MCAS

A Resource for Instructors - Fractions

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University of Massachusetts Lowell

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Focus on Mathematics

Boston University – Education Development Center, Inc. – Lesley University
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UMass Lowell mathematicians have joined mathematicians from Boston University, Worcester Polytechnic Institute, and Education Development Center in this effort. A team from Lesley University conducts program evaluation. As part of the program, each mathematician works with teachers in a school in one of the FoM districts: Arlington, Chelsea, Lawrence, Waltham and Watertown.

Preparation of this MCAS instructors' resource document dealing with fractions began as a follow-up to the well received document prepared June 2006 that was used in study groups for middle school teachers in Lawrence. The previous document focused on middle school geometry and measurement MCAS related topics. Seven years of MCAS examination problems and resources from referenced texts have been used to supplement developed materials for this fractions document. References for all MCAS problems are provided.

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The authors represent the Department of Mathematical Sciences at UMass Lowell. David Stanley was a mathematics graduate student when the document was written and is now an instructional technology integration specialist with JFY Networks. Marvin Stick is a member of the UMass Lowell mathematics department faculty.

Preface

This instructors' resource was developed to provide teachers with a concise reference for middle school related fractions topics, as well as professional development material. MCAS problems were included to help improve student achievement.

We hope this document provides you with a valuable resource. On-line versions are available at http://www.focusonmath.org/ and at http://faculty.uml.edu/mstick.

David Stanley, Marvin Stick

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Dimensional Fractions

The dimensions in dimensional fractions relate to length, time, weight, etc.

Examples of length dimensions are inches, feet, and miles.

Examples of weight dimensions are pounds and grams.

Examples of time dimensions are seconds, minutes and days.

Dimensional fractions are fractions composed of just dimensions like those mentioned above. The term "miles per hour" can be shown as a dimensional fraction:

$$\frac{miles}{hour}$$

Other terms like "miles per gallon" or "dollars per pound" can be shown similarly:

$$\frac{miles}{gallon} \qquad\qquad \frac{\$}{pound} \text{ or } \frac{\$}{lb.}$$

The use of dimensional fractions can be helpful in solving various word problems.

Example 1

Suppose we are told that a certain car gets 30 miles per gallon, and we need to find out how many gallons will be needed to go 200 miles. We can write 30 miles per gallon like this: $\frac{30\ miles}{gallon}$

We know that our answer must be a number of gallons, so we need to cancel miles from the above fraction. We can do this by dividing by 200 miles:

$$\frac{30\ miles}{gallon} \div 200\ miles = \frac{30\ \cancel{miles}}{200\ \cancel{miles}\ gallon} = \frac{3}{20\ gallons}$$

Since we want gallons in the numerator, we just invert the above fraction:

$$\frac{20\ gallons}{3} = 6\frac{2}{3}\,gallons$$

So it takes 6 2/3 gallons to go 200 miles.

Another way to look at this problem is to consider

$$200\ \cancel{miles} \Big/ 30\frac{\cancel{miles}}{gallon} = \frac{200}{30}\,gallons = 6\frac{2}{3}\,gallons.$$

We can also use dimensional fractions to convert between different units of measure. Sometimes we may want to go from pounds to ounces or even pounds to grams. Other times we may want to convert feet per second to miles per hour.

Example 2

A runner is measured going 90 feet in 10 seconds. How fast is this runner in miles per hour?

Going 90 feet in 10 seconds is the same as $\frac{90\text{ feet}}{10\text{ seconds}}$ or $\frac{9\text{ feet}}{\text{second}}$.

We know that there are 60 minutes in an hour which is $\frac{60\text{ minutes}}{\text{hour}}$.

We also know that there are 60 seconds in a minute which is $\frac{60\text{ seconds}}{\text{minute}}$.

We can cancel minutes by multiplying these two fractions:

$$\frac{60\ \cancel{\text{minutes}}}{\text{hour}} \times \frac{60\text{ seconds}}{\cancel{\text{minutes}}} = \frac{3{,}600\text{ seconds}}{\text{hour}}$$

Now, since there are 5,280 feet in a mile, we can write that as $\frac{5{,}280\text{ feet}}{\text{mile}}$.

The runner went $\frac{9\text{ feet}}{\text{second}}$, so we want to cancel feet and seconds.

First, we'll cancel feet:

$$\frac{\frac{9\text{ feet}}{\text{second}}}{\frac{5{,}280\text{ feet}}{\text{mile}}} = \frac{9\ \cancel{\text{feet}}}{\text{second}} \times \frac{\text{mile}}{5{,}280\ \cancel{\text{feet}}} = \frac{9\text{ miles}}{5{,}280\text{ seconds}}$$

Next, we'll cancel seconds:

$$\frac{9\text{ miles}}{5{,}280\ \cancel{\text{seconds}}} \times \frac{3{,}600\ \cancel{\text{seconds}}}{\text{hour}} = \frac{9 \times 3{,}600\text{ miles}}{5{,}280\text{ hours}} = \frac{6.1\text{ miles}}{\text{hour}}$$

So the runner was going 6.1 miles per hour.

Example 3

At the meat department of a supermarket, steak is priced at $7.99 per pound. The steak you want to buy is 12 ounces. How much will it cost?

We can write $7.99 per pound as $\frac{7.99 \text{ dollars}}{\text{pound}}$.

Since there are 16 ounces in a pound, we write this as $\frac{16 \text{ ounces}}{\text{pound}}$.

We want to buy 12 ounces so we need to find the number of pounds first:

$$12 \text{ ounces} \Big/ \frac{16 \text{ ounces}}{\text{pound}} = 12 \; \cancel{\text{ounces}} \times \frac{\text{pound}}{16 \; \cancel{\text{ounces}}} = \frac{12 \text{ pounds}}{16} = \frac{3}{4} \text{pound}$$

Now we can calculate the price:

$$\frac{7.99 \text{ dollars}}{\cancel{\text{pound}}} \times \frac{3}{4} \; \cancel{\text{pound}} = 7.99 \times \frac{3}{4} \text{ dollars} = 5.99 \text{ dollars}$$

So, 12 ounces of steak will cost $5.99.

Note that multiplying and dividing fractions are covered on pages 9 and 10. The emphasis in this example is to deal with ounce and pound dimensions.

MCAS Problems

2006

http://www.doe.mass.edu/mcas/2006/release/g8math.pdf

2006 Grade 8 #12

12 The formula below can be used to determine f, the total braking distance, in feet, that a car moving at n miles per hour will travel after the driver applies the brakes.

$$f = \frac{n^2}{20}$$

Using this formula, what is the total braking distance that a car moving at 60 miles per hour will travel after the driver applies the brakes?

A. 6 feet

B. 60 feet

C. 180 feet

D. 1800 feet

Answer: $f = \frac{n^2}{20} = \frac{60^2}{20} = \frac{3600}{20} = 180$, so the answer is C, 180 feet.

2004

http://www.doe.mass.edu/mcas/2004/release/g10math.pdf

2004 Grade 10 #8

8 A certain car averages 28 miles per gallon. Gasoline costs $1.11 per gallon. Which of the following is closest to the number of miles the car would be expected to go on $250 worth of gasoline?

A. 400 miles

B. 6,000 miles

C. 12,000 miles

D. 30,000 miles

$$\frac{\dfrac{1.11\text{ dollars}}{\text{gallon}}}{\dfrac{28\text{ miles}}{\text{gallon}}} = \frac{1.11\text{ dollars}}{\cancel{\text{gallon}}} \times \frac{\cancel{\text{gallon}}}{28\text{ miles}} = \frac{1.11\text{ dollars}}{28\text{ miles}}$$

$$\frac{250\text{ dollars}}{\dfrac{1.11\text{ dollars}}{28\text{ miles}}} = 250\ \cancel{\text{dollars}} \times \frac{28\text{ miles}}{1.11\ \cancel{\text{dollars}}} = \frac{250 \times 28\text{ miles}}{1.11} = 6,306.3\text{ miles}$$

Answer: B. 6,000 miles

2004 Grade 10 #9

9 The Sun is approximately 93,000,000 miles from Earth. Light travels approximately 186,000 miles per second.

$$\left(\text{time} = \frac{\text{distance}}{\text{speed}}\right)$$

Which of the following is **closest** to the number of seconds it takes light to travel from the Sun to Earth?

A. 0.005 second

B. 0.050 second

C. 500 seconds

D. 5000 seconds

$$93{,}000{,}000 \text{ miles} \Big/ \frac{186{,}000 \text{ miles}}{\text{second}} = 93{,}000{,}000 \ \cancel{\text{miles}} \times \frac{\text{second}}{186{,}000 \ \cancel{\text{miles}}}$$

$$= \frac{93{,}000{,}000 \text{ seconds}}{186{,}000}$$

$$= 500 \text{ seconds}$$

Answer: C. 500 seconds

Fraction Arithmetic

Adding and Subtracting Fractions

When adding or subtracting fractions, there are two important things to note:

1. Only the numerator should be added or subtracted
2. The denominators must be the same

Examples

$\frac{1}{2}+\frac{3}{2}=\frac{1+3}{2}=\frac{4}{2}=2$, $\frac{2}{5}+\frac{1}{5}=\frac{2+1}{5}=\frac{3}{5}$, $\frac{3}{4}-\frac{1}{4}=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}$

Common Denominator

If the fractions to be added or subtracted do not have the same denominator, then the *least common denominator* (LCD) must be found first. The least common denominator is the smallest number that both denominators can divide evenly.

Example

For the fractions $\frac{1}{3}$ and $\frac{1}{4}$, the least common denominator is 12 since that is the smallest number that both 3 and 4 can divide evenly.

The denominator and numerator of $\frac{1}{3}$ must be multiplied by 4:

$$\frac{1}{3}\cdot\frac{4}{4}=\frac{4}{12}$$

The denominator and numerator of $\frac{1}{4}$ must be multiplied by 3:

$$\frac{1}{4}\cdot\frac{3}{3}=\frac{3}{12}$$

Finding the Least Common Denominator

One way to find the least common denominator is to use the following steps:

1. Factor the denominator of each fraction into prime numbers.
2. Count the number of times each prime number occurs.

3. For each prime number, choose the largest count then multiply to get the least common denominator.
4. Multiply the denominator and numerator by the same number so that you have a new fraction with the least common denominator.

Example 1

Find the least common denominator of $\frac{2}{15}$ and $\frac{3}{8}$, then add the two fractions.

1. Factor the denominator of each fraction into prime numbers

   - The denominators are 15 and 8.

   - 15 factors into $3 \cdot 5$

   - 8 factors into $2 \cdot 2 \cdot 2$

2. Count the number of times each prime number occurs

   - For the first fraction, 3 occurs once and 5 occurs once

   - For the second fraction, 2 occurs three times.

3. For each prime number, choose the largest count then multiply to get the least common denominator.

   - The least common denominator is $3 \cdot 5 \cdot 2 \cdot 2 \cdot 2 = 120$

4. For $\frac{2}{15}$, multiply by $\frac{8}{8}$ since $15 \times 8 = 120$:

   $$\frac{2}{15} \times \frac{8}{8} = \frac{16}{120}$$

   For $\frac{3}{8}$, multiply by $\frac{15}{15}$ since $8 \times 15 = 120$:

   $$\frac{3}{8} \times \frac{15}{15} = \frac{45}{120}$$

Then $\frac{2}{15} + \frac{3}{8} = \frac{16}{120} + \frac{45}{120} = \frac{61}{120}$

Example 2

Find the least common denominator of $\frac{1}{12}$ and $\frac{3}{8}$, then add the two fractions.

1. Factor the denominator of each fraction into prime numbers

   - The denominators are 12 and 8.

   - 12 factors into $2 \cdot 2 \cdot 3$

   - 8 factors into $2 \cdot 2 \cdot 2$

2. Count the number of times each prime number occurs

   - For the first fraction, 2 occurs twice, 3 occurs once

   - For the second fraction, 2 occurs three times.

3. For each prime number, choose the largest count then multiply to get the least common denominator.

   - The largest count of 2 is three

   - The largest count of 3 is one

   - The least common denominator is $2 \cdot 2 \cdot 2 \cdot 3 = 24$

4. For $\frac{1}{12}$, multiply by $\frac{2}{2}$ since $12 \times 2 = 24$:

   $$\frac{1}{12} \times \frac{2}{2} = \frac{2}{24}$$

   For $\frac{3}{8}$, multiply by $\frac{3}{3}$ since $8 \times 3 = 24$:

   $$\frac{3}{8} \times \frac{3}{3} = \frac{9}{24}$$

Then $\frac{1}{12} + \frac{3}{8} = \frac{2}{24} + \frac{9}{24} = \frac{11}{24}$

Multiplying Fractions

To multiply fractions, the numerators and the denominators must be multiplied separately. After completing the multiplication, the fraction must be simplified to lowest terms.

Lowest Terms

To simplify a fraction to lowest terms, factor both the numerator and denominator to their prime factors then look for portions of the fraction that can be grouped to equal 1. The remaining fraction is in lowest terms.

Example

Simplify $\frac{20}{75}$ to lowest terms.

$$\frac{20}{75} = \frac{2\cdot 2\cdot 5}{3\cdot 5\cdot 5} = \frac{2\cdot 2}{3\cdot 5}\cdot\frac{5}{5} = \frac{2\cdot 2}{3\cdot 5}\cdot 1 = \frac{4}{15}$$

Multiplication Examples

$$\frac{2}{3}\times\frac{1}{4} = \frac{2}{12} = \frac{1}{6} \qquad \frac{3}{5}\times\frac{1}{9} = \frac{3}{45} = \frac{1}{15} \qquad \frac{4}{9}\times\frac{3}{4} = \frac{12}{36} = \frac{1}{3}$$

Another way to solve this problem is to cancel factors:

$$\frac{\cancel{2}}{3}\times\frac{1}{\cancel{2}\cdot 2} = \frac{1}{6} \qquad \frac{\cancel{3}}{5}\times\frac{1}{\cancel{3}\cdot 3} = \frac{1}{15} \qquad \frac{\cancel{2}\cdot\cancel{2}}{\cancel{3}\cdot 3}\times\frac{\cancel{3}}{\cancel{2}\cdot\cancel{2}} = \frac{1}{3}$$

Mixed Fractions

If the fractions are mixed with whole numbers like $2\frac{1}{2}$, convert the number to be a fraction only before multiplying. So for $2\frac{1}{2}$, convert this to $\frac{5}{2}$.

Examples

$$2\frac{1}{2}\times 1\frac{1}{4} = \frac{5}{2}\times\frac{5}{4} = \frac{25}{8} = 3\frac{1}{8} \qquad 1\frac{1}{3}\times\frac{1}{4} = \frac{4}{3}\times\frac{1}{4} = \frac{4}{12} = \frac{1}{3}$$

Dividing Fractions

Dividing is the inverse operation to multiplication. That means that dividing by a number is the same as multiplying by the inverse of that number. The inverse of a number is what you multiply that number by to get 1.

Example

Find the inverse of 6.

Let x be the inverse of 6. Then $6x = 1 \rightarrow x = \frac{1}{6}$

We can use the inverse for division such as $5 \div 6$ which is the same as $5 \times \frac{1}{6}$.

Dividing by a fraction is the same as multiplying by the inverse of that fraction. We find the inverse of a fraction in the same way as before.

Example

Find the inverse of the fraction $\frac{3}{4}$

Let x be the inverse of $\frac{3}{4}$

Then $\frac{3}{4}x = 1 \rightarrow x = \frac{4}{3}$

You can see that the inverse of a fraction is just switching the numerator and denominator. So now all we have to do when dividing by a fraction is find the inverse of that fraction and then multiply.

Examples

$\frac{1}{8} \div \frac{2}{3} = \frac{1}{8} \times \frac{3}{2} = \frac{3}{16}$ $\frac{2}{5} \div \frac{4}{5} = \frac{2}{5} \times \frac{5}{4} = \frac{10}{20} = \frac{1}{2}$ $\frac{3}{7} \div \frac{4}{5} = \frac{3}{7} \times \frac{5}{4} = \frac{15}{28}$

MCAS Problems

2006

http://www.doe.mass.edu/mcas/2006/release/g6math.pdf

2006 Grade 6 #15

15 Henry had a piece of rope that was $23\frac{1}{2}$ inches long. Henry cut the rope into two pieces so that one piece was $8\frac{1}{4}$ inches long. What was the length of the other piece of rope?

A. $15\frac{1}{4}$ inches

B. $15\frac{1}{2}$ inches

C. $31\frac{1}{3}$ inches

D. $31\frac{3}{4}$ inches

$$23\frac{1}{2}-8\frac{1}{4}=23\frac{2}{4}-8\frac{1}{4}=15\frac{1}{4}$$

Answer: A. $15\frac{1}{4}$ inches

2006 Grade 6 #32

32 What is the value of the expression below when $\triangle = 6$?

$$2 + \frac{\triangle}{3}$$

A. 4

B. 5

C. 11

D. 20

$$2+\frac{6}{3}=2+2=4$$

Answer: A. 4

http://www.doe.mass.edu/mcas/2006/release/g8math.pdf

2006 Grade 8 #3

3 What is the value of the expression below when $x = 10$?

$$\frac{4x + 5}{8x + 5}$$

A. $\frac{1}{2}$

B. $\frac{9}{17}$

C. $\frac{9}{13}$

D. $\frac{3}{2}$

$$\frac{4x+5}{8x+5}=\frac{4(10)+5}{8(10)+5}=\frac{40+5}{80+5}=\frac{45}{85}=\frac{9\cdot 5}{17\cdot 5}=\frac{9}{17}$$

Answer: B. $\frac{9}{17}$

2006 Grade 8 #4

4 The Enescu family rented 4 movies at a video store. The length, in hours, of each of the movies is shown below.

$2\frac{1}{4}$	$1\frac{3}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$

What is the total length of the 4 movies?

A. $5\frac{8}{14}$ hours

B. $6\frac{1}{4}$ hours

C. $6\frac{4}{6}$ hours

D. $7\frac{1}{4}$ hours

$$
\begin{aligned}
2\frac{1}{4}+1\frac{3}{4}+1\frac{1}{2}+1\frac{3}{4} &= 2+1+1+1+\frac{1}{4}+\frac{3}{4}+\frac{1}{2}+\frac{3}{4} \\
&= 5+\frac{1}{4}+\frac{3}{4}+\frac{2}{4}+\frac{3}{4} \\
&= 5+\frac{9}{4} \\
&= 5+2\frac{1}{4} \\
&= 7\frac{1}{4}
\end{aligned}
$$

Answer: D. $7\frac{1}{4}$

2006 Grade 8 #5

5 The length of Ann's bedroom is $5\frac{3}{4}$ yards. What is the length, in feet, of her bedroom?

A. $11\frac{1}{2}$ feet

B. $15\frac{3}{4}$ feet

C. $17\frac{1}{4}$ feet

D. $23\frac{1}{4}$ feet

$$5\frac{3}{4}\text{yards} = 5\frac{3}{4}\,\cancel{\text{yards}} \cdot \frac{3\text{ feet}}{\cancel{\text{yard}}} = 15\frac{9}{4}\text{feet} = 17\frac{1}{4}\text{feet}$$

Perhaps an easier way to solve this problem without the compound fractions is to consider:

$$5\frac{3}{4}\text{yards} = \frac{23}{4}\,\cancel{\text{yards}} \cdot \frac{3\text{ feet}}{\cancel{\text{yard}}} = \frac{69}{4}\text{feet} = 17\frac{1}{4}\text{feet}$$

Answer: C. $17\frac{1}{4}$feet

http://www.doe.mass.edu/mcas/2006/release/g10math.pdf

2006 Grade 10 #2

2 What is the value of the expression below?

$$\frac{15(4+8)}{2(2+1)-1}$$

A. 17

B. 36

C. 45

D. 60

$$\frac{15(4+8)}{2(2+1)-1}=\frac{15(12)}{2(3)-1}=\frac{15(12)}{6-1}=\frac{15(12)}{5}=3(12)=36$$

Answer: B. 36

2004

http://www.doe.mass.edu/mcas/2004/release/g6math.pdf

2004 Grade 6 #30

30 Kathryn walked $1\frac{1}{2}$ miles per day for 30 days. What was the total number of miles she walked?

$$1\frac{1}{2}\frac{\text{miles}}{\cancel{\text{day}}}\cdot 30\ \cancel{\text{days}}=30\frac{30}{2}\text{miles}=45\text{ miles}$$

2004 Grade 6 #38

38 Derek is making hot fudge sauce using the recipe shown below.

Hot Fudge Sauce

- 12 ounces of chocolate chips
- $\frac{3}{4}$ cup of heavy cream
- 1 tablespoon of butter

If Derek is going to double the recipe, how many cups of heavy cream will he need?

A. $2\frac{3}{4}$

B. $1\frac{1}{2}$

C. $1\frac{1}{4}$

D. $\frac{3}{8}$

$$2 \cdot \frac{3}{4} = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$$

Answer: B. $1\frac{1}{2}$

http://www.doe.mass.edu/mcas/2004/release/g8math.pdf

2004 Grade 8 #3

3 Which of the following is equivalent to the equation below?

$$\frac{n}{6} = 30$$

A. $n = 30 \cdot 6$

B. $6 = 30 \cdot n$

C. $n = \frac{30}{6}$

D. $6 = \frac{30}{n}$

$$\frac{n}{6} = 30$$

$$\cancel{6} \cdot \frac{n}{\cancel{6}} = 30 \cdot 6$$

$$n = 30 \cdot 6$$

Answer: A. $n = 30 \cdot 6$

2004 Grade 8 #8

8 Compute:

$$\frac{1}{3} \times 1\frac{1}{2}$$

$$\frac{1}{3} \times 1\frac{1}{2} = \frac{1}{3} \times \frac{3}{2} = \frac{3}{6} = \frac{1}{2}$$

Answer: $\frac{1}{2}$

2004 Grade 8 #31

31 A sugar cookie recipe calls for $3\frac{1}{2}$ cups of sugar to make 6 dozen cookies. Based on the recipe, how many cups of sugar must be used to make 20 dozen sugar cookies?

A. $9\frac{1}{2}$ cups

B. $11\frac{2}{3}$ cups

C. 61 cups

D. 70 cups

$$\frac{3\frac{1}{2}\text{ cups}}{6\text{ dozen}} = \frac{x\text{ cups}}{20\text{ dozen}}$$

$$20\ \cancel{\text{dozen}} \cdot \frac{\frac{7}{2}\text{ cups}}{6\ \cancel{\text{dozen}}} = x\text{ cups}$$

$$\frac{20 \cdot \frac{7}{2}\text{ cups}}{6} = x\text{ cups}$$

$$\frac{\frac{140}{2}\text{ cups}}{6} = x\text{ cups}$$

$$\frac{70\text{ cups}}{6} = x\text{ cups}$$

$$11\frac{4}{6}\text{ cups} = x\text{ cups}$$

$$11\frac{2}{3}\text{ cups} = x\text{ cups}$$

Answer: B. $11\frac{2}{3}$ cups

2004 Grade 8 #36

36 Mr. Lowery has $8\frac{3}{4}$ pounds of ground beef that he will use to make hamburgers for a picnic. What is the maximum number of quarter-pound hamburgers he can make?

A. 9

B. 17

C. 35

D. 36

$$8\frac{3}{4}\ \cancel{\text{pounds}} \times 4\frac{\text{hamburgers}}{\cancel{\text{pound}}} = 32\frac{12}{4}\text{hamburgers}$$

$$= 35 \text{ hamburgers}$$

Answer: C. 35 hamburgers

2002

http://www.doe.mass.edu/mcas/2002/release/g6math.pdf

2002 Grade 6 #24

Sylvia and Tomás are playing a game of chess. Each player began the game with 16 pieces. Now Sylvia has $\frac{1}{4}$ of her pieces remaining on the board and Tomás has $\frac{1}{8}$ of his pieces remaining on the board. How many total pieces remain on the board?

A. 4

✔ B. 6

C. 8

D. 12

$$\frac{1}{4}\cdot 16 \text{ pieces} = \frac{16}{4} \text{ pieces} = 4 \text{ pieces}$$
$$\frac{1}{8}\cdot 16 \text{ pieces} = \frac{16}{8} \text{ pieces} = 2 \text{ pieces}$$
$$4 \text{ pieces} + 2 \text{ pieces} = 6 \text{ pieces}$$

http://www.doe.mass.edu/mcas/2002/release/g8math.pdf

2002 Grade 8 #11

11 Huey is reading a book that is 697 pages long. He tells a friend that he is about $\frac{3}{4}$ of the way done. About how many more pages must Huey read before he finishes the book?

A. 150 pages

B. 160 pages

✔ C. 175 pages

D. 250 pages

Pages read so far:

$$\frac{3}{4} \cdot 697 \text{ pages} = \frac{2091}{4} \text{ pages} = 522\frac{3}{4} \text{ pages}$$

Pages left to read:

$$697 \text{ pages} - 522\frac{3}{4} \text{ pages} = 174\frac{1}{4} \text{ pages}$$

2002 Grade 8 #36

36 Which is equivalent to $p \div \frac{1}{10}$?

✔ A. $10p$

B. $0.1p$

C. $\frac{p}{10}$

D. $0.01p$

$$p \div \frac{1}{10} = p \cdot \frac{10}{1} = 10p$$

Ratios

Ratios are a comparison of numbers or amounts of things, like the ratio of boys to girls in a classroom. This is a comparison of part to part. Ratios can also compare numbers or amounts of things to the larger group that these things belong, like the ratio of boys to total number of students in a classroom. This is a comparison of part to whole.

Comparison of Part to Part

In the case of comparing part to part, we are comparing part of the whole to another part of the whole. Boys are one part of the whole classroom and girls are the other part of the whole classroom. Ratios of this type are written with a ":" between the two numbers. The numbers are simplified to lowest terms so the ratio 4:2 would be simplified to 2:1. This is read as "2 to 1".

Example 1

> A math class has 20 boys and 30 girls. The ratio of boys to girls is 20 to 30. This ratio is written 2:3.

Example 2

> The lunch room has 25 tables and 125 chairs. The ratio of tables to chairs is 25 to 125. This ratio is written 1:5.

More than two sets of things can be compared in part to part ratios. A colon (:) is placed between all of the numbers being compared. The numbers are simplified to lowest terms just like the comparison of two numbers. A ratio like 5:10:15 would be simplified to 1:2:3 and a ratio like 3:4:5:2 is already in lowest terms. A recipe is an example of a ratio that has multiple parts.

Example 3

> A recipe calls for 2 cups of flour, ½ cup of sugar and 3 cups of hot sauce. This ratio is written as 2:½: 3 or 4:1:6. In this case we have all numbers as whole numbers and multiplied by 2 to convert ½ to a whole number.

Comparison of Part to Whole

This type of ratio compares a part of the whole to the whole, like the ratio of boys to the total number of students in the classroom mentioned earlier. This type of ratio is the same as a fraction.

Example 4

There are 20 boys and 30 girls in a classroom. The total number of students is 50. The ratio of boys to the total number of students is 20:50 or 2:5. Since this is a comparison of part to whole, this can be written as $\frac{2}{5}$.

In this example we had to add the number of boys to the number of girls to get the total number of students. The number of boys is part of the whole and the total number of students is the whole.

Example 5

A bag of red, blue and yellow marbles has ratio of 2:3:1. What fraction are the blue marbles of the total number of marbles?

$$\frac{3}{2+3+1}=\frac{3}{6}=\frac{1}{2}$$

Equations from Ratios

Ratios can be written as an equation when solving certain types of problems. We learned earlier to write ratios in lowest terms. A ratio like 3:6:9 is written in lowest terms as 1:2:3. The ratio 5:10:15 is also written in lowest terms as 1:2:3. So the ratios 3:6:9, 5:10:15 and 1:2:3 are equivalent to each other. We can think of these ratios as $1x:2x:3x$ where x is equal to 1 in the lowest terms form and equal to 3 in the first ratio and equal to 5 in the second ratio.

Example 6

A bag of red, blue and yellow marbles has ratio of 2:3:1. There are a total of 18 marbles in the bag. How many red marbles are there?

We can write the ratio 2:3:1 as $2x:3x:1x$. The total number of marbles can be written as the equation $2x+3x+1x=18$.

Solve for *x*:

$2x+3x+1x=18$

$6x=18$

$x=3$

So there are 2*x* or 6 red marbles in the bag.

Example 7

A school has 500 students where the ratio of boys to girls is 2:3. How many girls are in this school?

Write the ratio 2:3 as $2x:3x$. The total number of students can be written as the equation $2x+3x=500$.

Solve for *x*:

$2x+3x=500$

$5x=500$

$x=100$

The number of girls in the school is 3*x* or 300.

We can get another kind of equation from ratios that compare two things. If we know the ratio of two things and we know the number of one of them, we can find the number of the other. This type of ratio equation is called a proportion. A proportion is an equation with two ratios. In the equation, the two ratios are equal to each other. For example, $\frac{1}{2}=\frac{3}{6}$. If we are solving a proportion with an unknown, the equation would look like $\frac{1}{2}=\frac{x}{6}$. We can find that $x=3$.

Example 8

The ratio of cats to dogs in the neighborhood is 2:3. If there are 6 cats in the neighborhood, how many dogs are there?

Since the number of dogs is unknown, we call this *x*.

We can write a second ratio as $6:x$.

Since these two ratios must be equal, we can write the proportion equation:

$$\frac{2}{3}=\frac{6}{x}$$

Now we just solve for *x*: $\frac{2}{3}=\frac{6}{x}\Rightarrow x=6\cdot\frac{3}{2}\Rightarrow x=9$

The number of dogs in the neighborhood is 9.

Example 9

The ratio of ducks to geese on a pond is 3:7. If there are 21 geese on the pond, how many ducks are there?

Since the number of ducks is unknown, we call this *x*.

We can write a second ratio as *x*: 21.

Since these two ratios must be equal, we can write the proportion equation:

$$\frac{3}{7}=\frac{x}{21}$$

Now we just solve for *x*: $\frac{3}{7}=\frac{x}{21}\Rightarrow x=9$

The number of ducks on the pond is 9.

MCAS Problems

2006

http://www.doe.mass.edu/mcas/2006/release/g8math.pdf

2006 Grade 8 #27

27 Mona counted a total of 56 ducks on the pond in Town Park. The ratio of female ducks to male ducks that Mona counted was 5:3. What was the total number of female ducks Mona counted on the pond?

A. 15

B. 19

C. 21

D. 35

$5x + 3x = 56$

$8x = 56$

$x = 7$

Female ducks: $5x = 5(7) = 35$

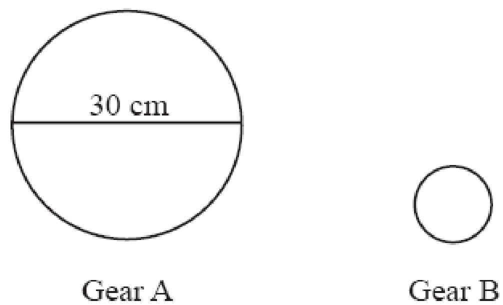
Answer: D. 35

2004

http://www.doe.mass.edu/mcas/2004/release/g8math.pdf

2004 Grade 8 #2

2 The circles below represent the gears of a bicycle. The diameter of Gear A is 30 centimeters. The ratio of the diameter of Gear A to the diameter of Gear B is 3:1.



What is the circumference, in centimeters, of Gear B?

A. 5π cm

B. 10π cm

C. 15π cm

D. 30π cm

Let *x* be the diameter of Gear B.

$$\frac{x}{30\text{ cm}} = \frac{1}{3}$$
$$x = \frac{30\text{ cm}}{3}$$
$$x = 10\text{ cm}$$

The circumference of Gear B is $\pi \cdot \text{diameter} = \pi \cdot 10$ cm

Answer: B.

2004 Grade 8 #7

7 Acme Doll Company makes the Super Hero Doll that sells for $10. The table below shows that the profit the company earns is based proportionally on the number of dolls sold.

Super Hero Doll Profits

Sales (number of dolls)	Profits
1,000	$50
1,500	$75
3,000	$150
4,500	$225

What is the profit for sales of 15,000 Super Hero Dolls?

Since the number of dolls sold is 15,000, we use the ratio $\frac{\$225}{4,500}$.

We can now form the proportion equation:

$$\frac{x}{15,000} = \frac{\$225}{4,500}$$
$$x = \frac{\$225 \cdot 15,000}{4,500}$$
$$x = 750$$

The profit for 15,000 dolls is $750

http://www.doe.mass.edu/mcas/2004/release/g10math.pdf

2004 Grade 10 #34

34 The force needed to stretch a spring is directly proportional to the amount the spring is to be stretched. If a force of 100 pounds stretches a certain spring 8 inches, how much force is needed to stretch the spring 12 inches?

A. 25 pounds

B. 50 pounds

C. 100 pounds

D. 150 pounds

We have the ratio $\frac{100 \text{ pounds}}{8 \text{ inches}}$.

We can form the proportion $\frac{100 \text{ pounds}}{8 \text{ inches}} = \frac{x \text{ pounds}}{12 \text{ inches}}$.

Solving the proportion we find that $x = 150 \text{ pounds}$.

Answer: D.

2002

http://www.doe.mass.edu/mcas/2002/release/g6math.pdf

2002 Grade 6 #12

12 The approximate costs of running an automobile in 1994 are shown in the chart below.

Automobile Costs in 1994

Item	Amount
Gas and Oil	\$750
Other	\$2,250
Total Cost	\$3,000

Correct Answer:

$\frac{1}{4}$

What fraction would represent the ratio of the cost of gas and oil to the total cost of running a car in 1994? Write your fraction in simplest form.

$$\frac{\text{Gas and Oil}}{\text{Total Cost}} = \frac{\$750}{\$3,000} = \frac{1}{4}$$

2000

http://www.doe.mass.edu/mcas/2000/release/g10math.pdf

2000 Grade 10 #26

26. Four hundred deer were captured in Milltown Forest, tagged, and released back into the forest. Several weeks later, a forest ranger captured a number of deer at a random location in Milltown Forest, recorded the number of tagged and nontagged deer, and released the deer back into the forest. She did this over two trials as shown below.

Record of Deer Captured in Milltown Forest

	Total of Deer	Tagged	Nontagged
Trial 1	65	10	55
Trial 2	75	15	60

Approximately how many deer could you expect to find in the entire forest?

A. 2,600

B. 1,600

✓ C. 2,300

D. 1,000

Answer: The correct answer is C. In each trial, the number of tagged deer is proportional to the total number of deer. For trial 1, we get: $\frac{10}{65} = \frac{400}{x_1}$ where $x_1 = 2600$ is an estimate of the total number of deer in the forest. For trial 2, $\frac{15}{75} = \frac{400}{x_2}$ and $x_2 = 2000$ is an estimate of the total number of deer in the forest.

Averaging estimates x_1 and x_2, $\bar{x} = \frac{x_1 + x_2}{2} = \frac{2600 + 2000}{2} = 2300$ which represents a better estimate for the number of deer we could expect to find in the forest.

Decimal Notation

Terminating Decimals

Any terminating decimal can be expressed as a rational number $\frac{p}{q}$ where *p* and *q* are both integers, $q \neq 0$. A rational number is a fraction.

Example 1

0.55=$\frac{55}{100}$ or simplified it can be expressed as $\frac{\cancel{5}*11}{\cancel{5}*20}=\frac{11}{20}$.

Example 2

0.625=$\frac{625}{1000}=\frac{\cancel{5}*125}{\cancel{5}*200}=\frac{\cancel{5}*25}{\cancel{5}*40}=\frac{\cancel{5}*5}{\cancel{5}*8}=\frac{5}{8}$. This process of canceling took a bit of time.

A simpler way is $\frac{625}{1000}=\frac{\cancel{125}*5}{\cancel{125}*8}=\frac{5}{8}$ but this means that you have to recognize that 125 is a factor of both the numerator and denominator. Use the approach that you find easier to deal with.

A *terminating decimal* can also be expressed as a sum of fractions. For example, $0.37=\frac{3}{10}+\frac{7}{100}$. We already know from Example 1 that we can express 0.37 as $\frac{37}{100}$. The question now is how to combine $\frac{3}{10}+\frac{7}{100}$ to get the result $\frac{37}{100}$. The least common denominator (LCD) as discussed on page 7 is a way to combine fractions with unlike denominators. The LCD is the least common multiple of the denominators. For the numbers 10 and 100, the LCD is 100, and we get $\frac{3}{10}+\frac{7}{100}=\frac{3*10}{10*10}+\frac{7}{100}=\frac{37}{100}$.

Example 3

Express the terminating decimal 0.573 as a sum of fractions and then combine the fractions. First of all, 0.573=$\frac{573}{1000}$. As a sum of fractions, 0.573=$\frac{5}{10}+\frac{7}{100}+\frac{3}{1000}$. In order to combine the fractions, we need the LCD which is 1000. Then adding the fractions, we find that $\frac{5}{10}+\frac{7}{100}+\frac{3}{1000}=\frac{5*100}{10*100}+\frac{7*10}{100*10}+\frac{3}{1000}=\frac{573}{1000}$.

The TI-84 calculator has as an option to enable conversion of a terminating decimal to a fraction. In order to convert 2.3 to a fraction on the TI-84, type in 2.3 on the home screen, the press MATH and option 1 ($\triangleright Frac$) followed by ENTER and the result on the screen will appear as $\frac{23}{10}$. This feature can be used to help check your future work when manipulating decimals and fractions.

The procedures to add and subtract like fractions and unlike fractions were presented in the Fraction Arithmetic section (page 7). On pages 9-10 of that section, multiplying and dividing fractions were discussed.

Repeating Decimals

Any repeating decimal can be expressed as a rational number $\frac{p}{q}$ where *p* and *q* are both integers, $q \neq 0$.

Example 4A

Let's find the rational number for the repeating decimal 0.333…

Let x = .333…, then 10x = 3.333… and this implies that 10x – x = 9x = 3. Dividing, we find that x = $\frac{3}{9}=\frac{1}{3}$, or .333… = $\frac{1}{3}$.

Geometric Series

Revisiting the previous problem, we will show that 0.333… is equal to $\frac{1}{3}$, but now we will use geometric series. A geometric series takes the form $a + ar + ar^2 + ar^3 + \cdots$ where *r* is called the ratio. The geometric series converges, i.e. approaches a finite value, when the absolute value of the ratio is less than 1, i.e. $|r| < 1$. The sum of a convergent geometric series is $\frac{a}{1-r}$. For those of wondering where this result came from, just divide *a* by 1-*r* and the result after the long division is $a + ar + ar^2 + ar^3 + \cdots$.

Example 4B

0.333… = $\frac{3}{10}\left(1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right)$. The ratio r = $\frac{1}{10}$ and a = $\frac{3}{10}$. Since the sum is $\frac{a}{1-r}$, we see that 0.333… = $\frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{9} = \frac{1}{3}$. After doing a few of these geometric series problems, they get easy. The goal is to factor out the common term *a* and then all you are left with is $1 + r + r^2 + r^3 + \cdots$ in parentheses.

Example 5

Show that 0.999… is equal to 1 using geometric series. In this case 0.999…=$\frac{9}{10}\left(1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right)$. The ratio r = $\frac{1}{10}$ and a = $\frac{9}{10}$. Then $\frac{a}{1-r}$ becomes $\frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$. Try to show that the repeating decimal 0.999… = 1 using the same approach as used in Example 4A.

Example 6

Find a rational number for the repeating decimal 1.231231…

As done in Example 4A, let's just look at the portion 0.231231… and let x = .231231…, so 1000x = 231.231231… and this implies that 1000x – x = 999x = 231. Solving, we get $x = \frac{231}{999} = \frac{77}{333}$. Then using a bit of algebra, 1.231231… is equal to 1 + $\frac{77}{333}$ = $\frac{1*333+77}{333} = \frac{410}{333}$. It's easier to do Example 6 this way than as a geometric series.

Do you think we can express π as a rational number? The answer is no since π is not a repeating decimal. People at times use the rational number $\frac{22}{7}$ as an estimate for π, but $\frac{22}{7}$ is only an approximation for π and not its exact value. To the 8th decimal place, π =3.14159265, and none of the decimal places will repeat. Dividing 22 by 7, we find that to the 8th place, $\frac{22}{7} = 3.14285714$. Its approximation for π is accurate to only the second decimal place and if we check carefully, the decimal portion of $\frac{22}{7}$ begins to repeat after the 6th decimal place.

Fractions to Decimals

We can use a calculator to convert a rational number to a decimal, but let's examine the mathematics.

Example 7

Write $21\frac{2}{5}$ as a decimal. Just working on the fractional part, $\frac{2}{5} \rightarrow 5\overset{\displaystyle 0.4}{\overline{)\ 2}}$, so $21\frac{2}{5}$ is equal to 21 + 0.4 or 21.4.

Example 8

Write $2\frac{1}{3}$ as a decimal. We already know from Example 4A that $\frac{1}{3}$ is the repeating decimal 0.333…, but what if we didn't know that. We would then do a long division as in the previous example and find that $\frac{1}{3} \to 3\overline{)\ \ 1}^{\,0.333}$ and the 3's in the quotient continue to repeat. We sometimes write 0.333… as $0.\overline{3}$ meaning that we have a repeating decimal. Anyway, 2 + $\frac{1}{3}$ is equal to 2 + 0.333… or 2.333… or $2.\overline{3}$. Using a calculator, we can use the mode and float options to round a decimal to a certain number of places. If we selected the option on a TI-84 calculator to round $2.\overline{3}$ to two decimal places, we would get a result of 2.33.

Examples

The following examples can be used to test the concepts presented.

1. Write 0.753 as a sum of fractions and then combine the fractions.

Answer: $0.753 = \frac{7}{10} + \frac{5}{100} + \frac{3}{1000} = \frac{7*100}{10*100} + \frac{5*10}{100*10} + \frac{3}{1000} = \frac{753}{1000}$.

2. Express the repeating decimal 4.232323… as a fraction.

Answer: Just work on the decimal portion .232323… and let x=.232323…, so 100x=23.232323… and subtracting we find that 99x=23. Solving for x we find that $x = \frac{23}{99}$ and 4.232323… is equal to 4+$\frac{23}{99}$. The LCD is 99, so combining the rational numbers, $\frac{4}{1} + \frac{23}{99} = \frac{4*99}{1*99} + \frac{23}{99} = \frac{419}{99}$.

3. Write $14\frac{1}{7}$ as a decimal.

Answer: $\frac{1}{7} \rightarrow 7\overline{)\,1\,}^{\,0.142857}$ and then repeats, i.e. $\frac{1}{7}$ =.142857142857… so there are 6 digits after the decimal point before the repetition begins. $14\frac{1}{7}$ is equal to 14+0.142857142857… or 14.142857142857… . which can be written as $14.\overline{142857}$.

4. Using geometric series, show that the repeating decimal 0.222… is equal to $\frac{2}{9}$.

Answer: $0.222\ldots=\frac{2}{10}\left(1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right)$. The ratio $r=\frac{1}{10}$ and $a=\frac{2}{10}$.

Then $\frac{a}{1-r}=\frac{\frac{2}{10}}{1-\frac{1}{10}}=\frac{\frac{2}{10}}{\frac{9}{10}}=\frac{2}{9}$.

Adding and Subtracting Decimals

Adding and subtracting decimals are done much like the addition and subtraction of whole numbers with the exception that the decimal point must be retained. Rewrite the quantities as necessary to enable decimal point and integer alignment.

Example 9A

Add the quantities 31 and 56.7. For this example, rewrite the addition problem as 31.0 + 56.7 and then add resulting in a sum of 87.7.

Example 9B

Perform the subtraction 98 – 24.3. Rewrite the problem as 98.0 – 24.3 yielding a result of 73.7.

Example 9C

Add the quantities 31.23 and 56.7. For this example, rewrite the addition problem as 31.23 + 56.70 and then add resulting in a sum of 87.93.

Multiplying Decimals

Multiply decimals just as you would multiply whole numbers, but the number of decimal places must be counted and adjusted for in the final result or product.

Example 10A

Perform the multiplication 7.3 x 8.2. The answer is 59.86 but let's show why we have two places after the decimal point. If we were to only consider the whole numbers and multiply 73 x 82, the product is 5986. However, in our problem, we actually have $(7+0.3)*(8+0.2)$. Expanding the factors, we get 56 + 7*0.2 + 8*0.3 + 0.3*0.2. We could go back to fraction representation as in Example 3 and write 7*0.2 as $7*\frac{2}{10}=\frac{14}{10}=1.4$. Likewise, 8*0.3 can be written as $8*\frac{3}{10}=\frac{24}{10}=2.4$ and 0.3*0.2 can be written as $\frac{3}{10}*\frac{2}{10}=\frac{6}{100}=.06$. Now add the individual results to get 56 + 1.4 + 2.4 + .06 or 59.86. The digits in our answer are of course the same as the original product 5986, and the two decimal places are the result of one decimal place for 7.3 and one for 8.2.

Example 10B

Perform the multiplication 2.1 x 4.3. Without the detailed explanation as in the previous example, 21 x 43 is 903 and we have two decimal places to consider so the final answer is 9.03. We can quickly approximate the final answer which helps us to align the decimal point. Since 2.1 x 4.3 is actually (2+0.1)*(4+0.3) resulting in 8 plus some other "stuff", the final result when multiplying 2.1 x 4.3 should be fairly close to 8. Adding the other "stuff", the final result is 9.03.

Dividing Decimals

When dividing a decimal by a decimal, the division is actually involving fractions. If the denominator (divisor) is a decimal, manipulate the division problem so that the denominator becomes a whole number. We will then be able to proceed with the division as done for whole numbers. The rules following Example 11 will explain the proper placement of the decimal point in the result.

Example 11

Compute $\frac{.23}{1.2}$. We want to make only the denominator a whole number. The initial problem takes the form $\frac{.23}{1.2}*\frac{10}{10}=\frac{2.3}{12}$. Following the approach in Example 7, $\frac{2.3}{12} \rightarrow 12\overline{\smash{)}\quad 2.3}^{\,0.19166}$ and the result or quotient is equal to 0.192 rounded to three decimal places.

Making the denominator a whole number results in the following rules to remember when dividing decimals:

- Make the denominator (divisor) a whole number;
- Count the number of places after the decimal point in the divisor and move the decimal point in the numerator (dividend) the appropriate number of places to the right required to make the divisor a whole number.

Example 12

Perform the division $37 \div 0.23$. Following the rules which are a result of making the denominator a whole number, we get $\frac{3700}{23} \rightarrow 23\overline{\smash{)}\ 3700}^{\,160.87}$ rounded to two decimal places.

References

Calculus, from Graphical, Numerical, and Symbolic Points of View, Ostebee and Zorn, 2nd edition, Houghton Mifflin, 2002.

Pre-Algebra, An Integrated Transition to Algebra and Geometry, Glencoe/McGraw-Hill, 2001.

Some of the MCAS problems listed in the following material relate to the specific topics discussed in the previous decimal notation sections while others serve as enrichment.

MCAS Problems

2006

http://www.doe.mass.edu/mcas/2006/release/g6math.pdf

2006 Grade 6 #19

19 Which of the following is equivalent to 6.25?

A. $6\frac{1}{5}$

B. $6\frac{1}{4}$

C. $6\frac{2}{5}$

D. $6\frac{3}{4}$

Answer: B

2006 Grade 6 #25

25 Which of the following shows 0.56 written in expanded notation?

A. $(5 \times 10) + (6 \times 100)$

B. $(5 \times 100) + (6 \times 1000)$

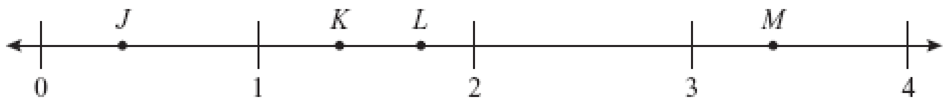
C. $(5 \times 0.1) + (6 \times 0.01)$

D. $(5 \times 0.01) + (6 \times 0.001)$

Answer: C

2006 Grade 6 #36

36 Which point on the number line shown below appears to be located at $1\frac{3}{8}$?



A. *J*
B. *K*
C. *L*
D. *M*

Answer: Since point K lies between 1 and 2, and is closer to 1 than it is to 2, the correct answer is B.

http://www.doe.mass.edu/mcas/2006/release/g8math.pdf

2006 Grade 8 #15

15 Sandra bought $2\frac{5}{8}$ pounds of apples.
Which of the following shows $2\frac{5}{8}$ written in decimal notation?

A. 2.375
B. 2.580
C. 2.625
D. 2.875

Answer: C.

2006 Grade 8 #32

32 Which of the following is represented by the expression below?

$$5x + 2$$

A. two more than $\frac{1}{5}$ of a number

B. two more than five times a number

C. five more than $\frac{1}{2}$ of a number

D. five more than twice a number

Answer: B

http://www.doe.mass.edu/mcas/2006/release/g10math.pdf

2006 Grade 10 #4

4 Which of the following expressions has the **greatest** value?

A. $(6 + 6) \cdot 2 \div 3 - 1$

B. $6 + 6 \cdot 2 \div 3 - 1$

C. $6 + 6 \cdot 2 \div (3 - 1)$

D. $6 + 6 \cdot (2 \div 3 - 1)$

Answer: The values of the expressions are: A=7, B= 9, C=12, D=4, so C has the greatest value.

2006 Grade 10 #7

7 Laura correctly used a property of real numbers to calculate the exact value of the product shown below.

$$(840)(998)$$

Which of the following demonstrates a property that Laura could have used?

A. (838)(1000)

B. (838)(1000) + 2

C. (840)(1000) − (840)(2)

D. (840)(1000) − (2)(1000)

Answer: C

2006 Grade 10 #15

15 What is the value of the expression below?

$$(8 - 4)^2 + 8 \div 4$$

Answer: $4^2 + \frac{8}{4} = 16 + 2 = 18$

2005

http://www.doe.mass.edu/mcas/2005/release/g6math.pdf

2005 Grade 6 #1

1 Bonnie bought a 13-pound turkey for \$0.85 per pound. How much money did she pay for the turkey?

A. \$11.05

B. \$13.85

C. \$33.00

D. \$110.05

Answer: 13*0.85=\$11.05, so A is the correct answer.

2005 Grade 6 #5

5 Which of the following is closest to the product 298.7×10.1?

A. 300

B. 2,000

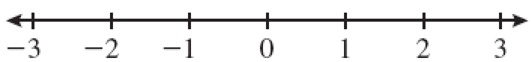
C. 3,000

D. 20,000

Answer: 298.7 x 10.1 is approximately 300 x 10 = 3,000 and C is the correct answer.

2005 Grade 6 #20

20 Marta is plotting points on the number line below.



Between which two numbers should Marta plot $-2\frac{1}{2}$?

A. 1 and 2

B. 2 and 3

C. −2 and −1

D. −3 and −2

Answer: D

2005 Grade 6 #23

23 Which of the following shows the numbers in order from **least** to **greatest**?

A. 0.765, 0.82, 0.791

B. 0.765, 0.791, 0.82

C. 0.791, 0.82, 0.765

D. 0.791, 0.765, 0.82

Answer: B

http://www.doe.mass.edu/mcas/2005/release/g8math.pdf

2005 Grade 8 #6

6 The first number in a pattern is 50. To go from one number in the pattern to the next number, the rule is to **divide by 5**. What is the fourth number in the pattern?

A. $\frac{1}{5}$

B. $\frac{2}{5}$

C. $\frac{3}{2}$

D. $\frac{5}{2}$

Answer: The numbers in the pattern are 50, 10, 2, 2/5 and B is the fourth number.

2005 Grade 8 #36

36 The chart below shows the cost of the four different-sized boxes of chicken nuggets that are available at The Chicken Shack.

Box Sizes and Costs of Chicken Nuggets

Box Size	Number of Nuggets in Box	Cost of Box (dollars)
Kid	5	1.29
Small	9	2.09
Medium	15	3.19
Large	22	4.99

Which of the following box sizes has the **least** cost per nugget?

A. Kid

B. Small

C. Medium

D. Large

Answer: The costs per nugget rounded to 3 decimal places are: Kid Size – 0.258, Small – 0.232, Medium – 0.213, Large – 0.227. C is the best buy and has the least cost per nugget.

http://www.doe.mass.edu/mcas/2005/release/g10math.pdf

2005 Grade 10 #6

6 The Golden Ratio is defined by the expression shown below.

$$\frac{1+\sqrt{5}}{2}$$

Which of the following is closest to the value of the ratio?

A. 1.1

B. 1.6

C. 2.1

D. 2.9

Answer: $\frac{1+\sqrt{5}}{2} \approx \frac{1+2.236}{2} = 1.618$, so B has the closest value.

2005 Grade 10 #11

11 At a fish market, Mr. Estes bought several pounds of cod that was on sale for $3.59 per pound. The total cost of the cod that he bought was $28.63.

Which of the following is closest to the amount of cod that Mr. Estes bought?

A. 6 pounds

B. 7 pounds

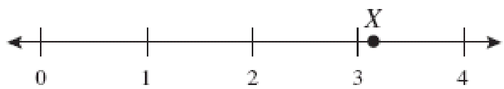
C. 8 pounds

D. 9 pounds

Answer: $\frac{28.63}{3.59} = 7.975$ rounded to 3 places, so C is the correct answer. Without a calculator, we could estimate the total costs as 6*3.60, 7*3.60, 8*3.60 and 9*3.60. In that case, 8*3.60=$28.80 which is closest to the amount paid.

2005 Grade 10 #12

12 Point X is graphed on the number line as shown below.



Which of the following numbers is closest to the location of point X?

A. $\sqrt{6}$

B. $\sqrt{8}$

C. $\sqrt{11}$

D. $\sqrt{13}$

Answer: X must be larger than 3. $\sqrt{8} \approx 2.83$ and $\sqrt{11} \approx 3.32$ so C is the correct answer.

2005 Grade 10 #31

31 A designer at Royal Jewelers wants to create a 10-ounce necklace that will be made of gold and silver. The necklace will have a total value of \$206.50.

a. Write an equation that represents the total weight of the 10-ounce necklace if it contains g ounces of gold and s ounces of silver.

b. Given that the value of gold is \$318 per ounce and the value of silver is \$5 per ounce, write an equation in terms of g and s that represents the total value of the 10-ounce necklace.

c. The two equations from parts a. and b. form a system. Solve the system of equations for g and s. Show all of your work.

d. What will be the value, in dollars, of the gold in the 10-ounce necklace? Show or explain how you got your answer.

Answer: a) 10=g + s b) 206.50= 318g + 5s c) Solve for g in part a) and substitute g=10 – s into part b) resulting in s=2973.5/313=9.5 ounces of silver. Then g=0.5 ounces of gold. d) Value of gold is 318 * 0.5=318/2=\$159.00.

2005 Grade 10 #33

33 Deborah decided to mow lawns to earn the \$280 she needs for a school orchestra trip. If she earns \$18 per lawn, what is the **minimum** number of lawns she needs to mow to earn the money for the trip?

A. 15

B. 16

C. 18

D. 20

Answer: 280/18 is approximately 15.56 so 16 lawns have to be mowed and the correct answer is B.

2005 Grade 10 #41

41 In a report on the history of irrational numbers, Celine compared three different values that have been used to approximate π. The values are listed below.

$\left(\frac{4}{3}\right)^4$ Egyptian approximation

$\frac{355}{113}$ Chinese approximation

$\frac{22}{7}$ Archimedes' approximation (Greek)

a. Celine compared $\left(\frac{4}{3}\right)^4$, the approximation used by the Egyptians, to $\frac{22}{7}$, a value that she often uses for π. She converted both $\left(\frac{4}{3}\right)^4$ and $\frac{22}{7}$ to decimals rounded to four decimal places (nearest ten-thousandth). To the nearest ten-thousandth, what is the absolute value of the difference between $\left(\frac{4}{3}\right)^4$ and $\frac{22}{7}$? Show or explain how you got your answer.

b. Celine also compared $\frac{355}{113}$, the approximation used by the Chinese, to $\frac{22}{7}$. She converted $\frac{355}{113}$ to a decimal rounded to four decimal places (nearest ten-thousandth). To the nearest ten-thousandth, what is the absolute value of the difference between $\frac{355}{113}$ and $\frac{22}{7}$? Show or explain how you got your answer.

c. Celine knows that $\pi \approx 3.1415927$. Place the four numbers, $\left(\frac{4}{3}\right)^4$, $\frac{355}{113}$, $\frac{22}{7}$, and π in order from **least** to **greatest**. Explain your reasoning.

Answer: a) To the nearest ten-thousandth, $\left(\frac{4}{3}\right)^4 = 3.1605$ and $\frac{22}{7} = 3.1429$. The absolute value of the difference is 0.0176.

b) $\frac{355}{113} = 3.1416$ and the absolute value of the difference with $\frac{22}{7} = 3.1429$ is 0.0013.

c) Since the approximation used by the Chinese to 7 decimal places is 3.1415929,

$$\pi < \frac{355}{113} < \frac{22}{7} < \left(\frac{4}{3}\right)^4.$$

Percents and Fractions

Conversions between Fractions and Percent

Just as quantities can be converted from decimals to fractions and from fractions to decimals, fractions can also be converted back and forth to percents. A percent is a ratio that compares a number to 100 and its symbol is %.

Example 1

Assume I got 42 of 50 multiple choice items on an exam correct. To find the percent correct, set up the ratio $\frac{42}{50}=\frac{x}{100}$, x=84 and I got 84% correct.

Example 2

Express the fraction $\frac{2}{5}$ as a percent.

a) One way to solve the problem is to represent the ratio $\frac{2}{5}=\frac{x}{100}$, so x=40 and $\frac{2}{5}$ is equal to 40%.

b) Another way to express $\frac{2}{5}$ as a percent is to first convert it to decimal, i.e. $\frac{2}{5}$=0.40. Since 0.40 is written to the hundredths place, it is equal to 40%.

We now see that we can move easily between fractions, decimals and percent.

Example 3

Express $\frac{5}{3}$ as a percent.

a) Using the ratio method, $\frac{5}{3}=\frac{x}{100}$, x=166.66… and as a percent $\frac{5}{3}$ is approximately equal to 167% or with a bit more accuracy, $\frac{5}{3}$ is approximately equal to 166.7%. As a fraction, 166.7% is $\frac{166.7}{100}$ and is 1.667 as a decimal to 3 places.

b) If we used the alternate method and first converted $\frac{5}{3}$ to a decimal, then $\frac{5}{3}$=1.66666… and as a percent it could be rounded to 167% or 166.7% if we want more accuracy.

Example 4

A baseball batter gets 27 hits in 85 times at bat. As a decimal accurate to 3 places, $\frac{27}{85}$=0.318 and we report the average as ".318" which is a pretty good average. We could say that the baseball batter's average is 31.8%, but that is not the customary way to report baseball averages and sports enthusiasts wouldn't know what we meant.

Example 5

2000 students attend ABC College. 1300 students live at college and 700 students commute from home.

a) What percent of the students commute?

Answer: 700/2000=0.35 so 35% commute.

b) Next year, a 5% increase in total enrollment is expected and it is expected that the college will be able to house 1350 students. What percent of students will live at school and what percent will commute?

Answer: 0.05*2000=100, so next year there will be 2100 students attending ABC College. 1350/2100=0.6428 or approximately 64.3% will live at school and 35.7% will commute.

Percent of Change

Often, businesses give percentage discounts during sales or percentage increase for services they render.

Example 6

Best Buy decreased the price on a HD Sony TV from \$1500 to \$1250. What is the discount rate?

Answer: The discount is \$250. $\frac{250}{1500} = \frac{1}{6} = 0.1666\ldots$ or approximately 17%.

Example 7

I am now paying \$100 a month to NSTAR for gas to heat my home. Rates are scheduled to increase 3% for 2008. How much more will I have to spend for heat in 2008?

Answer: Each month during 2008, it will cost me and additional 0.03*100 or \$3 for heat.

My total increase for the year will be 12 *months* * 3 $\frac{dollars}{month}$ = \$36.

Example 8

Assume that the present seating capacity at Fenway Park is 36,000 and that at Gillette Stadium is 60,000, and that they both increase their seating capacities by 4,000 seats. What is the percentage increase at each place?

Answer: The percentage increase at Fenway Park is $\frac{4000}{36000} = \frac{1}{9} = 0.111\ldots$ or slightly more than 11%. The percentage increase at Gillette Stadium is $\frac{4000}{60000} = \frac{1}{15} = 0.0666\ldots$ or just under 7%.

Example 9

Assume you are a professional athlete and are offered two contract options. The first option is to get \$600,000 the first year with an increase of \$200,000 each year for the next 3 years. The second option is to get \$800,000 the first year and a 10% increase yearly for the next 3 years. Which option is better?

Answer:

a) With the first option, yearly salaries are $600,000, $800,000, $1,000,000 and $1,200,000 for a total during the 4 years of $3,600,000.

b) With the second option, yearly salaries are $800,000, (800,000+0.10*800,000) or $880,000, (880,000+0.10*880,000) or $968,000, (968,000+0.10*968,000) or $1,064,800. The total for the 4 years is $3,712,800.

It is pretty close but the second option is better if you just consider total income during the 4 years. However, if we extended the same scenario one more year, then the first option would yield $1,400,000 for year 5 with a total of $5,000,000 for the 5 year period. The second option would yield $1,171,280 (1,064,800+0.10*1,064,800) during year 5 with a total of $4,884,080 for the 5 years. Now the decision is getting interesting since the first option is generating considerably higher incomes during years 4 and 5 and inflation becomes a consideration. Wouldn't it be nice to have these problems!

References

Comparing and Scaling, *Ratio, Proportions, and Percent*, Lappan et al, Prentice Hall, Connected Mathematics (grade 7), 2002

Pre-Algebra, An Integrated Transition to Algebra and Geometry, Glencoe/McGraw-Hill, 2001

Some of the MCAS problems listed in the following material relate to the specific topics discussed in the previous percents and fractions sections while others serve as enrichment.

MCAS Problems

2006

http://www.doe.mass.edu/mcas/2006/release/g6math.pdf

2006 Grade 6 #26

26 Muriel has 20 flowers in her garden. Exactly 16 of the flowers are tulips. What percent of the flowers in Muriel's garden are tulips?

A. 4%

B. 16%

C. 40%

D. 80%

Answer: D. The fraction is $\frac{16}{20} = 0.80 = 80\%$.

http://www.doe.mass.edu/mcas/2006/release/g8math.pdf

2006 Grade 8 #11

11 When Lisa first started exercising, she could exercise for only 8 minutes. Yesterday, she exercised for 15 minutes. Which of the following proportions could be used to determine the percent increase in Lisa's exercise time?

A. $\frac{x}{100} = \frac{8}{7}$

B. $\frac{8}{x} = \frac{7}{100}$

C. $\frac{100}{x} = \frac{7}{8}$

D. $\frac{x}{100} = \frac{7}{8}$

Answer: D: Lisa increased her exercise time by 7 minutes. The percent increase is 7/8 or 87.5%.

2006 Grade 8 #16

16 Martin correctly answered 90% of the questions on a math test that contained exactly 40 questions. How many of the questions did he answer **incorrectly**?

A. 4
B. 10
C. 44
D. 90

Answer: A. Martin answered 36 correctly and 4 incorrectly. 4/40 is 10% incorrectly.

2006 Grade 8 #37

37 There are a total of 500 students at Lincoln Middle School. The table below shows the number of students who are members of 0, 1, 2, 3, or 4 clubs.

Members of Student Clubs

Number of Clubs (n)	Number of Students Who Are Members of n Clubs
0	300
1	110
2	60
3	20
4	10

Based on the table, what percent of the 500 students are members of 2 or more clubs?

A. 12%

B. 18%

C. 90%

D. 94%

Answer: B. 90 students are members of 2 or more clubs. 90/500 is equal to 18%.

http://www.doe.mass.edu/mcas/2006/release/g10math.pdf

2006 Grade 10 #30

30 A local car dealership has 100 vehicles on its lot. The chart below shows the numbers of cars, vans, and trucks, both new and used.

Vehicles at Dealership

	Number of Cars	Number of Vans	Number of Trucks
New	4	7	9
Used	36	21	23

Based on the chart, what percent of the 100 vehicles are either new cars or new trucks?

A. 11%

B. 13%

C. 20%

D. 59%

Answer: B: 13 vehicles are either new cars or new trucks and 13/100=13%.

2005

http://www.doe.mass.edu/mcas/2005/release/g6math.pdf

2005 Grade 6 #8

8 Judith has a total of 8 fish in her aquarium. Exactly 6 of the fish are guppies. What percent of the fish in the aquarium are guppies?

A. 48%

B. 60%

C. 68%

D. 75%

Answer: D. 6/8=75%.

http://www.doe.mass.edu/mcas/2005/release/g8math.pdf

2005 Grade 8 #19

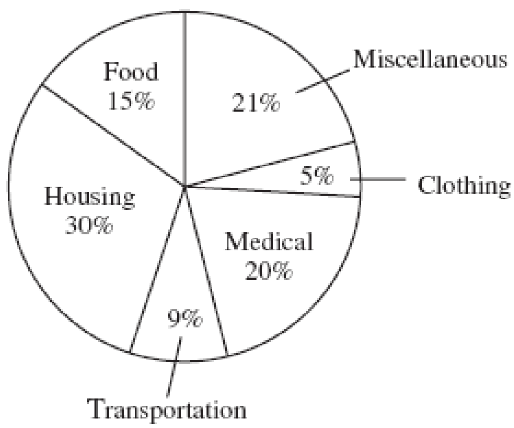
19 What is 30% of 600?

Answer: 0.30*600=180

2005 Grade 8 #23

23 The graph below shows a family's annual expenses.

Family Annual Expenses



If the family's annual income is $40,000, what is the family's annual expense for housing?

A. $120

B. $3,000

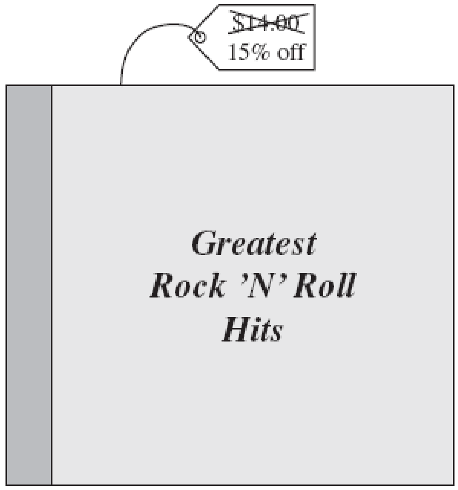
C. $12,000

D. $30,000

Answer: C. Housing is 30% of the total, i.e. 0.30*40,000=12,000.

2005 Grade 8 #39

39 Andrea works as a cashier in a music store. A customer wants to pay for a CD that is on sale for 15% off the regular price of $14.00.



The cash register is broken, and Andrea must calculate the price of the CD using only a calculator

a. What is the sale price of the CD? Show or explain how you got your answer.

b. Andrea needs to add 7% sales tax to the sale price of the CD. What should Andrea charge the customer for the CD, including tax? Show or explain how you got your answer.

c. The customer told Andrea that she could save time by just taking 8% off the regular price of the CD, because 15% − 7% is 8%. Is the customer right? Explain your reasoning.

Answer: a) Sale price is 14 - .15*14 = $11.90 b) Final price is 11.90 + .07*11.90 = $12.73 c) The customer is wrong since the regular price is $14 and 14 - .08 * 14 = $12.88, not $12.73. 15% of $14 does not equal 7% of $11.90 + 8% of $14.

http://www.doe.mass.edu/mcas/2005/release/g10math.pdf

2005 Grade 10 #16

16 On an airline, approximately 10% of the airline passengers who are booked for a flight do not show up for the flight. The airline has booked 160 passengers for a flight with maximum seating of 135. How many of the 160 passengers booked for this flight will **not** have a seat, assuming 10% of the booked passengers do not show up?

Answer: 0.10 * 160=16 so 144 passengers will show up and 9 will not have a seat.

2005 Grade 10 #37

37 The chart below separates the number of students majoring in math/science from students pursuing other majors at a state college.

Students' Majors by Class

	Freshmen	Sophomores	Juniors	Seniors
Math/Science Majors	260	310	200	330
Other Majors	1390	1510	1450	1550

What percent of the math/science majors are seniors?

A. 43%

B. 30%

C. 21%

D. 5%

Answer: There are 1100 Math/Science majors and 330/1100=0.3 so the correct answer is B.

2005 Grade 10 #39

39 Kelly wants to buy a tool set that is on sale at a hardware store. The price of each tool set will be decreased by 8% each morning just before the store opens. The sale will last for 7 days, or until all the sets are sold.

After the first reduction on Monday, the price of each set was $135.

If Kelly wants to wait until the first day that the price is $100 or less, on which day should she buy her tool set, if one is still available?

A. Wednesday

B. Thursday

C. Friday

D. Saturday

Answer: On Tuesday the price will be 135 – 0.08 *135=$124.20. On Wednesday the price is 124.20 – 0.08 * 124.20 or approximately $114.26. Since the price just dropped about $10 from Tuesday to Wednesday and it will drop a bit less (8% of the new daily price) each following day, a good guess is that the price will be less than $100 in 2 more days or on Friday. Let's do the math. On Thursday the price will be approximately $105.12 and on Friday it will be approximately $96.71. The correct answer is C.

Probability and Fractions

The probability of any event must lie between zero and one inclusive. If we are sure that an event will occur, its probability is one. Likewise if we are sure that an event won't occur, its probability is zero. The probability of an event occurring is the total number of favorable outcomes divided by the total number of outcomes.

Example 1A

There are 10 red marbles and 5 blue marbles in an urn. If we don't look, the probability of selecting a red marble on a single selection is 10/15 or 2/3 since there are 10 red marbles and 15 total marbles in the urn. Another way of looking at probability is to consider the results in terms of odds or chance. For this example with a probability of 2/3, there are 2 favorable outcomes and one unfavorable, so the odds or chance of winning when trying to select a red marble are 2 to 1 in our favor. In terms of a wager, that means I must bet $2 to win back $3. Looked at another way, the probability of selecting a blue marble is 5/15 or 1/3. The odds are 2 to 1 against us winning, so the wager must be $1 to win back $3 if in fact a blue marble is selected.

Example 1B

The same urn exists with 10 red and 5 blue marbles. The probability of selecting a green marble is 0/15 or 0 since there are no favorable outcomes. On the other hand, the probability of selecting on one selection either a red or blue marble is 15/15 or 1. This is called the sure event.

Rather than consider only two possible choices, let's consider a situation with several choices. Probability is still defined the same way.

Example 2

Fifty students in the King Middle School were surveyed about their favorite sandwich. Ten like tuna fish, 25 like turkey, 8 like jelly, and 7 like cheese. If a student is selected at random (closing your eyes and arbitrarily picking someone), the probability that the student's favorite sandwich is turkey is 25/50 or ½. The probability that the sandwich is tuna fish is 10/50 or 1/5. The odds that the sandwich is neither turkey nor tuna is 1 – (1/2 + 1/5). Remember that the sure event has a probability of 1. This last example represents some manipulation that we have to do with fractions. First of all, 1/2 + 1/5 has a lowest common denominator or 10 and equals $\frac{1*5}{2*5}+\frac{1*2}{5*2}=\frac{7}{10}$. The probability that the sandwich is neither turkey nor tuna is $1-\frac{7}{10}=\frac{1*10}{10}-\frac{7}{10}=\frac{3}{10}$ and the odds of this occurring are 7 to 3. That means, a wager of $3 that a random student eats neither turkey nor tuna would return $10 if we were correct.

The situation presented in Example 2 can be used to predict outcomes based on other populations sizes. If we assume that the results from the King School can be used in a school with 500 students, then we can expect that 500*1/5 or 100 students will select tuna fish. Mathematically, we are using the same probabilities in Example 2 and then saying that the expected outcome is equal to 500 times the appropriate probability. In a similar way, I would expect that 500*(8/50) at the larger school like jelly. This fraction simplifies to 80 students preferring jelly sandwiches.

Let's now look at independent events. In that situation, the probability of events A and B occurring is the product of their probabilities. Mathematically, we say that the $P(A \cap B) = P(A)P(B)$. An example of this occurs when tossing a fair coin. The probability of getting heads or getting tails on any toss of the coin equals ½, i.e. each outcome is equally likely to occur.

Example 3A

A fair coin is tossed twice. What is the probability of getting exactly 2 heads?

Answer: On each toss there 2 possible outcomes, so in all there are 2*2 or 4 possible outcomes. The only favorable outcome is getting heads on the first and the second toss, so the desired probability equals ¼. The odds against this happening are 3 to 1.

If we change the problem slightly and ask what is the probability of getting exactly 1 head and 1 tail, there are 2 favorable outcomes. They are HT which means heads on the first toss and tails on the second toss, and the second favorable outcome is TH. The probability is then 2/4 or ½. In other words the odds of this happening are 1 to 1. We sometimes refer to these odds as 50:50.

This type of problem dealing with independent events can be extended to 3 tosses of the coin. In that case there are 2*2*2 or 8 outcomes.

Example 3B

A fair coin is tossed 3 times. What is the probability of getting exactly 2 heads?

Answer: There 2*2*2 or 8 total outcomes. Listing the favorable outcomes, we get HHT, HTH, and THH. (Notice what was done when doing the listing. We kept H in the first position and listed the different possibilities. Then we move T to the first position.) There are 3 favorable outcomes and the probability of getting exactly 2 heads on 3 tosses is 3/8. The odds are 5 to 3 against this.

We could extend Example 3B and check for all the probabilities. See if you can verify that the probability of getting exactly 3 heads equals 1/8, the probability of getting exactly 2 tails equals 3/8 and is the same as that found for exactly 2 heads, and finally the probability of getting exactly 3 tails is 1/8. Look at what happens when adding all possible results. (The sum of all probabilities equals one since we covered all possible outcomes.)

These types of probability problems can easily be extended to *n* tosses of a fair coin but the listing of favorable outcomes would become tedious. Fortunately the binomial distribution formula covers such cases. The implication of that formula deals with combinations and is beyond the scope of this document, so for now we must resort to listing the outcomes.

Example 4

Another problem dealing with independent events is tossing a pair of dice. What is the probability of getting a sum of 7 or 11 when tossing dice?

Answer: Each die is a cube with 6 sides. The numbers on each of the sides are one of: 1, 2, 3, 4, 5 or 6. There are 6*6 total outcomes when tossing the 2 dice. To get a sum of 11, there are only 2 favorable outcomes. They are 5 and 6 (5 on the first die and 6 on the second die) or 6 and 5. To get a sum of 7 requires a bit more work. Let's start with 1 on the first die, and then we must get 6 on the second die. If we get 2 on the first die, then we must get 5 on the second die. The following table lists all favorable outcomes when getting a sum of 7.

Die 1	**Die 2**	**Sum**
1	6	7
2	5	7
3	4	7
4	3	7
5	2	7
6	1	7

There are 6 ways of getting a sum of 7. Finally, the probability of getting a sum of 7 or 11 is 6/36 + 2/36 or 8/36 which simplifies to 2/9. The chance of this happening is 7 to 2 against you. A fair wager would be $2 in order to win back $9 if you win.

Now that we have discussed various types of probability problems, let's return to the first example where we had 10 red and 5 blue marbles in an urn. We will now investigate conditional probability in which we will select 2 marbles from the urn, but without replacing the first marble selected. Mathematically, we say the $P(A \cap B) = P(A)P(B/A)$. This means that the probability of event A occurring first and B occurring second is the probability of A times the probability of B given that A has just occurred, i.e. $P(B/A)$. What has happened is that for cases without replacement, $P(B/A) \neq P(B)$, but for cases with replacement as will be shown in Example 5B $P(B/A) = P(B)$. This may seem confusing but the next examples will show the difference.

Example 5A

On 2 selections without replacement from an urn with 10 red and 5 blue marbles, what is the probability of selecting a red marble and then a blue marble?

Answer: On the first selection, the probability of selecting a red marble is 10/15 or 2/3. Now there are only 14 marbles left and the probability of selecting a blue marble is 5/14. Finally, the desired probability is $\frac{2}{3} * \frac{5}{14} = \frac{10}{42} = \frac{5}{21}$. The chance of this happening is 16 to 5 against it.

Example 5B

If we change Example 5A and make it with replacement, then we have independent events again and the probability of selecting a red marble is always 2/3 and that for the blue marble is always 1/3. What is the probability of selecting a red marble and then a blue marble from an urn with 10 red and 5 blue marbles if a marble is replaced after being selected?

Answer: Mathematically $P(R \cap B) = P(R)P(B)$ and the desired probability is $\frac{2}{3} * \frac{1}{3} = \frac{2}{9}$. The odds are 7 to 2 against this.

Probability problems without replacement can be extended to other situations and there is a formula to cover such cases. It is called the hypergeometric distribution which deals with combinations and, as mentioned in Example 3B, is beyond the scope of this document.

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Statistics for the social sciences, William L. Hays, Holt, Rinehart, Winston, 2nd ed. 1973

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MCAS Problems

2006

http://www.doe.mass.edu/mcas/2006/release/g6math.pdf

2006 Grade 6 #28

28 Marcus has a bag of 20 table tennis balls. The probability of selecting a yellow table tennis ball, without looking, is $\frac{3}{10}$. What is the total number of yellow table tennis balls in the bag?

Answer: There are (3/10)*20=6 yellow balls in the bag.

2005

http://www.doe.mass.edu/mcas/2005/release/g10math.pdf

2005 Grade 10 #26

26 Of the people in attendance at a recent baseball game,

- one-third had grandstand tickets,
- one-fourth had bleacher tickets, and
- the remaining 11,250 people in attendance had other tickets.

What was the total number of people in attendance at the game?

A. 27,000

B. 20,000

C. 16,000

D. 18,000

Answer: The remaining 11,250 people comprise 1- (1/3 +1/4) or 1 – 7/12 = 5/12 of those in attendance. The answer is 11,250/(5/12)=27,000 so select A.

2004

http://www.doe.mass.edu/mcas/2004/release/g10math.pdf

2004 Grade 10 #40

40 Bailey noticed that many of the students at her school had red hair. She randomly chose 25 of the students in her school and found that 2 of them had red hair. If Bailey's sample is representative, which of the following is closest to the number of the 2200 students at her school who have red hair?

A. 44

B. 88

C. 176

D. 200

Answer: (2/25)*2200=2*88=176. The correct answer is C.

2003

http://www.doe.mass.edu/mcas/2003/release/g6math.pdf

2003 Grade 6 #1

1. Johannah collects posters. She has 3 animal posters, 4 posters of sports teams, and 2 posters of musical bands. What fraction of her posters is of sports teams?

A. $\frac{2}{9}$

B. $\frac{3}{9}$

C. $\frac{4}{9}$

D. $\frac{5}{9}$

Answer: C

http://www.doe.mass.edu/mcas/2003/release/g8math.pdf

2003 Grade 8 #32

32 The chart below shows the number of restaurant advertisements in a city directory.

Restaurant Advertisements

Type of Restaurant	Number
Chinese	10
Mexican	17
Italian	8
French	5

The telephone numbers of these restaurants are written on separate slips of paper and one is selected at random. What is the probability that the telephone number of an Italian restaurant is selected?

A. 0.025

B. 0.125

C. 0.2

D. 0.8

Answer: 8/40=1/5=0.2 so the answer is C

2002

http://www.doe.mass.edu/mcas/2002/release/g8math.pdf

2002 Grade 8 #15

15 A bag contains 3 blue, 4 red, and 2 white marbles. Karin is going to draw out a marble without looking in the bag. What is the probability that she will **not** draw a red marble?

A. $\frac{1}{3}$

✔ B. $\frac{5}{9}$

C. $\frac{2}{3}$

D. $\frac{4}{9}$

Answer: The probability of drawing a red marble is 4/9. The probability of not drawing a red marble is 1 - 4/9 = 5/9.

2002 Grade 8 #22

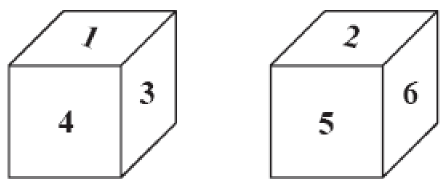
22 Lionel and Tracy are playing a game using two six-sided number cubes. The faces of each cube are numbered as shown below.

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$

Lionel has a red cube and Tracy has a green cube. To play the game they both roll their cubes at the same time.

- The numbers that show face up when the cubes stop rolling are used to make a fraction.
- The number on the red cube is used for the numerator and the number on the green cube is used for the denominator.

For example, the results shown below would make the fraction $\frac{1}{2}$.



red cube   green cube

- Lionel wins 1 point if the fraction formed has a value less than one.
- Tracy wins 1 point if the fraction has a value greater than one.
- No one gets a point if the fraction is equal to one.

a. Make a list or a table in your Student Answer Booklet of all of the fractions possible from rolling 1 red and 1 green cube. How many total different fractions are there?

b. If Lionel (red cube) rolls a 3, what is the probability that Tracy (green cube) wins 1 point? Show your work or explain how you obtained your answer.

c. Using your table, what is the probability of each player winning a point on a given turn? Do you think this game is fair to both players? Show your work or explain how you obtained your answer.

Answer:

a) The fractions are 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 1/2, 2/2, 3/2, 4/2, 5/2, 6/2 and so on. Since there are 6 fractions for each number in the denominator, there are 36 total different fractions.

b) Winning fractions for Tracy are the 2 outcomes 3/1 and 3/2. Since there are 36 total possible fractions from part a), the probability that Tracy wins 1 point is 2/36 or 1/18.
c) If Tracy's die shows a "1", she has 5 chances to win that occur when Lionel's die is 2, 3, 4, 5, or 6. Tracy loses only if Lionel's die is "1". Using this approach, we can set up the following tables to examine the probability of each player winning:

Tracy's die	1	2	3	4	5	6
Tracy's probability of winning	5/6	4/6=2/3	3/6=1/2	2/6=1/3	1/6	0

Lionel's die	1	2	3	4	5	6
Lionel's probability of winning	5/6	4/6=2/3	3/6=1/2	2/6=1/3	1/6	0

Examining the tables, the game is fair to both.

http://www.doe.mass.edu/mcas/2002/release/g10math.pdf

2002 Grade 10 #6

6 Lani had a box that contained

- 1 blue marble;
- 1 green marble;
- 1 purple marble;
- 1 yellow marble; and
- 2 red marbles.

Lani removed one marble without looking, and she recorded the result. She placed the marble back in the box and repeated the procedure one more time. What is the probability that Lani removed a red marble followed by a blue marble?

A. $\frac{1}{36}$

✔ B. $\frac{1}{18}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

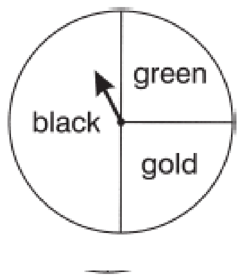
Answer: This situation is with replacement so the events are independent. The probability of a red marble is 2/6 and the probability of a blue marble is 1/6. The probability of a red followed by a blue is (2/6)*(1/6) = 1/18.

2001

http://www.doe.mass.edu/mcas/2001/release/g6math.pdf

2001 Grade 6 #12

Use the spinner shown below to answer question 12.



12 Melinda and Henry are playing a game with this three-color spinner.

a. Henry thinks that the probability of landing on gold is $\frac{1}{3}$. Is Henry correct?

- If he is correct, explain how you know.
- If he is **not** correct, give the correct probability and explain how you know it is correct.

b. If Melinda and Henry will spin the spinner 60 times in the game, about how many times can they expect it to land on each of the three colors? Explain or show how you found your answer.

c. Melinda and Henry started playing the game, and after 30 spins the spinner had landed on black 10 times. Henry told Melinda that this shows that the probability of landing on black must be $\frac{10}{30} = \frac{1}{3}$. Is Henry correct?

- If he is correct, explain how you know.
- If he is **not** correct, tell what is the probability of landing on black. Explain how it is possible that the spinner could have landed on black 10 times out of a total of 30 spins.

Answer: a) Since gold cover 25% of the circle, the probability of landing on gold is 25/100=1/4. b) 15 times on gold, 15 times on green, 30 times on black. c) The probability of landing on black is 50% or 1/2. The sample of only 30 spins does not represent a large enough sample to justify the true probability of landing on black.

2001 Grade 6 #16

16 Jerrod has a bag with 14 green marbles, 8 white marbles, 4 red marbles, and 4 black marbles. If he wants the probability of picking a green marble to be $\frac{1}{2}$, which should Jerrod do?

✓ A. add two green marbles

B. remove two white, two red, and two black marbles

C. remove two green marbles

D. add two white, two red, and two black marbles

Answer: There are 30 marbles, 14 of which are green. By adding two green marbles, 16 of the 32 marbles will be green.

http://www.doe.mass.edu/mcas/2001/release/g8math.pdf

2001 Grade 8 #33

33 Of the 12 songs on Ella's new CD, 3 are her favorites. If her CD player chooses one song at random, what is the probability that it will be one of her favorite songs?

✓ A. $\frac{1}{4}$

B. $\frac{3}{4}$

C. $\frac{1}{3}$

D. $\frac{2}{3}$

Answer: 3 favorites and 12 songs total implies 3/12=1/4 so A is the correct answer.

http://www.doe.mass.edu/mcas/2001/release/g10math.pdf

2001 Grade 10 #26

26 A set of 36 cards is numbered with the positive integers from 1 to 36. If the cards are shuffled and one is chosen at random, what is the probability that the number on the card is a multiple of **both 4 and 6**?

✓ A. $\frac{1}{12}$

B. $\frac{1}{6}$

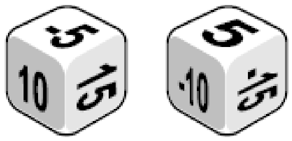
C. $\frac{5}{12}$

D. $\frac{2}{3}$

Answer: There are 3 multiples of both 4 and 6. They are 12, 24 and 36. The probability is 3/36=1/12.

2001 Grade 10 #37

37 Joseph has two number cubes, each with faces labeled by the numbers -15, -10, -5, 5, 10, and 15.



If Joseph rolls the two cubes and adds the resulting numbers, what is the probability that the sum will be 0?

A. $\frac{1}{36}$

B. $\frac{1}{12}$

C. $\frac{1}{4}$

✓ D. $\frac{1}{6}$

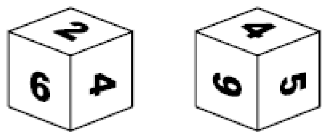
Answer: Each cube has 6 sides so there are 36 possible outcomes when rolling the 2 dice. A sum of 0 can be obtained by -15 and 15 (2 ways), -10 and 10 (2 ways) and -5 and 5 (2 ways) for a total of 6 favorable outcomes. The probability that the sum will be 0 is 6/36=1/6.

2000

http://www.doe.mass.edu/mcas/2000/release/g8math.pdf

2000 Grade 8 #8

8. John is playing a board game that uses a pair of number cubes with sides numbered 1 to 6.



To find how many spaces he can move on the board, he adds the two numbers he rolls. The possible sums are

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.

a. Are all the sums John can roll equally likely? Explain your reasoning in detail.

b. John needs to roll a sum of exactly 11 in order to get another turn. What is the probability that he will roll a sum of exactly 11? Explain your reasoning in detail.

Answer:

a) The sums are not equally likely. There is one way to get a sum of 2 (1 on each die), there are two ways to get a sum of 3, three ways to get a sum of 4, etc.
b) There are two ways of rolling a sum of 11, a 5 on one die and a 6 on the second die or a 6 on the first die and a 5 on the second die. There are 36 total outcomes, so the probability of rolling a sum of exactly 11 is 2/36 or 1/18.

2000 Grade 8 #16

16. A bag contains 2 blue, 6 black, and 4 white socks. Paula is going to draw out a sock without looking in the bag. What is the probability that she will draw either a blue or a black sock?

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

✓ D. $\frac{2}{3}$

Answer: 8 socks are either blue or black. There are 12 socks in the bag. The desired probability is 8/12 or 2/3.

2000 Grade 8 #29

29. Chris selected 50 students at random and asked them who they want for class president. The results are shown in the table below.

Candidate	Frequency
Jessica	30
Jeremy	4
Monique	16

Which statement is true regarding the probability that at least 5 of the next 10 students interviewed will want Jeremy for president?

A. It is impossible.

✓ B. It is unlikely.

C. It is likely.

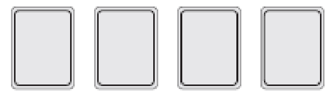
D. It is certain.

Answer: The probability that a single student selected at random will want Jeremy for president is 4/50 or 2/25 = 0.08. This probability is quite small and the event is rather unlikely to occur. To get at least 5 of the next 10 students to select Jeremy will make the probability much less than 0.08 and will be very unlikely, but not impossible. The formulas to find the exact probability for this example involve the binomial distribution as discussed in Example 3B on page 65. For the interested reader, the probability that at least 5 of the next 10 will want Jeremy is $1-\sum_{i=0}^{4} P(i)$ where $P(i)$ is the probability that exactly *i* students out of 10 want Jeremy. $P(i) = C(n,i) * (.08)^{i} * (.92)^{n-i}$ where $C(n,i)$ is the number of combinations of *i* things out of *n*. Computations yield P(0)=.4344, P(1)=.3777, P(2)=.1478, P(3)=.0343, P(4)=.0052 and $1-\sum_{i=0}^{4} P(i)$ =.0006 which confirms that the event is unlikely to occur.

http://www.doe.mass.edu/mcas/2000/release/g10math.pdf

2000 Grade 10 #19

Use the picture of the cards to answer question 19.



19. Each of the letters M, A, T, and H appear on the reverse side of one of the four cards on the right (one letter per card), but not necessarily in that order. If the cards are turned over, what is the probability that they will be ordered so that they spell the word MATH?

A. $\frac{1}{4}$

B. $\frac{1}{12}$

✓ C. $\frac{1}{24}$

D. $\frac{1}{48}$

Answer: There are 24 ways of arranging the 4 letters. Only one of the arrangements will have the correct order. The probability that the letters will spell MATH is 1/24. (Note that this problem involves permutations which are arrangements with order.)

$$\frac{1}{4!} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{24}$$

Advanced Fraction Topics

Division Algorithm

When dividing a number a by another number b, the result is a quotient q and a remainder r. So if we divide 13 by 3, the quotient is 4 and the remainder is 1. Looking at this another way, we can say $13 = 3 \cdot 4 + 1$. Written generally, this is $a = b \cdot q + r$. This is the division algorithm equation.

Example 1

Find the quotient and remainder for 13 divided by 5.

$$a = 13,\ b = 5$$
$$a = b \cdot q + r$$
$$13 = 5 \cdot q + r$$
$$13 = 5 \cdot 2 + 3$$
$$q = 2,\ r = 3$$

Example 2

Find the quotient and remainder for 53 divided by 17.

$$a = 53,\ b = 17$$
$$a = b \cdot q + r$$
$$53 = 17 \cdot q + r$$
$$53 = 17 \cdot 3 + 2$$
$$q = 3,\ r = 2$$

Euclidean Algorithm

The Euclidean Algorithm is used to find the greatest common divisor (GCD) of two integers. The Euclidean Algorithm is a repeated use of the Division Algorithm. We want to find the GCD of the two numbers a and b described in the Division Algorithm above. First we find the quotient q and the remainder r of a and b. Next we use the Division Algorithm again with $a = b$ from the first step and $b = r$. We continue this process until $r = 0$. The GCD is b from the last step.

Example 3

Find the GCD of 18 and 10.

First use the Division Algorithm with $a = 18,\ b = 10$:

$18 = 10 \cdot q + r$

$18 = 10 \cdot 1 + 8$

Since $r \neq 0$, use the Division Algorithm again with $a = 10,\ b = 8$:

$10 = 8 \cdot q + r$

$10 = 8 \cdot 1 + 2$

Since $r \neq 0$, use the Division Algorithm again with $a = 8,\ b = 2$:

$8 = 2 \cdot q + r$

$8 = 2 \cdot 4 + 0$

Since $r = 0$, the GCD of 18 and 10 is 2 which is *b* from this last step.

Example 4

Find the GCD of 13 and 5.

First use the Division Algorithm with $a = 13,\ b = 5$:

$13 = 5 \cdot q + r$

$13 = 5 \cdot 2 + 3$

Since $r \neq 0$, use the Division Algorithm again with $a = 5,\ b = 3$:

$5 = 3 \cdot q + r$

$5 = 3 \cdot 1 + 2$

Since $r \neq 0$, use the Division Algorithm again with $a = 3,\ b = 2$:

$3 = 2 \cdot q + r$

$3 = 2 \cdot 1 + 1$

Since $r \neq 0$, use the Division Algorithm again with $a = 2,\ b = 1$:

$2 = 1 \cdot q + r$

$2 = 1 \cdot 2 + 0$

Since $r = 0$, the GCD of 13 and 5 is 1 which is *b* from this last step.

Continued Fraction Expansion

The following expressions are called ***continued fractions*:**

$1+\dfrac{1}{x}$, $1+\dfrac{1}{1+\dfrac{1}{x}}$, $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$, $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}}$. Each simplifies to an expression of the form $\dfrac{ax+b}{bx+c}$. Trace the value of *a*, *b*, *c* as you continue the fraction. What is the pattern that these values follow?

Example 5

Convert $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$ to the form $\dfrac{ax+b}{bx+c}$.

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}=1+\frac{1}{1+\frac{1}{\left(\frac{x+1}{x}\right)}}=1+\frac{1}{1+\frac{x}{x+1}}=1+\frac{1}{\left(\frac{2x+1}{x+1}\right)}=1+\frac{x+1}{2x+1}=\frac{3x+2}{2x+1}$$

The fourth continued fraction before Example 5 can be converted to the form $\dfrac{5x+3}{3x+2}$. Try to verify this. Note that the results from Example 5 can be used to help.

Continued fraction expansion utilizes repeated application of the division algorithm until $r=0$ in a similar way as finding the GCD.

Example 6

Perform continued fraction expansion on the fraction $\dfrac{13}{5}$.

A fraction such as $\dfrac{13}{5}$ is the same as dividing 13 by 5. Using the division algorithm equation, we can represent this division as $13=5\cdot 2+3$ where *a* = 13 and b =5. Dividing both sides of this equation by b which is 5 gives us

$$\frac{13}{5}=2+\frac{3}{5}.$$

The fraction $\frac{3}{5}$ can be written $\frac{1}{5/3}$. So now we can write $\frac{13}{5}=2+\frac{1}{5/3}$. Now we apply the division algorithm to $\frac{5}{3}$ and get $5=3\cdot 1+2$. Dividing this last equality by 3 gives us $\frac{5}{3}=1+\frac{2}{3}$.

We go back to our earlier equation $\frac{13}{5}=2+\frac{1}{5/3}$ and substitute what we just found for $\frac{5}{3}$. This gives the equation $\frac{13}{5}=2+\cfrac{1}{1+\frac{2}{3}}$.

Now we repeat this process with the fraction $\frac{2}{3}$. Write $\frac{2}{3}$ as $\frac{1}{3/2}$, and rewrite the equation $\frac{13}{5}=2+\cfrac{1}{1+\frac{2}{3}}$ as $\frac{13}{5}=2+\cfrac{1}{1+\cfrac{1}{3/2}}$

Now apply the division algorithm to $\frac{3}{2}$. This gives us $3=2\cdot 1+1$. Next we divide by 2 and get $\frac{3}{2}=1+\frac{1}{2}$. Substituting in the earlier equation gives us

$$\frac{13}{5}=2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}.$$

Repeat the process with $\frac{1}{2}$. Write $\frac{1}{2}$ as $\frac{1}{2/1}$ and rewrite the equation

$$\frac{13}{5}=2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}} \text{ as } \frac{13}{5}=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2/1}}}$$

Apply the division algorithm to $\frac{2}{1}$ gives us $2 = 1 \cdot 2 + 0$. Since $r = 0$, this is our last step. So we really didn't need to apply the division algorithm to $\frac{2}{1}$. The full continued fraction expansion is $\frac{13}{5} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$. We can stop when we get a fraction that has a denominator of 1 as in $\frac{2}{1}$.

The continued fraction expansion for 53/17 in Example 2 is $3 + \cfrac{1}{8 + \cfrac{1}{2}}$. Try to verify this.

Example 7

Evaluate $3 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4 + \ldots}}}$ where the 3 dots indicate the pattern continues without end.

First set the continued fraction equal to x:

$$x = 3 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4 + \ldots}}} \Rightarrow x - 3 = \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4 + \ldots}}}$$

Note that $x - 3 = \frac{1}{4 + \ldots}$ and substitute into the equation above:

$$x - 3 = \cfrac{1}{4 + \cfrac{1}{3 + (x - 3)}} \Rightarrow x - 3 = \cfrac{1}{4 + \cfrac{1}{x}} \Rightarrow x - 3 = \cfrac{1}{\left(\cfrac{4x + 1}{x}\right)} \Rightarrow x - 3 = \frac{x}{4x + 1}$$

$$(x - 3)(4x + 1) = x \Rightarrow 4x^2 - 12x + x - 3 = x \Rightarrow 4x^2 - 12x - 3 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 4 \cdot 4 \cdot -3}}{8} = \frac{12 \pm \sqrt{192}}{8} = \frac{12 \pm 8\sqrt{3}}{8} = \frac{3}{2} \pm \sqrt{3}$$

Examining the solutions $\frac{3}{2}\pm\sqrt{3}$, $\frac{3}{2}-\sqrt{3}$ is an extraneous solution since it is less than 0 and x must be > 3. So then from the original equation:

$x=3+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{4+...}}}=\frac{3}{2}+\sqrt{3}$ or approximately 3.23205 rounded to 5 decimal places.

References

College Algebra by Michael Sullivan, Prentice Hall, 6^{th} ed., 2002.
NCTM calendar problem favorites.

Generalization of Continued Fractions

Examples 5, 6, 7 in the previous section showed different cases for continued fractions. In general, continued fractions are just another way of writing fractions. For any fraction p/q where p and q are whole, positive numbers, a continued fraction representation is p/q=a+1/(b+1/(c+1/(d+1/(e+…)))). In this general expression, a, b, c, d, e, etc. are all whole numbers. The expansion stops when the last fraction in the expansion has a numerator or denominator equal to 1.

Example 1

$\frac{37}{16}=2+\cfrac{1}{3+\cfrac{1}{5}}$. In this example, we stopped since the last fraction in the expansion 1/5 has a numerator of 1. We could have tried to start another step to yield $\frac{37}{16}=2+\cfrac{1}{3+\cfrac{1}{5/1}}$ but the last fraction now has a denominator of 1, so it is time to stop. Make sure that you can verify the details for the continued fraction of $\frac{37}{16}$.

Example 2

Express 4/19 as a continued fraction.

Answer: $\frac{4}{19}=0+\frac{1}{19/4}=\frac{1}{4+3/4}=\frac{1}{4+\frac{1}{4/3}}=\frac{1}{4+\frac{1}{1+\frac{1}{3}}}$. Note that in this example, we included every step in the expansion. Often, all the detail is not necessary. An interesting observation in this example is that since $\frac{4}{19}<1$, the generalization p/q=a+1/(b+1/(c+1/(d+1/(e+…)))) simplifies to p/q=1/(b+1/(c+1/(d+1/(e+…)))). Whenever the fraction $\frac{p}{q}<1$, the first value “a” in the continued fraction will always be 0.

Excellent sources for continued fraction material can be found on the web. The generalization material was drawn from the site http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/cfINTRO.html.

Examples

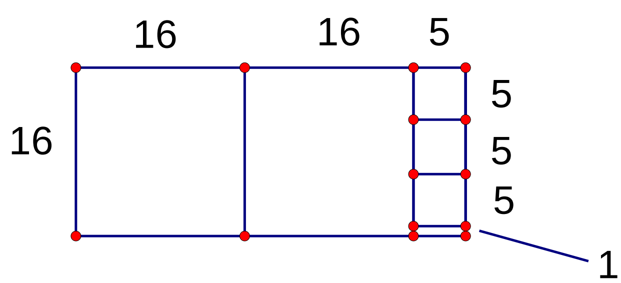
Show all work to express the following as continued fractions:

1. 21/26 Answer: $\frac{1}{1+\frac{1}{4+\frac{1}{5}}}$
2. 103/14 Answer: $7+\frac{1}{2+\frac{1}{1+\frac{1}{4}}}$

Splitting Rectangles into Squares

Another way to look at the generalization of continued fractions as outlined in this section is to use rectangles and squares for a visual interpretation of the division algorithms used.

Let's start with Example 1, $\frac{37}{16}$. We found that the continued fraction representation is 2+1/(3+1/5). For some, this representation may be easier to deal with when sketching a 37 by 16 rectangle and splitting it into squares.

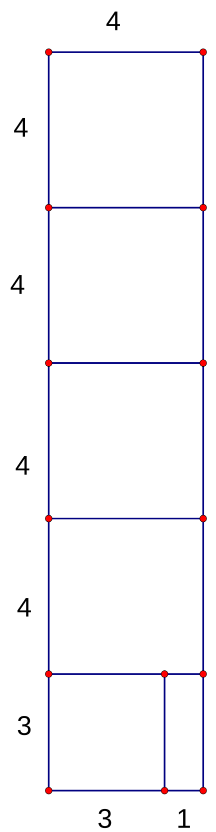


In this figure, we have created "2" 16x16 squares and "3" 5x5 squares within the 37x16 rectangle. We are still left with a 1x5 rectangle, but the process always stops with a final 1xn rectangle, in this case a 1x5 rectangle. The geometry we have shown directly relates to the continued fraction for $\frac{37}{16} = 2 + \cfrac{1}{3 + \cfrac{1}{5}}$.

Let's suppose that when splitting a rectangle into squares, we get "3" 17x17 squares, followed by "4" 4x4 squares, followed by a 1x4 rectangle. The longer side of the rectangle is 55 (3x17 + 4) and the shorter side is 17. The fraction is 55/17 and its continued fraction is $\frac{55}{17} = 3 + \cfrac{1}{4 + \cfrac{1}{4}}$. See if you can draw the figure to split the rectangle into squares. The geometry software Geometer's Sketchpad was used in this document, but even pencil and paper will suffice.

When setting up the rectangle to represent the fraction p/q, a guideline to follow is to let the numerator p be the horizontal part of the rectangle and the denominator q the vertical part of the rectangle. Following this approach will reaffirm that "a" in the general continued fraction expression is equal to 0 when the initial squares are generated along the vertical part of the rectangle.

In Example 2, we started with the fraction $\frac{4}{19}$. Splitting the rectangle into squares will take the following form:



In this figure, we have "4" 4x4 squares and "1" 3x3 square within the 4x19 rectangle. We still have a 1x3 rectangle left over. The geometry relates to the continued fraction form for $\frac{4}{19} = \cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{3}}}$. Since $\frac{4}{19} < 1$, "a" is equal to 0 and the initial squares are generated along the vertical part of the rectangle.

Examples

Use splitting rectangles into squares to generate figures for

1. 21/26
2. 103/14

Remember to let the numerator in each fraction be the horizontal part of the rectangle.

Fibonacci-related Continued Fractions

Continued fractions are another way of writing fractions as shown in the Division Algorithm and Euclidean Algorithm sections. They have interesting associations with various mathematical topics. We will now discuss their links with Fibonacci numbers.

The Fibonacci sequence is 0,1,1,2,3,5,8,13, … where F(n)=0 when *n=0*, F(n)=1 when *n=1*, and F(n)=F(n-1)+F(n-2) when $n \geq 2$. The sequence of continued fractions generated by

$$\frac{\mathbf{1}}{\mathbf{1}},\ 1+\frac{1}{1},\ 1+\frac{1}{1+1},\ 1+\cfrac{1}{1+\cfrac{1}{1+1}},\ 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}},\ 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}}$$

yields an interesting result. The evaluation of the terms follows:

Term	Value
1	$\frac{\mathbf{1}}{\mathbf{1}}=1$
2	$1+\frac{1}{1}=\frac{2}{1}$
3	$1+\frac{1}{2}=\frac{3}{2}$
4	$1+\cfrac{1}{1+\cfrac{1}{2}}=1+\cfrac{1}{\cfrac{3}{2}}=1+\frac{2}{3}=\frac{5}{3}$
5	$1+\cfrac{1}{1+\cfrac{2}{3}}=1+\cfrac{1}{\cfrac{5}{3}}=1+\frac{3}{5}=\frac{8}{5}$
6	$1+\cfrac{1}{1+\cfrac{3}{5}}=1+\cfrac{1}{\cfrac{8}{5}}=1+\frac{5}{8}=\frac{13}{8}$

The pattern hopefully is becoming apparent, and the 7th term if it were listed would be

$1+\dfrac{1}{1+\dfrac{5}{8}}=\dfrac{1}{\dfrac{13}{8}}=1+\dfrac{8}{13}=\dfrac{21}{13}$. The 8th term would be $1+\dfrac{1}{1+\dfrac{8}{13}}=1+\dfrac{1}{\dfrac{21}{13}}=1+\dfrac{13}{21}=\dfrac{34}{21}$.

The sequence we generated and will continue to generate by following the procedure described is the ratio of terms of the Fibonacci sequence

$\left\{\dfrac{1}{1},\dfrac{2}{1},\dfrac{3}{2},\dfrac{5}{3},\dfrac{8}{5},\dfrac{13}{8},\dfrac{21}{13},\dfrac{34}{21},\ldots,\dfrac{F(n)}{F(n-1)}\right\},\ n\geq 2$.

Example 1

Evaluate the continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}}}}}}}$$

In this example, there are 9 division symbols. In the sequence of continued fractions on page 93, the 4th term in the sequence has 2 division symbols. As the number of division symbols increases, we find that:

Division Symbols	**Value**	**F(n)/F(n-1)**
2	5/3	F(5)/F(4)
3	8/5	F(6)/F(5)
4	13/8	F(7)/F(6)

We can then generalize that for the example presented with 9 division symbols, the value of the continued fraction is F(12)/F(11). Assuming $n \geq 5$, **F(n)/F(n-1) as a continued fraction generates n-3 division symbols**. The terms in the Fibonacci sequence can be listed as 0,1,1,2,3,5,8,13,21,34,55,89,144,… so F(12)/F(11) is equal to 144/89. Setting up the following table may make it a bit easier to identify each term in the Fibonacci sequence.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	n
F(n)	0	1	1	2	3	5	8	13	21	34	55	89	144	F(n-1)+F(n-2)

An interesting result of the ratio F(n)/F(n-1) is that $\lim\limits_{n->\infty} \dfrac{F(n)}{F(n-1)} = phi = 1.6180339....$

This last statement involving *phi* which is often referred to as the *golden ratio* opens up another area of research for interested students. Note that 144/89 $\approx$ 1.618. Even though we have stopped at n=12 in the Fibonacci sequence of terms, the ratio is quickly converging to *phi*.

Example 2

What is the continued fraction represented by the ratio F(9)/F(8)?

In this case, there will be 6 divisions. We already know that F(n)/F(n-1) generates n-3 divisions when $n \geq 5$. The desired continued fraction is then

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}}}}$$

and its value F(9)/F(8) is equal to 34/21.

Example 3

Find the number n such that the error $\left|\frac{F(n)}{F(n-1)} - phi\right| < .00001$.

Answer: *Phi* is the positive solution to the quadratic $x^2 - x - 1 = 0$. The roots are $\frac{1 \pm \sqrt{5}}{2}$, so *phi* is equal to $\frac{1+\sqrt{5}}{2}$ (see page 47, Grade 10 #6). Since we already considered the case for n=12, let's start at that point and compute the errors.

n	$\frac{F(n)}{F(n-1)}$	*error*
12	144/89	.000056
13	233/144	.000022
14	377/233	.000008

We can see that for n=14, $\left|\frac{F(n)}{F(n-1)} - phi\right| < .00001$. Even without setting up an iterative process, the solution to find n was very quick using a TI-84 Plus for the repeated calculations.