

Material for Difference Quotient

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Preface

The following difference quotient material starts with the development of average slopes connecting points on a function. Then there is an algebra review focusing on factoring, rational expressions, radical notation and rational exponents. Examples are developed for each of the average slope and algebra review topics. Then the difference quotient topic is presented with examples of various complexity. The examples chosen use the concepts developed in the previous average slope and algebra review topics.

Finally, in the summary the rationalization for computing difference quotients is shown by taking the next step and computing the derivative of the functions.

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Material for Difference Quotient

Introduction

Now that we have discussed the slope of a line and least squares regression to find the line of best fit, let's introduce function notation and average rate of change. During this course, we are using the notation $y = f(x)$, which represents a function. By definition, $y = f(x)$ is a function of the independent variable x if for any x value there is only one y value. The y value is referred to as the dependent variable.

Some examples of functions are shown below.

$$\text{a) } y = 3x + 1 \quad \text{b) } y = x^2 - 1 \quad \text{c) } y = 7 \quad \text{d) } y = \sqrt{x}$$

For each of the examples, one value of x in turn will generate only one value of y . There are some other interesting points to mention about each of the examples, including the domain and range. The domain is the collection of possible x values that a function $y = f(x)$ can have, and the range is the collection of generated y values.

For example a), if we graph the line there is no restriction on the values for the independent variable x , so the domain is all real numbers or $-\infty < x < \infty$. We don't use the \leq symbol since x can never equal infinity. The range or possible y values for example a) is likewise all real numbers, i.e. $-\infty < y < \infty$.

For example b), graph the quadratic and you will no doubt be able to surmise intervals for the domain and range. The independent variable x can again be anything, so the domain is $-\infty < x < \infty$. However, the smallest y value is $y = -1$, so the range of the function is $-1 \leq y < \infty$.

In the third example c), y is a constant value regardless of the x value. It's like saying: "No matter who you are, I'll give you 7 extra points on your exam." The domain of the function is $-\infty < x < \infty$ and the range is $y = 7$.

For the last example d), the possible values of the independent variable x are limited to $x \geq 0$ since when dealing with real numbers we can't take the square root of a negative number. Likewise, the range of the function is $y \geq 0$. Another way to write the domain and range is $0 \leq x < \infty$ and $0 \leq y < \infty$. Some people use open and closed brackets to indicate intervals, but we'll concentrate on the interval notation just mentioned.

Assume we are walking up a hill defined by the equation $y = x^2 - 2x + 1$. So, this means as we walk up this hill, the slope changes as we move. This equation is graphed in figure 1.



Figure 1

1. Consider the points $(1, 0)$ and $(3, 4)$, which both lie on the curve shown in figure 13. An average rate of change $\frac{y_2 - y_1}{x_2 - x_1}$ is actually the slope of a secant line and the average rate of change for the points when x changes from $x = 1$ to $x = 3$ becomes $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{3 - 1} = 2$. A secant line “cuts” the curve at the two points. See if you can verify the computation.
2. Now let’s see what happens if the two x values get closer to each other. Using $x = 1$ and $x = 2$, the average rate of change is $\frac{1 - 0}{2 - 1} = 1$. Again, make sure you can verify this computation. An important item to note is that the average rate of change or slope of the secant line decreased from 2 to 1.
3. What happens if we pick a point still closer to $x = 1$? In this last scenario, let’s see what happens when we pick $x = 1.5$ as the other point. In this case the average rate of change becomes $\frac{y(1.5) - y(1)}{1.5 - 1} = \frac{1.5^2 - 2 * 1.5 + 1}{.5} = \frac{.25}{.5} = \frac{1}{2}$.

Summarizing the results of these computations, the average slope started out to be 2, when the x values were $x = 1$ and $x = 3$, then the slope became 1 as the x values got a bit closer when using the values $x = 1$ and $x = 2$. The last case showed that the slope of the secant line diminished to $\frac{1}{2}$ when the x values got even closer using $x = 1$ and $x = 1.5$.

We are getting closer and closer to introducing the idea of instantaneous slope, i.e. the slope of tangent line as the difference in the x values approaches zero, but we’re not quite there as yet. Slopes of tangent lines will be where we start differentiating functions and doing some more involved calculus work.

Algebra Review

Before introducing the difference quotient, let’s look at some rules of algebra needed to solve these problems.

1. Factoring

Problems have been selected from Algebra and Trigonometry, 2nd ed. By Beecher, Penna and Bittinger, Pearson Education, Inc., 2005, p.28.

Some steps to help with the factoring order of operations are:

1. Take out the GCF (greatest common factor).
2. Check for difference of squares. [Example: $a^2 - b^2 = (a - b)(a + b)$]
3. Check for Quadratic Trinomial [Example: $a^2 + 6a + 8 = (a + 4)(a + 2)$]
or Perfect Square Trinomial (Square of a Binomial).

$$\text{[Example: } a^2 + 2ab + b^2 = (a + b)^2 \text{]}$$

Note that $(a + b)^2 \neq a^2 + b^2$ unless either $a=0$ or $b=0$

4. Check for Sum of Cubes [Example: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$] or Difference of Cubes. [Example: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]
5. Grouping: Split the equation in two and fully factor the two separate equations. Then combine the two equations and factor them together.
6. If none of these work, the equation is fully factored.

Factoring Example 1 Factor $7y + 42$.

First, let's try to take a greatest common factor from the expression. The GCD of the two terms $7y$ and 42 is 7 . When we pull a 7 out of $7y$, we are left with a y . When we pull a 7 out of 42 , we are left with a 6 .

$$7y + 42 = 7(y + 6)$$

Keep in mind that an easy (and fast!) way to check work is to use the distributive property to multiply 7 by y plus 6 . If your answer does not come out to the original expression, there is a mistake somewhere!

Factoring Example 2 Factor $x^3 + 3x^2 + 6x + 18$.

Most times when there are four terms in an expression, the expression should be split in two and factored as two separate expressions. Let's try grouping. The GCD of x^3 and $3x^2$ is x^2 . Like example 1, the GCD of $6x$ and 18 is 6 .

$$x^3 + 3x^2 + 6x + 18 = x^2(x + 3) + 6(x + 3)$$

Next, we have two expressions connected by an addition sign. Both expressions have a GCD of x plus 3 . So, we have

$$x^3 + 3x^2 + 6x + 18 = x^2(x+3) + 6(x+3) = (x^2 + 6)(x+3).$$

2. Rational Expressions

Problems have been selected from Algebra and Trigonometry, 2nd ed. By Beecher, Penna and Bittinger, Pearson Education, Inc., 2005, p.35-37.

Some rules to help with factoring rational expressions (quotient of two polynomials) are:

1. $1 - y = -(y - 1)$
2. You can factor fractions, but you must factor the numerator and denominator separately.
3. To add/subtract fractions the denominator must be the same.
4. To find a common denominator, you must multiply the denominators by every unshared factor.
5. To multiply fractions, you multiply the numerator by the other numerator(s) and the denominator by the other denominator(s).
6. In order to divide fractions, you multiply the divisor by the reciprocal of the dividend. For example, the reciprocal of $\frac{1}{2}$ is $\frac{2}{1}$.
7. When you multiply and divide fractions, you can hopefully cancel out like factors (except when each factor is equal to zero). If one of the numerators shares a factor with one of the denominators, you can reduce the fraction by canceling those factors in the numerator and denominator. (Try to cancel as much as possible because it makes the fraction much simpler).
8. Always try to factor fractions!

Rational Expressions Example 3

Simplify the expression.

$$\frac{a^2 - a - 12}{a^2 - 6a + 8} \cdot \frac{a^2 + a - 6}{a^2 - 2a - 24}$$

Begin by factoring the numerator and denominator in both of the fractions.

$$\frac{a^2 - a - 12}{a^2 - 6a + 8} \cdot \frac{a^2 + a - 6}{a^2 - 2a - 24} = \frac{(a+3)(a-4)}{(a-4)(a-2)} \cdot \frac{(a+3)(a-2)}{(a-6)(a+4)}$$

Next, combine both numerators and both denominators to make one fraction. Also, since there is an $a - 2$ and an $a - 4$ on top and bottom, they cancel out.

$$\frac{a^2 - a - 12}{a^2 - 6a + 8} \cdot \frac{a^2 + a - 6}{a^2 - 2a - 24} = \frac{(a+3)(a-4)}{(a-4)(a-2)} \cdot \frac{(a+3)(a-2)}{(a-6)(a+4)} = \frac{(a+3)(a+3)}{(a-6)(a+4)}$$

Finally, since there is an $a + 3$ multiplied twice, it becomes $(a + 3)^2$.

$$\frac{a^2 - a - 12}{a^2 - 6a + 8} \cdot \frac{a^2 + a - 6}{a^2 - 2a - 24} = \frac{(a+3)(a-4)}{(a-4)(a-2)} \cdot \frac{(a+3)(a-2)}{(a-6)(a+4)} = \frac{(a+3)(a+3)}{(a-6)(a+4)} = \frac{(a+3)^2}{(a-6)(a+4)}$$

3. Radical Notation and Rational Exponents

Problems have been selected from Algebra and Trigonometry, 2nd ed. By Beecher, Penna and Bittinger, Pearson Education, Inc., 2005, p.44-46.

First, let's look at the rules involving radicals.

1. If n is even, $\sqrt[n]{a^n} = |a|$. For example, $\sqrt{(-11)^2} = |-11| = 11$.
2. If n is odd, $\sqrt[n]{a^n} = a$. In this case, $\sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$.
3. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$. For example, $\sqrt{60} = \sqrt{15}\sqrt{4} = 2\sqrt{15}$.

Next, let's look at the rules for rational exponents.

1. $a^{m/n} = \sqrt[n]{a^m}$. For example, $8^{2/3} = \sqrt[3]{8^2}$.
2. $a^{m/n} = \left(a^{1/n}\right)^m$. Using the example before, $8^{2/3} = \left(8^{1/3}\right)^2$.
3. $a^{-m/n} = \frac{1}{a^{m/n}}$. An example is $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{\left(8^{1/3}\right)^2} = \frac{1}{4}$.
4. $a^m a^n = a^{m+n}$. An example is $2^{1/3} 2^{2/5} = 2^{5/15} 2^{6/15} = 2^{11/15}$.
5. $\left(a^m\right)^n = a^{mn}$. An example is $\left(2^3\right)^2 = 2^6$.

Let's look at steps to rationalize the denominator. To rationalize the denominator, multiply the top and bottom by the conjugate. Consider the example $\frac{5}{2 + \sqrt{3}}$. The

conjugate of this expression is $2 - \sqrt{3}$, so let's multiply the top and bottom of the fraction by this expression. Rationalizing the numerator is very similar. To rationalize the numerator, multiply top and bottom by the conjugate of the numerator.

Radical Notation Example 4

Rationalize the denominator for $\frac{5}{2+\sqrt{3}}$.

The conjugate is $2-\sqrt{3}$, so multiplying the numerator and denominator of $\frac{5}{2+\sqrt{3}}$ by

$$2-\sqrt{3}, \text{ we get } \left(\frac{5}{2+\sqrt{3}}\right)\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right) = \frac{10-5\sqrt{3}}{4-3} = 10-5\sqrt{3}.$$

Radical Notation Example 5

Convert $a^{3/4}b^{-3/4}$ to radical notation and simplify.

$$a^{3/4}b^{-3/4} = \sqrt[4]{a^3} \frac{1}{\sqrt[4]{b^3}} = \sqrt[4]{\left(\frac{a}{b}\right)^3}.$$

Radical Notation Example 6

Simplify $\sqrt{2}\sqrt[4]{8}$.

$$\sqrt{2}\left(\sqrt[4]{8}\right) = \sqrt{2}\left(\sqrt[4]{2^3}\right) = 2^{1/2}2^{3/4} = 2^{2/4}2^{3/4} = 2^{5/4} = \sqrt[4]{2^5} = \sqrt[4]{32}.$$

Note that in this example we added the exponents when evaluating $2^{1/2}2^{3/4} = 2^{2/4}2^{3/4} = 2^{5/4}$ since $a^m a^n = a^{m+n}$.

Radical Notation Example 7

Convert $\sqrt[3]{\sqrt{11}}$ to exponential notation.

$$\sqrt[3]{\sqrt{11}} = \sqrt[3]{11^{1/2}} = \left(11^{1/2}\right)^{1/3} = 11^{1/6}.$$

Note that in this example we multiplied the exponents

when evaluating $\left(11^{1/2}\right)^{1/3} = 11^{1/6}$ since $(a^m)^n = a^{mn}$.

Difference Quotient

Let's now introduce the concept of the difference quotient.

Difference quotients are essentially average rates of change but we use functional notation instead of $\frac{y_2 - y_1}{x_2 - x_1}$ notation, as seen previously. The average rate of change of the function $f(x)$ with respect to the independent variable x is called the difference quotient and is defined as $\frac{f(x+h) - f(x)}{h}$ where h represents the change in x . Note that some textbooks use Δx instead of h to represent the change in x . Also, since h is in the denominator for the difference quotient, $h \neq 0$. Keep in mind that, in general, $f(x+h) \neq f(x) + f(h)$. Likewise, as we said earlier the algebraic expression $(a+b)^2 \neq a^2 + b^2$ so long as $a \neq 0$ or $b \neq 0$. For those who like to use numbers, $(3+2)^2 \neq 3^2 + 2^2$.

Consider the following example.

Difference Quotient Example 8

For $f(x) = x^2$, find the simplified form of the difference quotient. Then, find the value of the difference quotient when $x = 5$ and $h = 0.2$.

First, let's find $f(x+h)$.

$$f(x+h) = (x+h)^2 = (x+h)(x+h)$$

Next, we use the FOIL method to find $f(x+h) = x^2 + xh + xh + h^2 = x^2 + 2xh + h^2$.

Then, $f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$ and $\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h}$.

Now, we factor the numerator. The greatest common factor in the numerator is h , so we pull out an h from each term in the numerator and see what happens.

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x+h.$$

The result $2x+h$ geometrically represents the slope of a secant for the function $f(x) = x^2$ at a particular value for x with a gap h . Assume $x=1$ and $h=0.2$, then the slope of the secant line for $f(x) = x^2$ connecting the points $x=1$ and $x+h=1+0.2=1.2$ is $2x+h=2(1)+0.2=2.2$.

Difference Quotient Example 9

Next, let's consider the example $x - x^2$. Find $\frac{f(x+h) - f(x)}{h}$.

To start, let's find $f(x+h)$. We substitute $x+h$ wherever there is an x .

$$f(x+h) = (x+h) - (x+h)^2$$

$$= x+h - x^2 - 2xh - h^2$$

So,

$$\frac{f(x+h) - f(x)}{h} = \frac{x+h - x^2 - 2xh - h^2 - (x - x^2)}{h}$$

$$= \frac{x+h - x^2 - 2xh - h^2 - x + x^2}{h}$$

$$= \frac{h - 2xh - h^2}{h}$$

$$= \frac{h(1 - 2x - h)}{h} = 1 - 2x - h, \quad h \neq 0.$$

The slope of the secant line for $f(x) = x - x^2$ when $x=1$ and $h=0.2$ is $1 - 2x - h = 1 - 2(1) - 0.2 = -1.2$.

Difference Quotient Example 10

Let's look at a more involved example, $f(x) = \frac{1}{x}$. Like the previous example, first let's find $f(x+h)$.

We see that $f(x+h) = \frac{1}{x+h}$, so $\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$. To simplify, we must subtract fractions, but we have no common denominator for $\frac{1}{x+h} - \frac{1}{x}$. We have to find a common denominator and simplify. In this case, the common denominator is $(x+h)(x)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{\frac{1}{x+h} \left(\frac{x}{x} \right) - \frac{1}{x} \left(\frac{x+h}{x+h} \right)}{h}$$

$$= \frac{\frac{x}{(x+h)(x)} - \left(\frac{x+h}{(x+h)(x)} \right)}{h}$$

Now, we have a common denominator $(x+h)x$ and can subtract the fractions. The result

is $f(x+h) - f(x) = \frac{-h}{(x+h)x} = \frac{-1}{(x+h)x}$. When $x=1$ and $h=0.2$, the slope of the secant line for $f(x) = \frac{1}{x}$ is $\frac{-1}{(x+h)x} = \frac{-1}{(1+0.2)1} = \frac{-1}{1.2} = -0.833$ rounded to 3 decimal places.

Difference Quotient Example 11

Let's look at an example involving square roots to find the difference quotient.

Consider the function $f(x) = \sqrt{x}$.

We can see that $f(x+h) = \sqrt{x+h}$, so $\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$.

In order to further simplify and remove the h from the denominator, we need to rationalize the numerator. We multiply the top and bottom of the expression by the conjugate, or $\sqrt{x+h} + \sqrt{x}$.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{(\sqrt{x+h} + \sqrt{x})}. \end{aligned}$$

Now the slope of the secant line for $f(x) = \sqrt{x}$ at $x=1$ and $h=0.2$ is

$$\frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{1.2} + \sqrt{1}} = 0.477 \text{ rounded to 3 decimal places.}$$

Difference Quotient Example 12

Finally, consider $f(x) = \frac{1}{\sqrt{2x+3}}$. In this example $f(x+h) = \frac{1}{\sqrt{2(x+h)+3}}$. We will

have to rationalize the denominator when computing the difference quotient. Let's first begin with

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{\sqrt{2(x+h)+3}} - \frac{1}{\sqrt{2x+3}} = \frac{\sqrt{2x+3} - \sqrt{2(x+h)+3}}{(\sqrt{2(x+h)+3})(\sqrt{2x+3})}$$

Make sure that you can justify the steps required up to this point. We now have to multiply the numerator and denominator by the conjugate of $\sqrt{2x+3} - \sqrt{2(x+h)+3}$ which is $\sqrt{2x+3} + \sqrt{2(x+h)+3}$. This results in

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{2x+3} - \sqrt{2(x+h)+3}}{(\sqrt{2(x+h)+3})(\sqrt{2x+3})} = \left(\frac{\sqrt{2x+3} - \sqrt{2(x+h)+3}}{(\sqrt{2(x+h)+3})(\sqrt{2x+3})} \right) \left(\frac{\sqrt{2x+3} + \sqrt{2(x+h)+3}}{\sqrt{2x+3} + \sqrt{2(x+h)+3}} \right) \\ &= \frac{2x+3 - (2(x+h)+3)}{h(\sqrt{2(x+h)+3}\sqrt{2x+3})(\sqrt{2x+3} + \sqrt{2(x+h)+3})} = \frac{-2}{(\sqrt{2(x+h)+3}\sqrt{2x+3})(\sqrt{2x+3} + \sqrt{2(x+h)+3})} \end{aligned}$$

Wow, what a messy expression! Make sure you can justify the steps to get the above result. Don't worry, difference quotient problems that you will see usually won't be this demanding. Now when $x=1$ and $h=0.2$, we can find the slope of the secant line to

$$f(x) = \frac{1}{\sqrt{2x+3}} \text{ which results in } \frac{-2}{\sqrt{2(1.2)+3}\sqrt{2+3}(\sqrt{2+3} + \sqrt{2(1.2)+3})} = -0.084$$

rounded to 3 decimal places.

Summary

When we started the discussion in the introduction, we spoke of average rates of change with x values getting closer to each other. Since a difference quotient is essentially an average rate of change, but to a specified function, we can evaluate the slope of the secant lines at arbitrary x and h values based on the domain of the function $f(x)$. The derivative of a function $f(x)$ is the limit as $h \rightarrow 0$ for the difference quotient and this is called the slope of the tangent line to a curve $f(x)$ at some value x . The formal

definition of a derivative for $f(x)$ is $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. For each of the

difference quotient examples we did, let's now find the derivative.

a) Difference quotient example 8: $\lim_{h \rightarrow 0} (2x+h) = 2x+0 = 2x$ so the derivative (slope of the tangent line) of $f(x) = x^2$ is $f'(x) = 2x$.

b) Difference quotient example 9: $\lim_{h \rightarrow 0} (1-2x-h) = 1-2x-0 = 1-2x$ so the derivative (slope of the tangent line) of $f(x) = x - x^2$ is $f'(x) = 1 - 2x$.

c) Difference quotient example 10: $\lim_{h \rightarrow 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = -\frac{1}{x^2}$ so the derivative

(slope of the tangent line) of $f(x) = \frac{1}{x}$ is $f'(x) = -\frac{1}{x^2}$.

c) Difference quotient example 11: $\lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$ so the

derivative (slope of the tangent line) of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

d) Difference quotient example 12:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{-2}{\sqrt{2(x+h)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+h)+3})} \\ &= \frac{-2}{\sqrt{2(x+0)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+0)+3})} \\ &= \frac{-2}{(2x+3)2\sqrt{2x+3}} = \frac{-2}{2(2x+3)^{\frac{3}{2}}} = -\frac{1}{(2x+3)^{\frac{3}{2}}} \end{aligned}$$

So the derivative (slope of the tangent line) of $f(x) = \frac{1}{\sqrt{2x+3}}$ is $f'(x) = -\frac{1}{(2x+3)^{\frac{3}{2}}}$.

Fortunately we will develop formulae based on the limit of the difference quotient to make finding derivatives of functions and their applications much easier than the examples shown as we found the difference quotient and then took the limit as $h \rightarrow 0$. Remember however, that without going through the difference quotient process, we would never have been able to find derivatives and slopes of tangent lines to functions along with the many applications that are based on this technique.