Basics of Spectrum Analysis/Measurements and the FFT Analyzer
Transformation of Time to Frequency

Many times a transformation is performed to provide a better or clearer understanding of a phenomena. The time representation of a sine wave may be difficult to interpret. By using a Fourier series representation, the original time signal can be easily transformed and much better understood.

Transformations are also performed to represent the same data with significantly less information. Notice that the original time signal was defined by many discrete time points (ie, 1024, 2048, 4096 ...) whereas the equivalent Fourier representation only requires 4 amplitudes and 4 frequencies.
The FFT Analyzer can be broken down into several pieces which involve the digitization, filtering, transformation and processing of a signal.

Several items are important here:
- Digitization and Sampling
- Quantization of Signal
- Aliasing Effects
- Leakage Distortion
- Windows Weighting Functions
- The Fourier Transform
- Measurement Formulation
The Anatomy of the FFT Process

- Actual time signals
- Analog anti-alias filter
- Digitized time signals
- Windowed time signals
- Compute FFT of signal
- Average auto/cross spectra
- Compute FRF and Coherence
Aliasing (Wrap-Around Error)

Aliasing results when the sampling does not occur fast enough.

Sampling must occur faster than twice the highest frequency to be measured in the data - sampling of 10 to 20 times the signal is sufficient for most time representations of varying signals.

However, in order to accurately represent a signal in the frequency domain, sampling need only occur at greater than twice the frequency of interest.

Anti-aliasing filters are used to prevent aliasing.
These are typically Low Pass Analog Filters.
Anti-Aliasing Filters

Anti-aliasing filters are typically specified with a cut-off frequency. The roll-off of the filter will determine how quickly the signal will be attenuated and is specified in dB/octave.

The cut-off frequency is usually specified at the 3 dB down point (which is where the filter attenuates 3 dB of signal).

Butterworth, Chebyshev, elliptic, Bessel are common filters.
Digitization of a Signal

Sampling rate of the ADC is specified as a maximum that is possible. Basically, the digitizer is taking a series of “snapshots” at a very fast rate as time progresses.

Analog Signal $\rightarrow$ Digital Representation

ADC
Sampling

Each sample is spaced $\Delta t$ seconds apart. Sufficient sampling is needed in order to assure that the entire event is captured. The maximum observable frequency is inversely proportional to the $\Delta t$ time step used.

$$f_{\text{max}} = \frac{1}{2 \Delta t}$$
Sampling Theory

In order to extract valid frequency information, digitization of the analog signal must occur at a certain rate.

Shannon's Sampling Theorem states \( f_s > 2 f_{\text{max}} \)

That is, the sampling rate must be at least twice the desired frequency to be measured.

For a time record of \( T \) seconds, the lowest frequency component measurable is \( \Delta f = \frac{1}{T} \)

With these two properties above, the sampling parameters can be summarized as

\[
\begin{align*}
  f_{\text{max}} &= \frac{1}{2 \Delta t} \\
  \Delta t &= \frac{1}{2 f_{\text{max}}}
\end{align*}
\]
Sampling Parameters

Due to the Rayleigh Criteria and Shannon’s Sampling Theorem, the following sampling parameters must be observed.

With respect to the number of sample increments per period $N$:

\[ T = N \Delta t \quad \text{and} \quad BW = N \Delta f / 2 \]

where

- $\Delta t$ - sample interval; time resolution
- $N$ - # of data points
- $T$ - sample record length
- $f_{max}$ - highest desired frequency - BW
- $f_s$ - sampling frequency
- $\Delta f$ - frequency resolution

Note: Time Domain
N real data points
Frequency Domain
N/2 real data points
N/2 imaginary points
### Sampling Parameters

Due to the Rayleigh Criteria and Shannon’s Sampling Theorum, the following sampling parameters must be observed.

<table>
<thead>
<tr>
<th>PICK</th>
<th>THEN</th>
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<tr>
<td>Δt</td>
<td>f_{\text{max}} = 1 / (2 Δt)</td>
<td>T = N Δt</td>
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If we choose $Δf = 5 \text{ Hz}$ and $N = 1024$ then $T = 1 / Δf = 1 / 5 \text{ Hz} = 0.2 \text{ sec}$

$f_s = N Δf = (1024) (5 \text{ Hz}) = 5120 \text{ Hz}$

$f_{\text{max}} = f_s = (5120 \text{ Hz}) / 2 = 2560 \text{ Hz}$
Sampling Relationship

An inverse relationship between time and frequency exists

Given \( \delta t = 0.0019531 \) and \( N = 1024 \) time points, then \( T = 2 \) sec and \( BW = 256 \) Hz and \( \delta f = 0.5 \) Hz

Given \( \delta t = 0.000976563 \) and \( N = 1024 \) time points, then \( T = 1 \) sec and \( BW = 512 \) Hz and \( \delta f = 1 \) Hz

Given \( \delta t = 0.0019531 \) and \( N = 512 \) time points, then \( T = 1 \) sec and \( BW = 256 \) Hz and \( \delta f = 1 \) Hz
Quantization Error

Sampling refers to the rate at which the signal is collected. Quantization refers to the amplitude description of the signal.

A 4 bit ADC has $2^4$ or 16 possible values
A 6 bit ADC has $2^6$ or 64 possible values
A 12 bit ADC has $2^{12}$ or 4096 possible values
Quantization Error

Quantization errors refer to the accuracy of the amplitude measured. The 6 bit ADC represents the signal shown much better than a 4 bit ADC.
Quantization Error

Underloading of the ADC causes amplitude errors in the signal

All of the available dynamic range of the analog to digital converter is not used effectively.

0.5 volt signal

This causes amplitude and phase distortion of the measured signal in both the time and frequency domains.
**AC Coupling**

A large DC bias can cause amplitude errors in the alternating part of the signal. AC coupling uses a high pass filter to remove the DC component from the signal. All of the available dynamic range of the analog to digital converter is dominated by the DC signal. The alternating part of the signal suffers from quantization error. This causes amplitude and phase distortion of the measured signal.
Clipping and Overloading

Overloading of the ADC causes severe errors also.

The ADC range is set too low for the signal to be measured and causes clipping of the signal.

1.5 volt signal

This causes amplitude and phase distortion of the measured signal in both the time and frequency domains.

1 volt range on ADC
The Fourier Transform

Forward Fourier Transform

\[ S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} \, dt \]

and Inverse Fourier Transform

\[ x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} \, df \]
Discrete Fourier Transform

Even though the actual time signal is continuous, the signal is discretized and the transformation at discrete points is

\[ S_x(m\Delta f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi m\Delta f t} \, dt \]

This integral is evaluated as

\[ S_x(m\Delta f) \approx \Delta t \sum_{n=-\infty}^{+\infty} x(n\Delta t)e^{-j2\pi m\Delta f n\Delta t} \]

However, if only a finite sample is available (which is generally the case), then the transformation becomes

\[ S_x(m\Delta f) \approx \Delta t \sum_{n=0}^{N-1} x(n\Delta t)e^{-j2\pi m\Delta f n\Delta t} \]
Fourier Transform and FFT

Actual Time Signal

Captured Time Signal

Reconstructed Time Signal

Frequency Spectrum
Fourier Transform and FFT

Actual Time Signal

Captured Time Signal

Reconstructed Time Signal

Frequency Spectrum
Leakage

Periodic Signal
Non-Periodic Signal

Leakage due to signal distortion
Leakage

When the measured signal is not periodic in the sample interval, incorrect estimates of the amplitude and frequency occur. This error is referred to as leakage.

Basically, the actual energy distribution is smeared across the frequency spectrum and energy leaks from a particular $\Delta f$ into adjacent $\Delta f$ s.

Leakage is probably the most common and most serious digital signal processing error. Unlike aliasing, the effects of leakage can not be eliminated.
Windows - Minimize Leakage

In order to better satisfy the periodicity requirement of the FFT process, time weighting functions, called windows, are used. Essentially, these weighting functions attempt to heavily weight the beginning and end of the sample record to zero - the middle of the sample is heavily weighted towards unity.
Windows - Rectangular/Hanning/Flattop

In order to better satisfy the periodicity requirement of the FFT process, time weighting functions, called windows, are used. Essentially, these weighting functions attempt to heavily weight the beginning and end of the sample record to zero - the middle of the sample is heavily weighted towards unity.

**Rectangular** - Unity gain applied to entire sample interval; this window can have up to 36% amplitude error if the signal is not periodic in the sample interval; good for signals that inherently satisfy the periodicity requirement of the FFT process.

**Hanning** - Cosine bell shaped weighting which heavily weights the beginning and end of the sample interval to zero; this window can have up to 16% amplitude error; the main frequency will show some adjacent side band frequencies but then quickly attenuates; good for general purpose signal applications.

**Flat Top** - Multi-sine weighting function; this window has excellent amplitude characteristics (0.1% error) but very poor frequency resolution; very good for calibration purposes with discrete sine
Windows - Rectangular/Hanning/Flattop

Time weighting functions are applied to minimize the effects of leakage

Rectangular
Hanning
Flat Top
and many others

Windows DO NOT eliminate leakage !!!
Windows - Rectangular

The rectangular window function is shown below. The main lobe is narrow, but the side lobes are very large and roll off quite slowly. The main lobe is quite rounded and can introduce large measurement errors. The rectangular window can have amplitude errors as large as 36%.
The hanning window function is shown below. The first few side lobes are rather large, but a 60 dB/octave roll-off rate is helpful. This window is most useful for searching operations where good frequency resolution is needed, but amplitude accuracy is not important; the hanning window will have amplitude errors of as much as 16%.
The flat top window function is shown below. The main lobe is very flat and spreads over several frequency bins. While this window suffers from frequency resolution, the amplitude can be measured very accurately to 0.1%.
Windows

Rectangular  Hanning  Flat Top
Windows - Force/Exponential for Impact Testing

Special windows are used for impact testing

Force window
Windows - Force/Exponential for Impact Testing

Special windows are used for impact testing

Exponential window
Hammer Tips for Impact Testing

METAL TIP

TIME PULSE

FREQUENCY SPECTRUM

HARD PLASTIC TIP

TIME PULSE

FREQUENCY SPECTRUM

SOFT PLASTIC TIP

TIME PULSE

FREQUENCY SPECTRUM

RUBBER TIP

TIME PULSE

FREQUENCY SPECTRUM
Pretrigger Delay and Double Impacts

Pretrigger delay used to reduce the amount of frequency spectrum distortion

Double impacts should be avoided due to the distortion of the frequency spectrum and force dropout that can occur
Exponential Window

If the signal does not naturally decay within the sample interval, then an exponentially decaying window may be necessary.

However, many times changing the signal processing parameters such as bandwidth and number of spectral lines may produce a signal which requires less window weighting.
Measurement - Linear Spectra

\[ \begin{align*}
&x(t) \quad h(t) \quad y(t) \\
&S_x(f) \quad H(f) \quad S_y(f)
\end{align*} \]

- **x(t)** - time domain input to the system
- **y(t)** - time domain output to the system
- **S_x(f)** - linear Fourier spectrum of **x(t)**
- **S_y(f)** - linear Fourier spectrum of **y(t)**
- **H(f)** - system transfer function
- **h(t)** - system impulse response
Measurement - Linear Spectra

\[ x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} \, df \]
\[ S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} \, dt \]

\[ y(t) = \int_{-\infty}^{+\infty} S_y(f) e^{j2\pi ft} \, df \]
\[ S_y(f) = \int_{-\infty}^{+\infty} y(t) e^{-j2\pi ft} \, dt \]

\[ h(t) = \int_{-\infty}^{+\infty} H(f) e^{j2\pi ft} \, df \]
\[ H(f) = \int_{-\infty}^{+\infty} h(t) e^{-j2\pi ft} \, dt \]

Note: \( S_x \) and \( S_y \) are complex valued functions
**Measurement - Power Spectra**

- **Rxx(t)** - autocorrelation of the input signal \( x(t) \)
- **Ryy(t)** - autocorrelation of the output signal \( y(t) \)
- **Ryx(t)** - cross correlation of \( y(t) \) and \( x(t) \)

\[
\begin{align*}
R_{xx}(t) &= \text{autocorrelation of the input signal } x(t) \\
R_{yy}(t) &= \text{autocorrelation of the output signal } y(t) \\
R_{yx}(t) &= \text{cross correlation of } y(t) \text{ and } x(t)
\end{align*}
\]

- **Gxx(f)** - autopower spectrum of \( x(t) \)
- **Gyy(f)** - autopower spectrum of \( y(t) \)
- **Gyx(f)** - cross power spectrum of \( y(t) \) and \( x(t) \)

\[
\begin{align*}
G_{xx}(f) &= S_{x}(f) \cdot S_{x}^{*}(f) \\
G_{yy}(f) &= S_{y}(f) \cdot S_{y}^{*}(f) \\
G_{yx}(f) &= S_{y}(f) \cdot S_{x}^{*}(f)
\end{align*}
\]
**Measurement - Power Spectra**

\[ R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t + \tau) dt \]

\[ G_{xx}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j2\pi ft} d\tau = S_x(f) \cdot S_x^*(f) \]

\[ R_{yy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)y(t + \tau) dt \]

\[ G_{yy}(f) = \int_{-\infty}^{+\infty} R_{yy}(\tau) e^{-j2\pi ft} d\tau = S_y(f) \cdot S_y^*(f) \]

\[ R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)x(t + \tau) dt \]

\[ G_{yx}(f) = \int_{-\infty}^{+\infty} R_{yx}(\tau) e^{-j2\pi ft} d\tau = S_y(f) \cdot S_x^*(f) \]
The Frequency Response Function and Coherence

\[ S_y = HS_x \]

**H1 formulation**
- susceptible to noise on the input
- underestimates the actual \( H \) of the system

\[ S_y \cdot S_x^* = H S_x \cdot S_x^* \]

**COHERENCE**

\[ \gamma_{xy}^2 = \frac{(S_y \cdot S_x^*)(S_x \cdot S_y^*)}{(S_x \cdot S_x^*)(S_y \cdot S_y^*)} = \frac{G_{yx}}{G_{xx}} / \frac{G_{yy}}{G_{xy}} = \frac{H_1}{H_2} \]
Typical Measurements

Measurements - Auto Power Spectrum

- \( x(t) \)
- \( y(t) \)
- \( G_{xx}(f) \)
- \( G_{yy}(f) \)

Measurements - Cross Power Spectrum

- \( G_{xx}(f) \)
- \( G_{yy}(f) \)
- \( G_{yx}(f) \)

Measurements - Frequency Response Function

- \( G_{xx}(f) \)
- \( G_{yy}(f) \)
- \( G_{yx}(f) \)

- \( H(f) \)

Measurements - FRF & Coherence

- Frequency Response Function
- Coherence
Hammers and Tips

![Graph of Hammers and Tips]

- 

frf: Frequency Response Function

coherence: Coherence Function

input power spectrum: Input Power Spectrum
Leakage and Windows for Impact Testing

ACTUAL TIME SIGNAL

SAMPLED SIGNAL

WINDOW WEIGHTING

WINDOWED TIME SIGNAL
Simple time-frequency response relationship

Increasing rate of oscillation →

FORCE

RESPONSE

WOW !!!

Time

Frequency
Sine Dwell to Obtain Mode Shape Characteristics
Mode Shape Characteristics for a Simple Beam
**Mode Shape Characteristics for a Simple Plate**

**MODE 1**

**MODE 2**
Why and How Do Structures Vibrate?

Motor or disk unbalance can cause unwanted vibrations or worse

**Ouch!!!**

**Oops!!!**

INPUT TIME FORCE

\[ f(t) \]

FFT

INPUT SPECTRUM

\[ f(j\omega) \]

FREQUENCY RESPONSE FUNCTION

\[ h(j\omega) \]

OUTPUT SPECTRUM

\[ y(j\omega) \]

OUTPUT TIME RESPONSE

\[ y(t) \]
HP 35660 FFT Dual Channel Analyzer
HP 35660 FFT Dual Channel Analyzer