqrt{-w_s}$. Conversely, if h_s is the distance between $\mathbb{R}^d$ and a site s in $\mathbb{R}^{d+1}$ whose (not-weighted) Voronoi domain intersects $\mathbb{R}^d$, then the corresponding site on $\mathbb{R}^d$ has the same first d coordinates and weight $-|h_s|^2$. The following proposition is a direct consequence of the Ash and Bolker theorem and Theorem 5.1.

Proposition 7.4 *A decomposition of $\mathbb{R}^d$ by finite convex polyhedra has a d-tension if and only if it is the section of a $(d+1)$-dimensional Voronoi diagram.*

The following theorem shows that the classes of weighted Voronoi diagrams and Delaunay decompositions coincide (the author has not found this statement anywhere in literature).

Theorem 7.5 *A decomposition Δ of $\mathbb{R}^d$ by convex polyhedra is a weighted Voronoi diagram if and only if Δ constrainted to conv $Sk^0(\Delta)$ is a weighted Delaunay decomposition.*

Proof. If Δ is weighted Voronoi, it has the dual Delaunay decomposition Δ^* of *conv* $Sk^0(\Delta)$. A k-cell of this decomposition is the convex hall of the weighted Voronoi sites of d-cells making full contact in a $(d-k)$-face of Δ. $Sk^1(\Delta)$ is a reciprocal for Δ^*. We want to show that the vertices of $Sk^1(\Delta)$ can be taken for the sites of a weighted Voronoi diagram representing Δ^*. Fix zero weight for a vertex of $Sk^1(\Delta)$ and then define weights on the other vertices of $Sk^1(\Delta)$ by weight transfer via edge-paths on $Sk^1(\Delta)$. Let v and v_1 be adjacent vertices of $Sk^1(\Delta)$. If a vertex v is assigned weight $w(v)$ and x is the intersection point of the line spanned by $[vv_1]$ with the plane spanned by the corresponding facet F of Δ^*, then the site v_1 is assigned weight:

$$w(v_1) = \mathbf{d}^2(x, v_1) - \mathbf{d}^2(x, v) + w(v).$$

This assignment of weights makes the plane spanned by F equidistant (in terms of weighted distances) from the weighted sites $(v, w(v))$ and $(v_1, w(v_1))$. A simple check shows that such transfer of weights is well defined over the cycles of $Sk^1(\Delta)$ corresponding to the stars of all $(d-2)$-cells of Δ^*. Therefore by Lemma 4.2 weights can be assigned to all vertices of $Sk^1(\Delta)$ turning Δ^* into a weighted Voronoi diagram. Sites of a weighted Voronoi diagram are vertices for the dual weighted Delaunay diagram. Therefore vertices of Δ can be associated with weights so that D constrainted to *conv* $Sk^0(\Delta)$ can be regarded as a weighted Delaunay decomposition.

Let D be weighted Delaunay. It has the dual weighted Voronoi decomposition V. By Theorems 7.1 and 7.2 D is weighted Voronoi. (In the case when *conv* $Sk^0(\Delta) \neq \mathbb{R}^d$, D is a constrained weighted Voronoi diagram). □

The straightforward adaptation of the Euclidean theory of weighted diagrams, duality and tensions given in this section for the spherical case gives a somewhat more symmetric form of the theory for spherical complexes (see also Section 12 and [12] for the 2-dimensional case).

Theorem 7.6 *The following properties of a spherical d-complex Δ with convex cells are equivalent:*
1) Δ is a weighted Voronoi diagram in $\mathbb{S}^d$;
2) Δ is a weighted Delaunay decomposition of $\mathbb{S}^d$;
3) Δ is the central projection of a convex $(d+1)$-polytope in $\mathbb{R}^{d+1}$;
4) Δ has a "spherical" d-tension.

8 Combinatorics of $\mathcal{M}^d$ and $Lift(M^d)$.

The following obvious observations give important implications for the analysis of stresses and liftings of $(d-k)$-primitive manifolds.

Proposition 8.1 *Assume that the star $St(C^{d-k})$ of a $(d-k)$-cell C^{d-k} in a d-manifold contains $k+1$ $(d-1)$-cells. If $St(C^{d-k})$ is realized in $\mathbb{R}^d$ generically (no pair of $(d-1)$-cells lie on the same hyperplane) and the coefficient of d-stress for one of its $(d-1)$-cells is fixed, then the values of d-stresses for the other $(d-1)$-cells are uniquely determined. If the d-cells of $St(C^{d-k})$ are convex and $St(C^{d-k})$ is embedded, then all coefficients of stresses have the same sign.*

Proposition 8.2 *Assume the star $St(C^{d-k})$ of a $(d-k)$-cell C^{d-k} contains $k+1$ $(d-1)$-cells. If $St(C^{d-k})$ is realized in $\mathbb{R}^d$ generically and a lifting of the star of a $(d-1)$-cell of $St(C^{d-k})$ is fixed, then the lifting of $St(C^{d-k})$ is uniquely determined. If the d-cells of the star are convex and the star is embedded, then all dihedral angles of the lifting are either convex or concave.*

The star of a $(d-2)$-cell can be either embedded or realized with a self-intersection. If the star is $(d-2)$-primitive, then there is only one type of PL-realization with a self-intersection (under our definition of a PL-realization), namely, where two d-cells have the same orientation, and the third cell has the opposite one. In the latter case we will say that the star is folded.

Let $[x, y]$ be a segment in $\mathbb{R}^d$. A combinatorial path $[C_1, \dots, C_n]$ of d-cells of a PL-realization M^d is *strung* on $[x, y]$ if:

1. $x \in C_1$, $y \in C_n$;
2. $$[x, y] \subset \bigcup_{i=1}^{n} |C_i|$$
3. for any $i \neq j$ $|C_i| \bigcap |C_j|$ is either a common facet of C_i and C_j or empty.

Let $L(\mathbf{x}; C)$ be a lifting of M^d. If for any $x, y \in |M^d|$ a combinatorial path strung on $[x, y]$ is lifted convex up (down), then $L(\mathbf{x}; C)$ is called locally convex (concave).

Theorem 8.3 *Let $\mathcal{M}^d$ be $(d-2)$-primitive, and each d-cell of $\mathcal{M}^d$ has an internal $(d-2)$-face. Assume that for each d-cell C of $\mathcal{M}^d$ set $(\partial C \backslash \partial \mathcal{M}^d)$ is a strongly connected $(d-1)$-pseudomanifold. Let M^d be a PL-realization of $\mathcal{M}^d$ where all cells are convex. If there is a sharp lifting of M^d, then for any sub-complex of M^d which is actually embedded the lifting is either locally concave or convex. The lifting is unique up to the choice of a d-face and a dihedral angle.*

Proof. Let $[C_0, C_1, C_2]$ be a path of d-cells on $\mathcal{M}^d$ such that in the realization $M^d \cup_{i=0}^2 C_i$ is embedded into $\mathbb{R}^d$. Let $L(\mathbf{x}; C)$ be a sharp lifting. Assume $L(\mathbf{x}; C)$ is convex on $([C_0, C_1])$. It is enough to show that the convexity of $l([C_0, C_1])$ implies the convexity of $l([C_1, C_2])$. Denote by E_1 a $(d-2)$-cell shared by C_0 and C_1. Consider a d-cell sharing E_1 with C_0 and C_1. Let C_1 and this d-cell share a facet F_1 and let F_2 be the facet which is common for C_1 and C_2. Connect F_1 and F_2 by a combinatorial path $\mathfrak{p}$ on $\partial\, C_1$ which consists of internal facets of $\mathcal{M}^d$ sharing internal $(d-2)$-cells (such path exists by the conditions of the theorem). Let E_2 be a $(d-2)$-cell shared by F_2 and the preceding $(d-1)$-cell in $\mathfrak{p}$. Since $\mathcal{M}^d$ is $(d-2)$-primitive, the type of the realization of a star cannot switch on the surface of C_1. Therefore $St(E_2)$ has the same type of realization as $St(E_1)$. If $St(E_1)$ is embedded in $\mathbb{R}^d$, then the stars of all $(d-2)$-cells from $\mathfrak{p}$ are embedded in $\mathbb{R}^d$. Therefore, $St(E_2)$ is also embedded and both $St(F_1)$ and $St(F_2)$ are lifted in the same way. If $St(E_1)$ is folded, then the stars of all $(d-2)$-cells from $\mathfrak{p}$ are folded. An application of Proposition 8.2 shows that if $St(F_1)$ is lifted onto a convex dihedral angle, then $St(F_2)$ is lifted onto a convex dihedral angle too. In both cases $L(\mathbf{x}; C)$ has necessarily the same type on $[C_1, C_2]$ as on $[C_0, C_1]$. Therefore, $L(\mathbf{x}; C)$ is convex on $[C_0, C_1, C_2]$. □

Corollary 8.4 *[38] If a $(d-2)$-primitive decomposition of $\mathbb{R}^d$ by convex polyhedra has a non-trivial lifting, then this lifting is sharp and globally convex or concave. The lifting is unique up to the choice of a supporting plane and a dihedral angle.*

As an implication of the above theorem we have the following.

Corollary 8.5 *Let M^d be a PL-realization in $\mathbb{R}^d$ of a $(d-2)$-primitive manifold and $L^d = L(\mathbf{x}; C)$ be its sharp lifting to $\mathbb{R}^{d+1}$. If*
1) $|M^d|$ is convex;
2) for any $\mathbf{x}$ from $|M^d|$ there are exactly two points in L^d which project onto $\mathbf{x}$;
3) all cells of M^d are convex,
then L^d is a convex sphere in $\mathbb{R}^{d+1}$.

An analog of theorem 8.3 for stresses can be formulated as follows.

Proposition 8.6 *Let $\mathcal{M}^d$ be $(d-2)$-primitive, and each d-cell of $\mathcal{M}^d$ has an internal $(d-2)$-face. Assume for each d-cell C of $\mathcal{M}^d$ the set $\partial C \backslash \partial \mathcal{M}^d$ is a strongly connected $(d-1)$-pseudomanifold. If the stars of all internal $(d-2)$-cells of M^d are generic, then $\dim Stress_d(M^d)$ is equal to either 1 or 0.*

Notice that even though $Stress_d(M^d)$ and $Lift(M^d)$ are isomorphic to either $\mathbb{R}$ or 0 for any realization of a closed $(d-2)$-primitive manifold $\mathcal{M}^d$ with generic stars of $(d-2)$-cells, they need not coincide.

9 Sharp liftings of $(d-k)$-primitive manifolds

It is natural to ask when there is a sharp lifting of a "flat" PL-realization M^d, especially when M^d is a decomposition of $\mathbb{R}^d$ or a PL-realization of a sphere in $\mathbb{R}^d$. In this section we give an improvement on a well-known theorem of Davis [14] on the existence and uniqueness of a sharp lifting for a 0-primitive (simple) cell-decomposition of $\mathbb{R}^d$, $d > 2$. We also improve a theorem by Whiteley [47] on the existence and uniqueness of a lifting for PL-realizations of 0-primitive (simple) d-spheres ($d > 2$) in $\mathbb{R}^d$. If $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, then the existence of a sharp lifting is equivalent to the existence of an all-non-zero stress (see Section 5). As before, we assume that the stars of all $(d-2)$-cells are generic. We also impose a quite natural restriction on the combinatorics of $\mathcal{M}^d$.

Let C be a d-cell in a d-manifold $\mathcal{M}^d$ with boundary. We understand by *recl* $(\partial C \setminus (\partial\mathcal{M}^d \backslash \partial Sk_{d-3}(\mathcal{M}^d)))$ a cell-complex which is obtained from ∂C by removing open $(d-1)$ and $(d-2)$-cells which belong to $\partial\mathcal{M}^d$, and then augmenting all remaining $(d-1)$-cells of C by their relative boundaries. Informally it means that after removing boundary (with respect to $\mathcal{M}^d$) $(d-1)$-cells from ∂C we tear along those $(d-2)$-cells which occur on the boundary of $\mathcal{M}^d$ (e.g. see *figure 4*; in this example the manifold is a 3-cube decomposed into five simplexes and $C = (abcd)$).

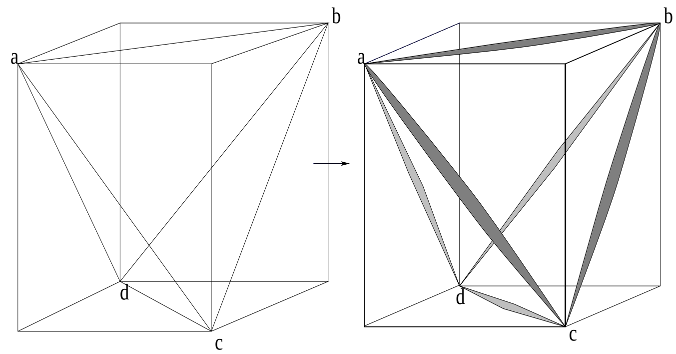

figure 4

Condition 9.1 *For any d-cell C the complex recl $(\partial C \setminus (\partial\mathcal{M}^d \backslash \partial Sk_{d-3}(\mathcal{M}^d)))$ can be represented as $\cup F_i(C)$, where*
1) each $F_i(C)$ is a $(d-1)$-manifold (possibly with boundary) with $H_1(F_i(C), \mathbb{Z}_2) = 0$.
2) $F_i(C)$ and $F_j(C)$ do not share internal $(d-2)$-cells of $\mathcal{M}^d$, if $i \neq j$;

Notice that some important classes of manifolds such as *cell-partitions of* $\mathbb{R}^d$, *closed compact manifolds, and convex tilings of convex regions in* $\mathbb{R}^d$ satisfy the above condition. If $d > 2$, the local geometric properties of M^d (i.e. geometry of the stars) have a stronger effect on $Stress_d(M^d)$ and $Lift(M^d)$ than in the planar case, as illustrated by the following theorem.

Theorem 9.2 *A PL-realization of a $(d-2)$-primitive manifold $\mathcal{M}^d$ $(d > 2)$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, satisfying Condition 9.1, admits an all-non-zero d-stress if and only if the star of each internal $(d-3)$-face has an all-non-zero d-stress. If the realization is an embedding and an all-non-zero stress exists, then there is a global tension.*

Theorem 9.3 *A PL-realization of a $(d-3)$-primitive manifold $\mathcal{M}^d$ $(d > 2)$ with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ and satisfying Condition 9.1 has an all-non-zero d-stress.*

It is easy to see that Theorem 9.3 easily follows from Theorem 9.2, since a generic realization of the star of a $(d-3)$-cell with only four d-cells always has an all-non-zero stress which is unique up to scale.
Proof of Theorem 9.2 We call a subset of facets of $\mathcal{M}^d$ an independent component if stresses on the facets of this subset do not depend on the choice of stresses for the facets which do not belong

to this subset. All internal facets can be partitioned into independent components I_k. Choose a facet in each component and fix an arbitrary non-zero stress for each chosen facet. We have to show that the fixed set of stresses uniquely determines an all-non-zero stress on all of M^d. To prove this we will translate stress via chains of adjacent facets. The stress transfer has to be defined independent of path. By Proposition 8.2 a stress on a facet belonging to the primitive star of a $(d-2)$-cell determines the coefficients of stresses of the other two facets. Therefore, if F and F' are two adjacent facets, then there are two non-trivial reciprocal linear maps $l([F, F'])$ and $l([F', F])$ associated with pairs $[F, F']$ and $[F', F]$. Consider a graph G whose vertices are internal facets of $\mathcal{M}^d$, and whose edges are internal $(d-2)$-cells of $\mathcal{M}^d$, where an edge exists between facet F and F' precisely when F and F' share a common internal $(d-2)$-cell. (Notice that G is not the dual graph $\mathcal{G}(\mathcal{M}^d)$.) The process of assigning coefficients of stresses to facets of $\mathcal{M}^d$ can be regarded as the process of assigning real numbers to vertices of G via the use of functions $l([F, F'])$. In this model, each edge of G is associated with a pair of reciprocal linear functions. G consists of connected components G_k corresponding to I_k (see above). For each G_k, Let's assign a real number $s_k \in \mathbb{R}$ to an arbitrarily chosen vertex of G_k. We have to show that real numbers can be assigned to all vertices of G_k so that the numbers corresponding to adjacent vertices are connected by the pair of linear maps assigned to their common edge. This is the same as to show that quality transfer over each cycle is well-defined. We can regard G as a sub-graph of the 1-skeleton of the dual (combinatorial) cell-decomposition of $\mathcal{M}^d$. Denote by $\mathcal{D}$ this dual decomposition. Since $\mathcal{M}^d$ is $(d-2)$-primitive, all 2-cells of $Sk^2(\mathcal{D})$ corresponding to internal $(d-2)$-cells are (combinatorially) triangles. Such 2-cell of $Sk^2(\mathcal{D})$ can be decomposed into four smaller triangles, one of them being a cycle of G. Let (ABC) be a triangle of $\mathcal{D}$ where the vertices correspond to d-cells A, B and C forming the star of an internal $(d-2)$-cell and the edges correspond to the facets of this star. If e, f and g are the "centers" (in the combinatorial sense) of the sides AB, BC and CA, then the sub-triangulation of (ABC) consists of (efg), (eAg), (eBf), (fCg), where (efg) is a cycle of G (see *figure 5*).

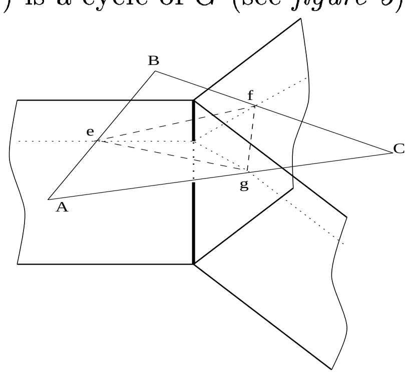

figure 5

In the similar way we decompose all 2-cells of $Sk^2(\mathcal{D})$ corresponding to boundary $(d-2)$-cells of $\mathcal{M}^d$. If the star of a boundary $(d-2)$-cell contains m d-cells, then the subdivision of the corresponding 2-cell of $Sk^2(\mathcal{D})$ consists of m triangles and one $(1+m)$-gon. Consider an arbitrary 1-cycle $\mathfrak{c}$ on G. Since $H_1(Sk^2(\mathcal{D}), \mathbb{Z}_\in) = /$, then $\mathfrak{c} = \partial\, \Delta^2$, where Δ^2 is a 2-chain from $\mathcal{C}_2(Sk^2(\mathcal{D}), \mathbb{Z}_\in)$. The chain Δ^2 can be represented as the sum of 2-cells from the subdivision of $Sk^2(\mathcal{D})$ described above. Cells of this subdivision of $Sk^2(\mathcal{D})$ can be conveniently partitioned into two groups, namely those which are of type (efg) (the boundaries of such cells are the links of internal $(d-2)$-cells of $\mathcal{M}^d$), and those which are of type (eAg) (see above). Therefore

$\Delta^2 = \Delta^2_1 + \Delta^2_2$, where Δ^2_1 is the sum of 2-cells of the first type and Δ^2_2 is the sum of 2-cells of the second type. The boundaries of 2-cells of the first type are actually cycles of G corresponding to internal stars of $(d-2)$-cells of M^d (clearly, 2-cells that correspond to boundary $(d-2)$-cells of $\mathcal{M}^d$ do not occur in Δ^2_1). We will refer to such cycles as *link-cycles*. Let $\partial\Delta^2_2 = \sum_k \mathfrak{c}(C^d_k)$, where $\mathfrak{c}(C^d_k)$ is a cycle connecting only the facets of the d-cell C^d_k. We will call such cycles *surface cycles*, for they can be thought of as lying on the surfaces of d-cells. Since the choice of the cycle $\mathfrak{c}$ was arbitrary, the link-cycles and the surface cycles form a generating system of $H_1(G, \mathbb{Z}_2)$. We only need to prove that quality transfer is well-defined over surface cycles. One can think of quality transfer over a surface cycle connecting facets of a d-cell C^d as of quality transfer over paths of $(d-1)$-cells of $\partial\, C^d$. Lemma 4.2 provides sufficient conditions that quality transfer on a manifold with $H_1 = 0$ over $\mathbb{Z}_2$ is well-defined. Before applying Lemma 4.2 we need to obtain some information on circuits over the links of $(d-3)$-cells on $\partial\, C^d$. By Condition 9.1 *recl* $(\partial\, C^d \backslash (\partial\mathcal{M}^d \backslash \partial Sk^{d-1}(\mathcal{M}^d))) = \cup F_m(C^d)$, where for each manifold $F_m(C^d)$ all $(d-3)$-cells internal with respect to $F_m(C^d)$ are internal with respect to $\mathcal{M}^d$. By hypothesis, there is an all-non-zero stress for the star of each internal $(d-3)$-cell of $\mathcal{M}^d$ which is, of course, unique up to scale. It implies that the quality transfer over the link of each internal $(d-3)$-cell of $\partial C^d \backslash \partial\mathcal{M}^d$ is well-defined. Quality transfer is well-defined on $\partial\, C^d \backslash \partial\mathcal{M}^d$, if and only if it is well-defined on each $F_m(C^d)$. By Condition 9.1, all such components are manifolds with $H_1 = 0$ over $\mathbb{Z}_2$. By Lemma 4.2, quality transfer over every circuit of $(d-1)$-cells of $F_m(C^d)$ is well-defined. Therefore, quality transfer is well-defined over surface cycles. Thus it is well-defined on G. It therefore follows that an all-non-zero stress on M^d exists and is unique up tao scale for each independent component I_k. □

Corollary 9.4 *A PL-realization of a $(d-2)$-primitive $\mathcal{M}^d$ $(d > 2)$ satisfying Condition 9.1 and $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, admits a sharp lifting if and only if the star of each $(d-3)$-face has a sharp lifting. If $\mathcal{M}^d$ is closed, then any non-trivial lifting is sharp and unique up to the choice of an affine function and a dihedral angle.*

Assume all cells of M^d be convex. Under the conditions of the preceding theorem, any sharp lifting is either convex or concave on each sub-complex of $\mathcal{M}^d$ which is embedded into $\mathbb{R}^d$.

Corollary 9.5 *Any PL-realization of a $(d-3)$-primitive $\mathcal{M}^d$ $(d > 2)$ satisfying Condition 9.1 and $H_1(\mathcal{M}^d, \mathbb{Z}^2) = 0$ has a sharp lifting. If the realization is an embedding and all cells are convex, then there is a convex lifting.*

These results improve a well-known theorem of Davis [14] on the existence and uniqueness of a convex lifting for a 0-primitive (simple) finite decomposition of $\mathbb{R}^d$ into convex polyhedra as well as a similar theorem by Whiteley [47] for spheres. Combining results of Theorem 8.3 and the above corollaries we obtain the following result generalizing a theorem by Whiteley [47], where the manifold is required to be 0-primitive.

Theorem 9.6 *Let M^d $(d > 2)$ be a PL-realization of a $(d-3)$-primitive manifold. If*
1) $|M^d|$ is convex;
2) the realization M^d covers each point of the interior of $|M^d|$ twice;
3) all cells of M^d are convex,
then M^d has a unique lifting (up to the choice of a supporting d-plane and a dihedral angle) onto a convex sphere in $\mathbb{R}^{d+1}$.

10 Algorithmic analysis of stresses and liftings.

It is useful to consider liftings from a somewhat more general point of view, adopted in a field of artificial intelligence called *scene analysis of polyhedral pictures* (see [41]).

A polyhedral incidence structure is a triple $S = (V, F, I)$: an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subset V \times F$, where $V \times F$ denotes the set of all ordered pairs whose first elements are taken from V and second elements from F.

A polyhedral d-picture is a pair $(S, \mathbf{p})$: an incidence structure S, and a location map $\mathbf{p} : V \to \mathbb{R}^d$ assigning coordinates to vertices. A lifting is a pair of "lifting" maps (LF, LP): a map $LF : F \to \mathbb{R}^{d+1}$, $f \mapsto A^f$ assigning affine functions on $\mathbb{R}^d$ to faces, and a map $LP : V \to \mathbb{R}$, $\mathbf{p}(v) \mapsto z^{\mathbf{p}(v)}$ assigning points in $\mathbb{R}^{d+1}$ to vertices of V, such that for every $v \in V$ and for every incidence (v, f) $z^{\mathbf{p}(v)} = A^f(\mathbf{p}(v))$. Liftings of $(S, \mathbf{p})$ form a linear space $Lift(S, \mathbf{p})$. A lifting is *sharp* if any two affine functions corresponding to faces with at least d common vertices are distinct.

It is worth mentioning that the problem of recognizing whether a planar polyhedral picture can be a interpreted as the projection of a spatial polyhedral scene satisfying given constraints is one of the central problems in computer line drawing [41]. Similar problems for parallel drawings occur in computer aided geometric design (see [49]). Since parallel drawings in $\mathbb{R}^d$ are equivalent to liftings from $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ (see [49]) the ability to find a lifting of a given type allows us to construct a parallel drawing having prescribed qualitative properties. Thus algorithms constructing liftings may be useful to computer aided design and computer vision [49, 41]. As an example of another application for algorithms of this sort one can mention perturbing a non-simple decomposition by convex polyhedra of a region in $\mathbb{R}^d$ $(d > 2)$. If we do not know of any sharp lifting for the decomposition, then we may not be able to perturb the decomposition so, that it will become 0-primitive (simple). However, if there is a method to find a sharp lifting, one can always perturb such a lifting and project it back to $\mathbb{R}^d$. The perturbed decomposition will be 0-primitive and the convexity of the cells will be preserved. By the results of Sections 7 and 9 a 0-primitive decomposition of $\mathbb{R}^d$ $(d > 2)$ is always a weighted Voronoi diagram; thus algorithms constructing sharp liftings can be used to approximate decompositions admitting sharp lifting by weighted Voronoi diagrams (see [42, 43] for applications of such approximations). Since $Tension_d(M^d) \cong WVor(M^d)$ for cell-decompositions of $\mathbb{R}^d$ by convex polyhedra, the problem of finding an all-non-zero stress is equivalent to the problem of recognition of weighted diagrams and sections of Voronoi diagrams. It can be useful in mathematical crystallography and other sciences dealing with space partitions [42, 43].

A picture $(S, \mathbf{p})$ is called *generic* if the dimension of the space of liftings of $(S, \mathbf{p})$ is minimal over all realizations $\mathbf{p}$ of the incidence structure S in $\mathbb{R}^d$. All other realizations are called *singular*. Generic realizations form an open dense subset of the space $\mathbb{R}^{|dV|}$ of all $\mathbb{R}^d$-realizations of S. Singular realizations correspond to points of an algebraic variety in the space of all $\mathbb{R}^d$-realizations. This variety is called the algebraic variety of singular realizations of S (for details see [51]).

There is a polynomial time combinatorial (i.e. not employing realization data) algorithm by Sugihara which determines whether a generic picture has a sharp lifting, and computes the dimension of the space of lifting in this case (see [27, 41, 50, 52]). In fact, one can modify this algorithm so that it will find $dim\ Lift(S, \mathbf{p})$ for any generic picture $(S, \mathbf{p})$. Imai's modification [27] of this algorithm has complexity $O(|I|^2)$. *It is an open question whether there is a polynomial* *algorithm which computes all maximal sub-pictures of* $(S, \mathbf{p})$ *which have sharp liftings.* Note that polyhedral pictures which occur in practical applications usually have symmetries or traces of symmetries which often make the use of combinatorial methods inappropriate for practical implementations.

In this paper we consider a special case, where S is the d-cell–vertex incidence structure of a fixed cell-decomposition of a manifold $\mathcal{M}^d$. Since we want every k-cells to be realized as a PL-ball living in an affine k-subspace of $\mathbb{R}^d$, it is natural to mean by the space of $\mathbb{R}^d$-realizations of $\mathcal{M}^d$ an affine subspace of $\mathbb{R}^{df_0}$ which consists of all realizations satisfying this requirement. If $\mathcal{M}^d$ is simplicial, then this space is all $\mathbb{R}^{df_0}$. Indeed, from the practical point of view the cases of orientable 2-manifolds and 3-manifolds are the most important ones (see [41, 49, 51]). Let's prescribe the type of dihedral angle of a lifting for some edges of a picture (e.g. convex versus concave, if two adjacent cells lie in different halfplanes) and ask whether there is a lifting which satisfies these conditions. More formally, the *lifting problem* is as follows.

1) Find $dim\, Lift(M^d)$.

2) Given a set of restrictions for some dihedral angles, determine whether a lifting of this type exists and construct such a lifting if it does.

For different generic realizations of $\mathcal{M}^2$ the answers to the lifting problem may be different, and therefore there are no purely combinatorial methods to answer the above questions for general instances of this problem. This is because the set of generic $\mathbb{R}^d$-realizations with the same type of lifting is a semialgebraic variety of full dimension in the space of all $\mathbb{R}^d$-realizations of $\mathcal{M}^d$. Notice that similar questions arise in the theory of tensegrity frameworks [10, 49].

One can express qualitative restrictions on dihedral angles in terms of C_1^0-cofactors (see Section 5). Let A and B be d-cells of M^d which make contact in a facet F. Notice that $c_1^0(A, B) > 0$ if and only if the affine function $L(A, \mathbf{x})$ corresponding to A is greater that the affine function $L(B, \mathbf{x})$ corresponding to B on the halfspace $< \mathbf{x}, \mathbf{n} > +c < 0$, where $< \mathbf{x}, \mathbf{n} > +c = 0$ is the supporting hyperplane of F and n is the outer normal to A at F. For example, Let's consider the case when d-cells A and B are convex (in fact, this is not important for our definitions). When A and B lie in different halfspaces with respect to the supporting hyperplane of F, $c_1^0(A, B) > 0$ if and only if the lifting of $St(F)$ is convex, and $c_1^0(A, B) < 0$ if and only if the lifting of $St(F)$ is concave. When A and B lie in the same halfspace, then $c_1^0(A, B) > 0$ if and only if the cell of the lifting corresponding to A lies above the cell of the lifting corresponding to B. The above restrictions (for $d = 2$) are standard in computer vision (see Sugihara [41]), and when $St(F)$ is embedded, are usually expressed with edge labels $+$ for convex lifting and $-$ for concave lifting (e.g. see *figure 6*). It is easy to see now that liftings which satisfy some fixed restrictions of the above type form an open convex polyhedral cone in the linear space of liftings.

If $\mathcal{M}^d$ is oriented, then the restrictions of the above type naturally give rise to dependent sets of an oriented matroid on internal facets of $\mathcal{M}^d$. Fix an orientation and a lifting for M^d. For a facet F shared by d-cells A and B define the sign of F in the *matroid of liftings* by $o(A)sgn(c_1^0(A, B))$. Facets with non-zero signs form a dependent set of the matroid. Hence, in the case of orientable M^d it can be said that the purpose of the algorithm is to determine whether a dependent set with given signing exists.

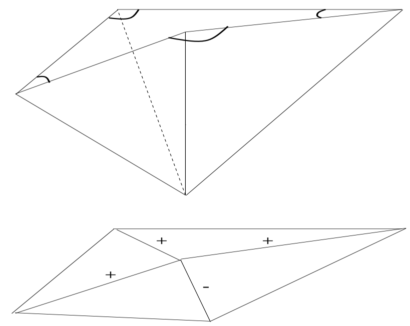

figure 6

Let $\mathcal{I}(\mathcal{M}^d)$ be the incidence graph of a *finite* cell-decomposition of a manifold $\mathcal{M}^d$ (see [20] for a description of incidence graphs), and let, as before, M^d be a PL-realization of $\mathcal{M}^d$ in $\mathbb{R}^d$. To store the parameters of the realization M^d with the incidence graph of combinatorial $\mathcal{M}^d$, $\mathcal{I}(\mathcal{M}^d)$ should be equipped with additional information which determines the positions of facets and d-cells of M^d (up to translation). Assume that in the incidence graph $\mathcal{I}(M^d)$ storing the PL-realization M^d each node representing a facet of M^d stores the coefficients of a normal to the hyperplane on which the PL-realization of this facet lies. Since non-simplicial, non-convex and non-compact cells are allowed, this information is not sufficient for efficient algorithms. If C is a d-cell and F is its facet, the data structure should allow determining in constant time from which side of the facet F the cell C makes contacts with F (when C is a simplex, it can always be determined in constant time). Let's modify $\mathcal{I}(M^d)$ by restructuring the layer of the nodes representing the facets. For a node representing a facet with normal $(a_1, \ldots, a_d)$ we create a node-antipode representing the same facet, but with normal $(-a_1, \ldots, -a_d)$. Let $n(F; (a_1, \ldots, a_d))$ be a node of $\mathcal{I}(M^d)$ representing a facet F with equation $a_1x^1 + \ldots + a_dx^d + c = 0$, and let F be shared by d-cells C_1 and C_2. The node $n(F; (a_1, \ldots, a_d))$ and a node that represents C_i are linked by an arc in the modified graph if and only if $\vec{a} = (a_1, \ldots, a_d)$ is an outer normal to C_i (i=1,2). In other words, in the modified incidence graph $\mathcal{I}(\mathcal{M}^d)$ (we do not change the notation) the layer which represents the facets contains two copies of each facet with different orientations of normals. In fact, the first algorithm uses only a subgraph of $\mathcal{I}(M^d)$ which stores the nodes representing d-cells, $(d-1)$-cells, and all incidences between them. Denote this sub-graph by $\mathcal{I}_{d-1}(M^d)$. The second algorithm uses $\mathcal{I}_{d-2}(M^d)$ which contains the nodes representing d-cells, $(d-1)$-cells, $(d-2)$-cells and all incidences between them. In the description of the algorithms we will also use the combinatorial dual graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of $\mathcal{M}^d$ introduced in Section 6. Whenever it is appropriate, we identify in our descriptions d-cells of $\mathcal{M}^d$ with vertices of $\mathcal{G}$, and facets of $\mathcal{M}^d$ with edges of $\mathcal{G}$.

11 Algorithm for $(d-2)$-primitive decompositions

11.1 Preliminaries to Algorithm 1

A reciprocal is called maximum if it has the maximum possible number of non-degenerate edges. The algorithm attempts to construct a maximum reciprocal for M^d and finds the dimension of $Rec(M^d)$ in linear running time in the number of internal facets (recall, that by the results of Section 6 $Rec(M^d) \equiv Lift(M^d)$).

Designing Algorithm 1 we were guided by the following analog of theorem 8.3 for reciprocals.

This proposition is a direct consequence of the Theorems 6.1 and 9.3.

Proposition 11.1 *Let $\mathcal{M}^d$ be $(d-2)$-primitive, and each d-cell of $\mathcal{M}^d$ have an internal $(d-2)$-face. Assume for each d-cell C of $\mathcal{M}^d$ the set recl $(\partial C \backslash \partial \mathcal{M}^d)$ is a strongly connected $(d-1)$-pseudomanifold. If the stars of all internal $(d-2)$-cells of M^d are generic, then $\dim Rec(M^d)$ is equal to either 1 or 0.*

Since the correctness of the algorithm can be guaranteed only for $(d-2)$-primitive decompositions, we assume that our input is a PL-realization of a finite $(d-2)$-primitive cell-decomposition of a manifold. We assume also that realization M^d has generic stars of $(d-2)$-cells and that the combinatorial $\mathcal{M}^d$ satisfies at least one of the following conditions:
1) $\mathcal{M}^d$ is a closed manifold,
2) $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$.

In fact one can adopt the algorithm so that it handles non-generic stars of $(d-2)$-cells, but we omit the proof. We pay attention to d-cells with parallel facets since such cells occur in tilings derived from lattices and other point systems with symmetries, and it seems quite natural to take them into account. The applicability of the algorithm to the above classes, its complexity and its robustness are discussed in Subsections 11.3, 11.4.

The above proposition implies that the space of liftings is determined by a system of at most $f^{\circ}_{d-1}(\mathcal{M}^d)$ linear equations like $x_i = cx_j$, where x_i, x_j are cofactors of the stars of facets, and k or $-k$ is the length of the corresponding edge of a maximal reciprocal. It is clear, however, that one can read off a system of equations determining $Lift(M^d)$ from a maximum reciprocal and $\mathcal{I}_{d-1}(M^d)$ (cf. Section 10) in time $O(f^{\circ}_{d-1}(\mathcal{M}^d))$.

In this coordinate system, the cone of liftings of a certain fixed type is determined by inequalities like $x_i < 0$ or $x_i < 0$, and therefore the question of the existence of a lifting of a certain type can be answered in time $O(f^{\circ}_{d-1}(\mathcal{M}^d))$. *Thus our algorithm solves the lifting problem (cf. Section 10) for $(d-2)$-primitive decompositions of closed manifolds and homology disks in optimal time.*

As in the previous section, we call a subset of internal facets of $\mathcal{M}^d$ an independent component if stresses on the facets of this subset do not depend on the choice of stresses for the facets from the complement of this subset. For example, if two facets in a $(d-2)$-primitive manifold can be connected by a chain of adjacent facets such that every two consecutive facets share an internal $(d-2)$-cell, then they belong to the same component. Often it is somewhat more convenient to describe the dependences of stresses via the use of the combinatorial dual graph $\mathcal{G}$. We will switch freely between vertices and edges of $\mathcal{G}$ and d-cells and facets of $\mathcal{M}^d$. Call a subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an independent component if the choice of stresses for facets from $\mathcal{E}'$ does not depend on the choice of stresses for facets from $\mathcal{E} - \mathcal{E}'$. All facets can be partitioned into disjoint independent components I_k in a unique way. This partition corresponds to an edge-partition of $\mathcal{G}$.

Starting with an arbitrary edge of $\mathcal{G}$, the algorithm processes an independent component of $\mathcal{G}$ to which this edge belongs. In the same way the algorithm processes all other independent components of $\mathcal{G}$ in a sequence. As the algorithm works, a dynamical data structure $LIST$ which contains all not-processed d-cells of $\mathcal{M}^d$ (i.e. their indexes) is maintained. Initially $LIST$ contains d-cells of $\mathcal{M}^d$. We maintain links (implemented as pointers in both directions) between facets of $\mathcal{M}^d$ and corresponding elements of $LIST$. Thus, we can assume that any d-cell which is known to be in $LIST$ can be removed from $LIST$ in constant time, and that for any d-cell of $\mathcal{M}^d$ one can check in constant time whether its index is in $LIST$. (When an element of

$LIST$ is deleted, its link is deleted too.) The algorithm uses $STACK$ where d-cells are put as the algorithm proceeds. Initially, $STACK$ is empty. The algorithm attempts to construct a reciprocal for each independent component of $\mathcal{G}$ using an inductive procedure which finds vertices of the reciprocal. Let $\mathcal{G}'$ be an independent component of $\mathcal{G}$, and $R'_k \subset R' \subset R$ ($k > 2$) be the subgraph of R corresponding to the first k constructed vertices of $\mathcal{G}'$. Let $v(A) \in \mathbb{R}^d$ denote the vertex of R'_{k+1} corresponding to a d-cell A. Assume that vertices $v(B)$ and $v(C)$ of R'_k corresponding to d-cells B and C that are adjacent to A have been constructed on the first k steps, and the construction can be extended to A. In this case one can find vertex $v(A)$ as the intersection point of the lines passing through $v(B)$ and $v(C)$, and perpendicular to the facets at which B and C contact A. Denote by $V(A; v(B), v(C))$ the function expressing this dependence (we write $V(A; v(B), v(C)) = \emptyset$ if these lines do not intersect). If $v(A)$ does not exist, R'_{k+1} and therefore reciprocal R' corresponding to $\mathcal{G}'$ collapses into a single point. After computing function $V(A; v(B), v(C))$ d-cell A is marked as *processed* (in both cases). If R' collapses into a point, all d-cells corresponding to its vertices are marked as processed. The algorithm produces the coordinates of vertices of a maximum reciprocal R and computes $dim\ Rec(M^d)$. It also formulates a system of linear equations Θ determining $Lift(M^d)$ whose variables are the C^0_1-cofactors (in our notation variable x_{AB} represents cofactor $c^0_1(A, B)$). Let $a(C)$ be the number of actually processed d-cells which are adjacent to C (this number changes as the algorithm proceeds). Initially, for all C the $a(C)$ are set equal to 0. Index j ranges over all independent components and is initialized by 1. The dimension of the j-th component is denoted by dim_j. Both $dim\ Rec(M^d)$ and dim_j are initialized to zero.

11.2 Algorithm 1

0 **while** $LIST$ is not empty;
1 **if** $STACK$ is not empty **then** take a cell A from $STACK$; mark A as processed;
    remove A from $LIST$;
2 **else if** $LIST$ is empty; **then** $dim\ Rec(M^d) := dim\ Rec(M^d) + dim_j$; **terminate**
  else remove a cell A from $LIST$;
    if there is a cell B adjacent to A in the $LIST$ **then** remove B from $LIST$;
      push A onto $STACK$;
      set $v(B)$ at the origin;
      mark B as processed;
    else mark A as processed; $dim\ Rec(M^d) := dim\ Rec(M^d) + 1$;
      j:=j+1; $dim_j := 1$; denote x_{AB} by x_j
    endif
  endif
endif
3 **if** $a(A) \geq 2$ **then** scan all d-cells adjacent to A and find two processed cells D and E
  that make contact with A in non-parallel facets;
  if $v(A; v(D), v(E))$ exists **then** add $''x_{AD} = \langle v(A)v(D), n(A, D)\rangle x_j''$
    and $''x_{AE} = \langle v(A)v(E), n(A, E)\rangle x_j''$ to Θ;
  else $dim_j := 0$; add $''x_j = 0''$ to Θ;
    for each C which is a processed neighbour of A
    add $''x_{AC} = 0''$ to Θ;

endfor

endif

else if $a(A) = 1$ **then** find the processed d-cell adjacent to A (denote it by F); find the common facet C^{d-1} of A and F; find a point in $\mathbb{R}^d$ at unit distance from $v(A)$ so that $[v(A)v(F)]$ is orthogonal to C^{d-1} and is cooriented with $n(A, F)$;

endif

endif

4 **for** all d-cells G adjacent to A **do**

if G has not been processed **then** $a(G) = a(G) + 1$

if G has two adjacent d-cells which make contact with G in two non-parallel facets **then** push G onto $STACK$

endif

elseif $v(A)$ does not exists **or** $[v(G)v(A)]$ is not orthogonal to the corresponding facet **then** $dim_j := 0$; add $''x_j = 0''$ to Θ; add $''x_{AG} = 0''$ to Θ;

endif

endfor

5 **end**

11.3 Analysis of Algorithm 1.

The algorithm is supposed to process all independent components in a sequence, and can give wrong results only if it does not recognize independent components properly, i.e. if it treats edges of $\mathcal{G}$ which belong to a component as edges from different components. It is enough to prove that $STACK$ does not become empty until all d-cells of a current component $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ have been processed. The algorithm recognizes $\mathcal{G}_1$ as processed before all d-cells corresponding to vertices of $\mathcal{E}_1$ have been processed only if cannot find a non-processed cell which has two processed neighbours in non-parallel facets. The construction of the algorithm guarantees that in this case there is a d-cell C with one processed neighbour corresponding to a vertex of $\mathcal{G}_1$. Let B be a processed neighbour of C, and A be a processed neighbour of B from $\mathcal{G}_1$. Note that B must have a processed neighbour, because of the assumptions about $\mathcal{G}_1$ and C.

First, consider the case when $\mathcal{M}^d$ is closed. Then, A and C can be connected by a path which consists of d-cells making full contact with B. Therefore one can find a non-processed d-cell which contacts two processed d-cells (one of them is A) and shares an internal $(d-2)$-cell with them. It contradicts to our assumption that the $STACK$ is empty.

Now assume that $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$. Since $C \in \mathcal{V}_1$, there is a path $\mathbf{p}$ in $\mathcal{G}_1$ which connects vertices B and C, but does not contain edge $[BC]$. We can assume that this path is the sum of two paths $\mathbf{p}_1 = [B, \ldots, C']$ and $\mathbf{p}_2 = [C', \ldots, C]$, where the first consists only of processed vertices except the last vertex, and the second contains only non-processed vertices. Therefore there is a non-trivial edge-cycle in $\mathcal{G}_1$ which contains $[BC]$. Since $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, $\mathbf{p}_1 + \mathbf{p}_2 + [CB] = \partial \sum_i \mathbf{t}_i \ (mod\ 2)$, where the $\mathbf{t}_i$ are triangles corresponding to the stars of internal $(d-2)$-cells of M^d (see Section 4). Let $\sum_i \mathbf{t}_i = \sum^1 + \sum^2$, where $\sum^1$ is the sum modulo 2 of the $\mathbf{t}_i$ with all processed vertices and $\sum^2$ is the sum modulo 2 of the $\mathbf{t}_i$ with at least one non-processed vertex. Assume that none of the triangles of the second type contains two processed vertices. Thus $\partial \sum^1 = \mathbf{p}_1$, which is impossible since $\mathbf{p}_1$ is not a cycle. Therefore there is a non-processed vertex of $\mathcal{G}_1$ which forms a triangle in $\mathcal{G}_1$ with two processed vertices. This contradicts our choice of C. Since the choice of $\mathcal{G}_1$ was arbitrary, one can conclude that the algorithm processes all independent components properly.

It is easy to see that if L is the binary size of the numerical input (an array of the coordinates of normals to facets), then the number of arithmetical operations which have to be performed to a precision of $O(L)$ is $O(f^{\circ}_{d-1}(\mathcal{M}^d))$.

11.4 Robustness and approximations

Notice, that even when $\mathcal{M}^d$ $(d > 2)$ satisfies all of the conditions requied for the successful performance of the algorithm (see Section 11.1), a real numerical experiment with not-exact machine arithmetic might give wrong results. When on the step 3 the algorithm checks whether two facet normals intersect (computing function $V(A; v(D), v(E))$, any loss of precision may result in the wrong conclusion (if $d > 2$). Therefore this algorithm is not numerically stable. When we know as a preliminary that a non-degenerate reciprocal exists, one can modify the algorithm so that it will construct an approximation of the reciprocal. Instead of computing the point of intersection of two lines (function $V(v(A); v(B), v(C))$, a numerically stable version of Algorithm 1 should compute the point which is equidistant from both lines and minimizes the sum of the distances between a point and the lines. It does not increase the complexity of the algorithm, because it can be done in time O(1). This can be also used for approximation of cell-decompositions of $\mathbb{R}^d$ which have a high proportion of primitive stars of $(d-2)$-cells by weighted Voronoi diagrams. Suzuki and Iri [43] report on how approximation of planar tesselations by Voronoi diagrams can be applied to urban planning and biological growth models.

What kind of manageable sufficient conditions can be used for determining whether a cell-decomposition of $\mathbb{R}^d$ $(d > 2)$ can be represented by a weighted diagram? Some such conditions are:

1) The decomposition is $(d-3)$-primitive (see Theorem 9.3);
2) The decomposition is $(d-2)$-primitive and for the star of every $(d-3)$-dimensional face there is an all-non-zero stress (see Theorem 9.2).

The second condition is not locally robust, but can be used for analysing tilings which consist of polytopes with symmetries arising from lattices (lattice polytopes, space-fillers) (see [38]).

12 Algorithm for general cell-decompositions

12.1 Preliminaries

Let M^d be stored as it is described in Section 10. The algorithm determines whether a lifting of a given type exists. Our strategy is to formulate a system of homogeneous linear equations and inequalities whose feasibility set coincides with the set of liftings of the prescribed type. For the variables of this system we take the C^0_1-cofactors of the stars of facets (see Sections 5, 10, 11). The variables corresponding to those facets which are not involved in cycles of the dual graph are indeed free. Let $A(C_j)$ be a family of affine functions corresponding to the d-cells of M^d. By the results of Sections 4 and 5 if for each cycle $\mathbf{c}_i = [C_1, \dots, C_{k-1}, C_1]$ from a generating system $\{\mathbf{c}_i\}$ of the cycle space of the dual graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ of $\mathcal{M}^d$ the affine functions corresponding to d-cells of this cycle determine a lifting of the sub-complex which consists of $C_1, \ldots, C_{k-1}$, then $A(C_j)$ is a lifting of M^d. In terms of reciprocals it means that we have to pick lengths and positions of edges for the geometrical rectilinear realization of $\mathcal{G}$ so that edges are perpendicular to corresponding facets and the realizations of cycles $\{\mathbf{c}_i\}$ are closed in $\mathbb{R}^d$. It takes $O(f_{d-1})$ operations to construct the dual graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ from the graph of incidences of $\mathcal{M}^d$. The cost of finding a basis of the space of cycles for a graph with $|\mathcal{E}|$ edges is $O(|\mathcal{E}|) = O(f^\circ_{d-1})$ operations (we find a spanning tree and then the basis associated with the tree). The number of cycles will be less than $|\mathcal{E}|$. In order to write equations expressing continuous gluing of d-cells of lifting over each cycle of the basis, we need to have these cycles oriented. It takes $O(|\mathcal{E}|^2) = O(f^{\circ\,2}_{d-1})$ operations. Note that by Lemma 4.2 if $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$, one can take the links of internal $(d-2)$-faces for a generating system of cycles. It requires $f^\circ_{d-2} f^\circ_{d-1}$ operations. This is reasonable when $d = 2$, since the number of vertices in a planar tiling is less than the number of edges. If the facets of M^d are situated in a general position relative to the coordinate system, the vanishing of the vector sum of cyclically ordered facet normals (scaled by the cofactors) in a $(d-2)$-cell C^{d-2} can be expressed with two linear homogeneous scalar equations which involve only the two first coordinates of normals to facets sharing C^{d-2} at C^{d-2}. This means that instead of taking df°_{d-2} scalar equations, we can take only $2f^\circ_{d-2}$. *In this case the time complexity of the algorithm does not depend on d at all.*

Denote by Θ the system of d $(dim\, H_1(\mathcal{G}, \mathbb{Z}_2))$ scalar homogeneous equations corresponding to the basis elements of the space of cycles $H_1(\mathcal{G}, \mathbb{Z}_2)$ augmented with inequalities expressing restrictions on the qualitative type of lifting (like $x_i > 0$ or $x_i < 0$ or $x_i = 0$). The complexity of constructing this system is $O(f^2_{d-1})$. Thus the problem of finding a lifting of a given type is reduced to the problem of finding a solution for a system of df°_{d-1} linear homogeneous equations and at most f°_{d-1} linear homogeneous strict inequalities with f°_{d-1} variables. These variables are C^0_1-cofactors of the stars of facets.

System Θ is a homogeneous feasibility problem in the standard Karmarkar form. Solutions to our original geometrical problem are represented by *interior* points of the cone Θ. An interior point for the cone Θ can also be found by recasting this problem to an auxiliary linear programming problem in the standard format and solving this auxiliary problem by a polynomial projective method (see [45]). A solution for Θ can also be effectively found via a modified Karmarkar algorithm for the homogeneous feasibility problem developed in [15]. The fact that the inequalities of Θ are strict does not affect the formal time complexity.

The time complexity of the algorithm will be given as a function of the number of $(d-1)$- and $(d-2)$-cells. This estimation will also take into account the binary size of numerical input. *Let L is the binary size of the array of coordinates of normals to the supporting hyperplanes of facets.* We assume that all operations are performed to a precision of $O(L\, lnL)$. By Vaidya's estimates [45] the complexity of finding a feasible solution for the resulting feasibility problem by a modified projective (Karmarkar-like) method is $O(f^{\circ\,3}_{d-1}\, L)$.

This leads to:

Theorem 12.1 *Let M^d be a PL-realization of a finite cell-decomposition of a manifold in $\mathbb{R}^d$. Algorithm 2 establishes whether M^d has a lifting of a given type and finds such lifting in the case of the existence in time*

$$O(f^{\circ\,3}_{d-1}\, L \; + \; f_{d-1}).$$

If $H_1(\mathcal{M}^d, \mathbb{Z}^2) = 0$, and the links of $(d-2)$-cells are used as a generating system, then the number of equations f°_{d-2} can be asymptotically larger than the number of variables f°_{d-1}. Removing redundant equations by Gaussian elimination costs $O(f^\circ_{d-2} f^{\circ\,2}_{d-1})$ operations. After such reduction the number of equations does not exceed f°_{d-1}. In this version the construction of Θ requires $O(f^\circ_{d-2} f^{\circ\,2}_{d-1} \; + \; f^\circ_{d-2} f^\circ_{d-1} \; + \; f_{d-1})$ operations. Thus the total time complexity of this version is

$$O(f^{\circ\,3}_{d-1} L \; + \; f^\circ_{d-2} f^{\circ\,2}_{d-1} \; + \; f_{d-1}),$$

where L is the binary size of the array of the first two coordinates of normals to the supporting hyperplanes of facets.

Let M^d be a realization of a homology sphere or a cell-decomposition of $\mathbb{R}^d$. It is interesting to estimate the complexity of the above algorithm in the case when the asymptotic upper bound theorem applies. Assume a sharp lifting for M^d exists. (It is equivalent to the existence of an all-non-zero stress). In this case one can apply to $\mathcal{M}^d$ the asymptotic upper bound theorem, because even if the cell-decomposition of $\mathcal{M}^d$ is not simplicial or 0-primitive (simple), one can prove the asymptotic upper bound via the perturbation of a sharp lifting. (For example, the asymptotic upper bound can be always used in the case $d = 2$, since any cell-decomposition of a 2-manifold can be made simple by a small perturbation.) Let $n = f_d$ be the number of d-cells of $\mathcal{M}^d$. We have $f_{d-2} = O(n^{min(\lfloor (d+1)/2 \rfloor, 3)})$, $f_{d-1} = O(n^{min(\lfloor (d+1)/2 \rfloor, 2)})$. Thus we have the following estimations for spherical manifolds: $O(n^3\, L)$ for $d = 2$, and $O(n^6\, L)$ for $d \geq 3$.

13 Fans

It is natural to interpret all obtained results in terms of fans. A *fan* in Euclidean space $\mathbb{R}^d$ is a finite collection of pointed polyhedral cones which is closed under taking faces and intersections, and which covers $\mathbb{R}^d$. A fan can be alternatively regarded as a spherical complex, namely a cell-decomposition of $\mathbb{S}^{d-1}$ where all cells are intersections of $\mathbb{S}^{d-1}$ and pointed polyhedral cones with centers at the center of $\mathbb{S}^{d-1}$. A convex polytope gives rise to its *normal* fan, whose cones are formed by normals to faces of the polytope. By analogy with the "flat" case, one can consider for a spherical complex K radial liftings, i.e. polytopes such that their radial projections from the center of $\mathbb{S}^{d-1}$ give K. A star polytope generates a special fan. The apex of this fan is visible from any point of the boundary, and the cones of this fan are based on the faces of the polytope. We call such fans *polytopical.* Notice that if a fan is the radial projection of a convex polytope if and only if this is a normal fan of a convex polytope. A fan is referred to as *k-primitive* if the star of each k-face has only $(d-k+1)$ d-cones (minimal possible number). Alternatively, a spherical complex on $\mathbb{S}^{d-1}$ is referred to as k-primitive if the star of each k-face has exactly $d-k$ $(d-1)$-cells.

Two questions are of our interest.

1. Under what conditions is a given finite spherical complex S^{d-1} the radial projection of the boundary complex of a convex polytope, or in other words, when is a fan normal?
2. Under what conditions is a given finite spherical complex S^{d-1} the radial projection of the boundary complex of a star polytope, or in other words, when is a fan polytopical?

Denote by $StPol(F)$ the manifold of star polytopes (up the choice of a hyperplane) that have F as their radial fan, and by $ConPol(F)$ the cone of convex polytopes (in the same sense) that have this F as their radial fan. These objects can be embedded into $\mathbb{R}^{f_{d-1}(F)-1}$.

All propositions in this section can be proved either via a straightforward adaptation of the arguments given for "flat" realizations to spherical complexes, or by regarding fans as finite cell-decompositions of $\mathbb{R}^d$ by convex polyhedra. In the latter case we lift a fan to a cone in $\mathbb{R}^{d+1}$, intersect this cone with a hyperplane parallel to $\mathbb{R}^d$, and project the intersection back onto $\mathbb{R}^d$. Depending on whether the cone is convex or not, the resulting polytope in $\mathbb{R}^d$ is either convex or star-like. Adopting the definition of a reciprocal (see Section 6) for the standard sphere $\mathbb{S}^{d-1}$ one can make use of this notion for spherical complexes.

Proposition 13.1 *A fan is polytopical if and only if the spherical complex of the fan has a non-degenerate reciprocal. Convex polytopes correspond to convex reciprocals.*

The theorem that a fan is normal if and only if the spherical complex of the fan has a convex reciprocal was proved earlier by McMullen [32]. We give here a characterization of fans of convex polytopes and star polytopes in terms of stresses. The definition of the k-stress for $k = 1, , \ldots , d-1$ (Section 3) works well for fans and spherical complexes. A k-stress on a fan corresponds to a $(k-1)$-stress on the corresponding spherical complex. Recall that for manifolds with $H_1(\mathcal{M}^d, \mathbb{Z}_2) = 0$ the coefficients of facet stresses can be regarded as C_1^0-cofactors of facets. In other words, they define affine forms which are the differences between functions representing a lifting on adjacent d-cells. The coefficients of a d-stress on facets of a fan can be interpreted exactly in the same way (see Theorem 5.1). Adopting the arguments from the proof of Theorem 5.1 for the spherical case, we have the following proposition.

Proposition 13.2 *Let F be a polyhedral fan in $\mathbb{R}^d$ $(d > 1)$. Then $dimStPol(F) = dimStress_d(F)$. Cones $ConPol(F)$ and $Tension_d(F)$ are isomorphic.*

In other words a fan in $\mathbb{R}^d$ is the radial fan of a convex (star) polytope if and only if there is a d-tension (all-non-zero d-stress) on this fan. The following propositions are adaptations of the results of Sections 3-11 for fans.

Proposition 13.3 *Every $(d-3)$-primitive fan $(d > 2)$ is the radial (normal) fan of a convex polytope which is unique up to the choice of a hyperplane and a dihedral angle.*

Proposition 13.4 *Let fan F be $(d-2)$-primitive $(d > 2)$. If there is an all-non-zero tension for each $(d-3)$-dimensional face of F, then F is the radial (normal) fan of a convex polytope.*

Proposition 13.5 *Let F be a $(d-2)$-primitive fan. There is an algorithm which determines whether F is the radial (normal) fan of a convex polytope. This algorithm has linear time complexity in the number of $(d-1)$-cells of F. If such convex polytope exists, it is unique up to the choice of a facet and a dihedral angle.*

Proposition 13.6 *There is a polynomial algorithm which determines whether a fan F in $\mathbb{R}^d$ is the radial (normal) fan of a convex polytope. Let n be the number of d-cones in F, and L be the binary size of an array of normals to $(d-1)$-cones of F. The time complexity of this algorithm is $O(n^3\, L)$ for $d = 3$, and $O(n^6\, L)$ if $d > 3$ and the asymptotic upper bound theorem applies.*

In [39] Shephard gave necessary and sufficient conditions in terms of Gale diagrams that a given finite spherical complex S^{d-1} is the radial projection of the boundary complex of a convex polytope. Aurenhammer [6, 7] worked out applications of this approach to scene analysis and Voronoi diagrams. We can interpret the Gale diagram technique in terms of computational complexity. Our analysis will be based on the assumption that the algorithm works with a data structure which is well adapted for it. For example, we can assume that the nodes representing the rays of a fan F in the incidence graph of F contain their coordinates. Notice that Algorithm 2 requires inequalities determining $(d-1)$-cells of F, whereas an algorithm which can be extracted from the Shephard theorem requires coordinates of rays of F. If f_k is the number of k-cells of S^{d-1} ($(k+1)$-cones of F), then the Shepard's condition is the existence of an interior point in the intersection of a finite number of convex polytopes in $\mathbb{R}^{f_0-d}$ (see [39, 40]). Each polytope of this family corresponds to a $(d-1)$-cell of S^{d-1}. If a $(d-1)$-cell C_i^{d-1} has $f_0(C_i^{d-1})$ vertices, then the corresponding polytope $G(C_i^{d-1})$ is the convex hall of $f_0 - f_0(C_i^{d-1})$ vertices. Switching to dual polytopes $G^*(C_i^{d-1})$ one can see that the problem has polynomial complexity. Let $\cap_{i=1}^{f_{d-1}} G^*(C_i^{d-1})$ be the intersection of polytopes $G^*(C_i^{d-1})$ over all $(d-1)$-cells. Notice that unlike system Θ from Algorithm 2 affine equations determining the polytope $\cap_{i=1}^{f_{d-1}} G^*(C_i^{d-1})$ in the Shephard's criterion are not homogeneous. The complexity of solving the feasibility problem for $\cap_{i=1}^{f_{d-1}} G^*(C_i^{d-1})$ can be formally estimated as $O(((f_0 f_{d-1} + f_0)f_0^2 + (f_0 f_{d-1} + f_0)^{1.5} f_0)L)$ in the worst case (L is the binary size of numerical input) [45]. For example, for $d = 3$ it gives $O(f_0^4 L) = O(f_{d-1}^4 L)$ which is asymptotically worse than $O(f_{d-1}^3 L)$ given by Proposition 13.6.

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